Analytic continuation and appearance of a new phase for $n<1$ in n-component $(\boldsymbol{\phi}^2)^2$ field theory

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We study the analytic continuation of the renormalized $(\boldsymbol{\phi}^2)^2$ field theory in the presence of a magnetic field H as a function of n, the number of components of the field $\boldsymbol{\phi}$, to all *real* $n\geq 0$. We focus our attention at low temperatures ($T<T_c$) for dimensionality d in the range $2<d\leq 4$. We show that while for $n\geq 1$ one can reach the limit $H\to 0$ below T_c without encountering any singularity in the analytic continuation due to $n-1$ transverse modes, these modes become catastrophic when n is less than unity for all *nonzero* H lying below some curve AC defined by $H=H_c(T)>0$ for $T<T_c$ in the H-T plane: The theory cannot be analytically continued below AC for $n<1$. The catastrophe is caused by n as soon as it becomes less than unity. We calculate the form of AC to first order near T_c. For $n=0$, the paramagnetic phase of the theory describes the dilute regime of the polymer solution as expected. However, we argue that the phase below AC must describe a new phase for polymers, and that there must be a transition across AC from a dilute system to the new phase.

I. INTRODUCTION

The n-component Euclidean $(\boldsymbol{\phi}^2)^2$ field theory has been of considerable interest in statistical mechanics because of its close connection with the classical n-vector model with an $O(n)$-symmetric interaction.[1] The Hamiltonian density $\mathscr{H}_0$ of the above field theory, also known as the linear σ model,[2,3] is given by the following form, defined in a d-dimensional Euclidean space:

$$\mathscr{H}_0=\tfrac{1}{2}(\partial\boldsymbol{\phi}_0)^2+\tfrac{1}{2}m_0^2\boldsymbol{\phi}_0^2+\frac{1}{4!}\lambda_0(\boldsymbol{\phi}^2)^2 . \tag{1}$$

Here, $\boldsymbol{\phi}_0$ represents an n-component classical real field $\boldsymbol{\phi}_0=\{\phi_0^{(\alpha)}\,|\,\alpha=1,2,\ldots,n\}$, and

$$\boldsymbol{\phi}_0^2=\sum_{\alpha=1}^{n}(\phi_0^{(\alpha)})^2,\quad (\partial\boldsymbol{\phi}_0)^2=\sum_{i=1}^{d}\sum_{\alpha=1}^{n}(\partial_i\phi_0^{(\alpha)})^2 . \tag{2}$$

The theory is mathematically well defined when the parameters n and d are positive integers. However, it is a standard practice in renormalization-group studies to treat d and n as continuous *real* variables.[1,4–7] Therefore, one must consider, at least in a formal sense, the above theory for all positive real values of n and d. In the following we will restrict ourselves to $2<d\leq 4$. Moreover, we will also restrict n to positive real values ($n\geq 0$) even though its extension to negative values is also of some physical significance.[6,7]

In order to define the theory for all $n\geq 0$, we must invoke an analytic continuation of the theory. The motivation for the analytic continuation is provided by an analogy between self-avoiding walks and the $n\to 0$ limit of the $(\boldsymbol{\phi}^2)^2$ field theory.[5,8,9] This analogy provides a certain "reality" to the theory when n is continued to zero. This analytic continuation raises an important question: Does there exist a sensible theory corresponding to (1) as $n\to 0$, or for that matter, any real value of n not necessarily an integer? This question must be faced whenever one tries to perform an analytic continuation of a theory which is mathematically well defined only for certain integer values of a certain parameter (for example, n or d in the present theory), to *all* real values of that parameter. It is primarily this question that we will attempt to answer here. Eventually, this will enable us to draw conclusions about the appearance of a "new" phase for $n<1$ at low temperatures. Throughout this paper we will consider the analytic continuation of (1) in n only, treating d as given and fixed at some real value. Thus we assume that we know how to continue the theory to arbitrary d and there are no peculiarities involved with this analytic continuation in d.

The low-temperature phase associated with $T<T_c$ and $H\to 0$, where T_c denotes the critical temperature and H is the external magnetic field, is physically relevant since, as $n\to 0$, this phase is supposed to describe the scaling limit of the semidilute regime of polymers, i.e., self-avoiding random walks.[8] In contrast, the high-temperature phase ($T>T_c$) is supposed to describe the dilute solution of polymers.[8] In the dilute solutions of polymers, the polymer chains are practically nonoverlapping, while they overlap strongly in the semidilute regime. It should be emphasized at this point that the analogy between the polymer system and the $n\to 0$ limit can be established[5,8,9] *only* in the high-temperature phase ($T>T_c$), where all the n components of the field are treated identically. Thus, each loop, which can be formed from any of the n equivalent field components, contributes a factor of n after a summation over the field components. The analogy is established by noting that as $n\to 0$, all loops disappear from the Feynman diagrams. On the other hand, at low temperatures, there is a *distinction* between the longitudinal and the $n-1$ transverse modes. Thus, the analogy *cannot* be explicitly established here by comparing the Feynman diagrams of the field theory as $n \to 0$ with those of the polymer system. One usually *assumes* implicitly that the analogy works even below T_c, just at it works above T_c. However, we wish to emphasize that it is not an obvious assumption. The rationale for this assumption is the hope that as one continues the theory along the path P_1P_2 (see Fig. 1) from $T > T_c$ to $T < T_c$, one does not encounter any singularity, except possibly at $H=0$ for any $n \geq 0$. Indeed, we will establish here that such analytic continuation *cannot* be carried through to $H=0$ below T_c for $n<1$. We do this explicitly by establishing the appearance of various pathologies as the theory is analytically continued below AC in Fig. 1(b), *the form of pathology depending on the scheme adopted for renormalization.* This indicates that the phase below AC *cannot* be obtained analytically from the phase above it. The appearance of the new phase below AC can be demonstrated *only* in the thermodynamic limit $V \to \infty$: The thermodynamic limit must be performed before the analytic continuation in n is performed and n is taken to be less than 1. The order of limit is *very* important.

The strength of our arguments lies in the following two observations:

(i) The need for the analytic continuation in n is evident since the theory is not defined physically for $n=0$, which is the case of prime interest here. As is common practice, the analytic continuation is defined perturbatively. (This is the *only* way known to the author that can clearly distinguish between the longitudinal and transverse modes.) It is natural to demand that the proposed analytic continuation satisfies certain *positivity* conditions for all real $n \geq 1$. (This is not necessary, but is sufficient to ensure the fulfillment of the positivity conditions for integer n.) Now it can be shown that under some mild assumptions[10] (which are usually assumed either implicitly or explicitly in most analytic continuations in physics) this analytic continuation is *unique.* What we discover in this work is that this unique analytic continuation becomes meaningless below AC, for $n<1$. We argue that the phase below AC must be distinct from the phase above AC. This is a very important observation since this implies that the corresponding polymer system $(n=0)$ must also *develop* a new phase below AC. This is a new and surprising result.

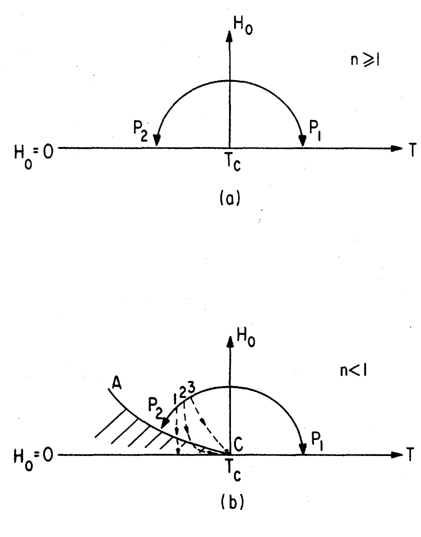

FIG. 1. Phase diagram for the $O(n)$ model. (a) $n \geq 1$. One can bypass the singularity at $H=0$, $T=T_c$ by following a path P_1P_2. (b) $n<1$. A new phase (hatched area) appears below AC. One cannot cross AC while continuing the theory along P_1P_2.

(ii) The catastrophe developed below AC is not due to the singularities induced by the Nambu-Goldstone modes in the theory, as it might appear at first sight. To appreciate this point, one should note that the catastrophe appears at *finite* magnetic field (but only for $n<1$), where the Nambu-Goldstone modes are not critical. This catastrophe is genuinely induced by n when it is less than unity and is, in fact, quite independent from the singularities due to the Nambu-Goldstone modes. Thus, there is *no* need to be careful about the effects of the Nambu-Goldstone singularities. As a matter of fact, we point out that the catastrophe *persists* for $n<1$, even when one carefully takes into account the Nambu-Goldstone singularities.[11,12] However, an independent and stronger argument in support of this claim comes from the observation that the polymer system that is identical (above AC) to the $n=0$ limit of the $O(n)$ model with continuous symmetry described here is *also* identical to the $n=0$ limit of another n-vector model but with *discrete* symmetry.[13] Since the polymer system has a transition along AC [or, since the $O(n)$ model has a transition along AC for $n=0$], the discrete model must also have a transition along AC $(n=0)$. However, this transition in the discrete model cannot be produced by any Nambu-Goldstone modes since they are absent when the symmetry is discrete. This justifies our claim.

We now summarize our results:

(i) We study the renormalizibility of the theory in the presence of an external magnetic field at low temperatures $(T<T_c)$. The infrared behavior of the theory is regulated by applying an external magnetic field H. We must make a distinction between the longitudinal and the $n-1$ transverse modes: These two modes have different masses as $H \to 0$. This is accomplished by carrying out an asymmetric renormalization which is *distinct* from the symmetric renormalization of the theory in the symmetric phase $(T > T_c)$, where the two modes have the same mass as $H \to 0$. This is in contrast with the result due to Lee[3] which states that the renormalization of the theory is identical in both phases. We also explain why we expect our result to be different from that due to Lee.

Using our asymmetric renormalization scheme, we find the following two pathologies depending on what is kept fixed in our scheme:

(a) Keeping the renormalized coupling λ *fixed* and *positive,* we establish by explicit calculation, to the two-loop level, that the theory exists for *all* real $n \geq 1$ as $H \to 0$. This means that the theory can be analytically continued through T_c along the path P_1P_2 all the way to $H \to 0$ [see Fig. 1(a)]. For $n<1$ the analytic continuation of the theory does *not* exist for H lying in the region below the curve AC in Fig. 1(b). This anomalous behavior means that one *cannot* analytically continue the theory from the phase above AC into the phase below AC for $n<1$. The new phase below AC must be quite distinct from the phase above it. However, it is not clear (i) what is the nature of the transition across AC, and (ii) what is the nature of the new phase below AC.

(b) If, instead, we keep the bare couplings *fixed* and *positive*, the anomaly appears in a different way. One of the renormalized couplings develops a *pole* on AC, indicating the failure of the perturbation expansion for $n<1$. Again, the theory cannot be continued analytically below AC.

(ii) In order to show that the existence of an anomaly is *not* due to our choice of an asymmetric renormalization, we consider the equation of state that is obtained in a symmetric renormalization scheme.[11,12] We find that the anomaly now appears in the form of a complex solution of the equation of state below AC, as soon as $n<1$, turning the polymer analogy into nonsense. For $n\geq 1$ the equation is physical all the way up to $H=0$.

(iii) We calculate the form of AC near $T=T_c$. On AC the magnetic field H_0 is related to $\tau=(T-T_c)/T_c$ by $H_0\sim\tau^{\gamma+\beta}$. This means that for $d<4$ any path on which one can define the semidilute limit of the polymer solution must necessarily cross the boundary AC [see curves 1 and 2 in Fig. 1(b)]. Any curve approaching the critical point such that it lies above AC must necessarily describe the dilute limit of the polymer solution [see curve 3 in Fig. 1(b)]. Thus, our study casts doubts on the identification of the semidilute regime of the polymer solution as proposed by des Cloizeaux.[8]

All of our conclusions are based on the fact that there is a *nonzero* spontaneous magnetization in the ordered, i.e., nonsymmetric, phase ($T<T_c$). For $n<1$ the analytic continuation of the theory becomes meaningless below AC. No such catastrophe is seen for any real $n\geq 1$. Our conclusion about the present catastrophe for $n<1$ does *not* apply to the symmetric phase that does not have a nonzero spontaneous magnetization: There is no anomaly in the high-temperature phase ($T>T_c$) for all $n>0$ (see Fig. 1). Since the existence of the spontaneous magnetization for arbitrary n is possible only in the infinite-volume limit ($V\to\infty$) for $d>2$, the present observation of the anomaly for $n<1$ at $T<T_c$ is distinct from observations made previously[10,14] about the violations of convexity that occur even for systems of finite size such as, for example, a single spin. A negative susceptibility does *not* imply the breakdown of the analytic continuation, as is evident from the single-spin case in a magnetic field.[14]

We should emphasize that the pathologies noted here are really due to the fact that n is less than unity, and *not* due to any infinities appearing in any Feynman diagrams. Throughout our analysis in the asymmetric renormalization scheme, the ultraviolet cutoff $\Lambda=(\text{lattice spacing})^{-1}$ is kept fixed and finite. Therefore, there are no ultraviolet divergences in the theory. Moreover, since the catastrophes appear at a nonzero value of the magnetic field, there are no infrared divergences in the theory either.

We do not intend to study here all the implications of the present catastrophe ($n=0$) for the corresponding polymer problem. However, we will quote the results of a preliminary study which suggests that polymers below the curve AC are in a "collapsed" phase ($\nu=1/d$), and are markedly different from polymers above AC that are in a "swollen" phase ($\nu>1/2$). The nonexistence of the analytic continuation of the $O(n)$ model for $n=0$ below AC is probably an indication of such a collapse transition. What is remarkable is that such a *collapse* occurs in a system that only has *repulsive interactions. A more detailed analysis is currently under way* and will be reported elsewhere.

The layout of the paper is as follows. We consider briefly, in Sec. II, the notion of the Nambu-Goldstone modes of the symmetry and their implications for the renormalizability of the theory. In Sec. III we develop our asymmetric renormalization scheme and express bare couplings in terms of renormalized parameters. The analysis is simplified by the fact that the perturbation series we obtain are well behaved. We show that a catastrophe appears in the analytic continuation of the theory for $n<1$ when H becomes sufficiently small [the phase below AC in Fig. 1(b)]. In Sec. IV we consider the problem of expressing the renormalized quantities in terms of bare quantities in our asymmetric renormalization scheme. The analysis here becomes very complicated as the perturbation series we obtain are not well behaved. However, we propose a method of expressing the above relations in a manner which remains sensible and consistent for $n\geq 1$. For $n<1$ these relations encounter poles on AC, again indicating the appearance of an anomaly below AC. In Sec. V we discuss the equation of state that is obtained in a symmetric renormalization scheme and show that the anomaly is present, albeit in a different form. This equation becomes meaningless below AC for $n<1$ and $T<T_c$. In Sec. VI we describe the analogy between the semidilute regime of the polymer solution and the $n\to 0$ limit as proposed by des Cloizeaux and show that this requires crossing the line AC for $d<4$ to describe the scaling limit of the semidilute regime. In the final section we discuss various implications of our results.

II. NAMBU-GOLDSTONE MODES AND ASYMMETRIC RENORMALIZATION

We now proceed with the description of the Nambu-Goldstone modes of $O(n)$ symmetry at low temperatures and the renormalizability of the theory. However, before considering the spontaneously-broken-symmetry phase and the Nambu-Goldstone modes, let us briefly consider what is known about the renormalizability of the theory described by (1) at high temperatures. One must renormalize the theory to make it insensitive to variations in lattice spacing $a=1/\Lambda$. Let us introduce the renormalized quantities $\boldsymbol{\phi}$, m, and λ. We rewrite (1) as follows:

$$\mathcal{H}_0=\mathcal{H}+\mathcal{H}_{\rm ct}\,, \tag{3}$$

where

$$\mathcal{H}=\tfrac{1}{2}(\partial\boldsymbol{\phi})^2+\tfrac{1}{2}m^2\boldsymbol{\phi}^2+\frac{1}{4!}\lambda(\boldsymbol{\phi}^2)^2\,, \tag{4}$$

and $\mathscr{H}_{ct}$ represents all the counterterms required to make the theory finite as the ultraviolet cutoff Λ tends to infinity.[15] It is well known[15] that to an arbitrary high order in λ only counterterms which match the original Hamiltonian in form are needed to render the theory finite.

The low-temperature phase $(T<T_c)$ is conveniently studied by applying an external magnetic field H_0 along the $\alpha=1$ direction.[2,3] The external field also acts as an infrared regulator in the theory by providing a mass to the $n-1$ transverse modes as we will see below. In the presence of the magnetic field, the new Hamiltonian is given by

$$\mathscr{H}'_0=\mathscr{H}_0-H_0\phi_0^{(1)} .$$

The presence of H_0 induces a nonzero magnetization $M_0=\langle\phi_0^{(1)}\rangle$ along the $\alpha=1$ direction. Here, $\langle A\rangle$ denotes the thermodynamic average of some quantity $A[\phi_0]$:

$$\langle A\rangle=\frac{\int\mathscr{D}[\phi_0]A[\phi_0]\exp(-\mathscr{H}'_0)}{\int\mathscr{D}[\phi_0]\exp(-\mathscr{H}'_0)} .$$

Following Gell-Mann and Lévy,[2] we subtract M_0 from $\phi_0^{(1)}$ and define

$$\phi_{L0}=\phi_0^{(1)}-M_0 .$$

Evidently, $\langle\phi_{L0}\rangle=0$. We denote the $n-1$ transverse components $\phi_0^{(2)},\phi_0^{(3)},\ldots,\phi_0^{(n)}$ collectively by $\boldsymbol{\phi}_{T0}$:

$$\boldsymbol{\phi}_{T0}=\{\phi_0^{(\beta)}\,|\,\beta=2,3,\ldots,n\} .$$

In terms of ϕ_{L0} and $\boldsymbol{\phi}_{T0}$, we can rewrite $\mathscr{H}'_0$ as follows:

$$\begin{aligned}\mathscr{H}'_0=&\ \tfrac{1}{2}[(\partial\phi_{L0})^2+(\partial\boldsymbol{\phi}_{T0})^2+m_{L0}^2\phi_{L0}^2+m_{T0}^2\boldsymbol{\phi}_{T0}^2]\\&+\frac{1}{3!}(\lambda_{10}\phi_{L0}^3+\lambda_{20}\phi_{L0}\boldsymbol{\phi}_{T0}^2)\\&+\frac{1}{4!}[\lambda_{L0}\phi_{L0}^4+2\lambda_{LT0}\phi_{L0}^2\boldsymbol{\phi}_{T0}^2+\lambda_{T0}(\boldsymbol{\phi}_{T0}^2)^2]\\&-(H_0-m_{T0}^2M_0)\phi_{L0} ,\end{aligned} \tag{5}$$

where

$$\begin{aligned}&m_{L0}=m_0^2+\tfrac{1}{2}\lambda_0M_0^2,\quad m_{T0}^2=m_0^2+\tfrac{1}{6}\lambda_0M_0^2 ,\\&\lambda_{10}=\lambda_{20}=\lambda_0M_0,\quad \lambda_{L0}=\lambda_{LT0}=\lambda_{T0}=\lambda_0 ,\end{aligned} \tag{6}$$

and where we have neglected constant terms that are not important for our discussion. To the lowest order, we find that

$$H_0=m_{T0}^2M_0 .$$

For $H_0=0$ and $M_0>0$, $m_{T0}^2=0$. This is possible if $m_0^2<0$. This case is the *Nambu-Goldstone modes* of the symmetry, with the $\boldsymbol{\phi}_{T0}$ field playing the role of the Nambu-Goldstone modes. The longitudinal mass $m_{L0}^2=-2m_0^2>0$. Thus, the two fields are nondegenerate.

Let us now introduce the renormalized quantities to be denoted by symbols without the subscript 0. We identify m_L^2 and m_T^2 as the inverse of the longitudinal and the transverse susceptibilities, respectively. It is easily shown[16] that $m_T^2=H/M$. Thus, as $H\to0$, $m_T^2\to0$ and causes infrared singularities in the theory. For example, we will see in Sec. IV [see (25)] that the renormalized longitudinal susceptibility $\chi_L=1/m_L^2$ is given by

$$\chi_L=\frac{1}{m_L^2}\simeq\chi_{L0}\left[1+\left[\frac{n-1}{18}\right]\chi_{L0}\lambda_{20}^2I(m_T)\right] , \tag{7}$$

where $\chi_{L0}=1/m_{L0}^2$, and $I(m_T)$ is given in (10) below. (Here, we have replaced m_{T0} by m_T in I to this order.) The function $I(m_T)$ diverges as m_T vanishes, i.e., as H vanishes. For $d=4$, $I(m_T)$ diverges[16] as $\ln(1/H)$, and for $d=4-\epsilon$, $\epsilon>0$, it diverges[16,17] as $H^{-\epsilon/2}$ as $H\to0$. The *important* observation is to note that as $H\to0$, m_T vanishes much *faster* than m_L. Therefore, the singularities caused by vanishing m_T are much stronger than those due to vanishing m_L. This distinction between the rates at which m_L and m_T tend to zero forces us to renormalize our theory in the nonsymmetric phase in a *different* way than that in the symmetric phase, as will be seen below. This contradicts the result due to the Lee which says that the renormalizibility of the theory is identical in both phases.[3] The reason is easily understood. In field theory we are interested in the behavior of the theory in the large-momentum region $\Lambda\to\infty$ and, therefore, the distinction between m_L and m_T, which are much smaller than Λ, is not important: We can treat the two fields as degenerate. Thus, one can use the counterterms of symmetric theory in the ordered phase. Since our interest is in studying the effect of the Nambu-Goldstone modes, we must treat the two masses on different footings. Thus, the renormalizability of the theory in the nonsymmetric phase turns out to be very different from that in the symmetric phase. We wish to emphasize that the new renormalization prescription is indeed the correct one if, for example, we wish to predict the divergence in χ_L for $n=2$ below T_c, which is consistent with the rigorous results due to Lebowitz and Penrose.[18] They have shown on rigorous grounds that the divergence of χ_L below T_c for $n=2$ is at least as strong as $\ln(1/H)$ in $d=4$ and as strong as $H^{-1/2}$ in $d=3$.

III. FIXED RENORMALIZED PARAMETERS: ASYMMETRIC RENORMALIZATION

We now consider the situation when the renormalized parameters m^2 and $\lambda>0$ are held fixed as the magnetic field H is varied. We will find that the analysis in this situation is simplified because the perturbation series we obtain here are well behaved. This simplification allows us to draw various conclusions about the nature of the analytic continuation and the renormalizability of the theory, even though it is the bare quantities we must keep fixed to draw any conclusions relevant for statistical mechanics. However, once we understand the nature of the analytic continuation and the renormalizability for fixed renormalized quantities m^2 and λ, we can use this information to study the situation in the next section, when the bare parameters m_0^2 and λ_0 are kept fixed.

We impose an external magnetic field H to control the infrared behavior of the theory at low temperatures $(T<T_c)$. The corresponding Hamiltonian $\mathscr{H}'$ in terms of

the renormalized quantities is obtained from (5) by deleting the subscript 0. This Hamiltonian is characterized by two nondegenerate masses m_L^2 and m_T^2; two cubic vertices, $\lambda_1\phi_L^3/3!$ and $\lambda_2\phi_L\phi_T^2/3!$; and three quartic vertices, $\lambda_L\phi_L^4/4!$, $2\lambda_{LT}\phi_L^2\phi_T^2/4!$, and $\lambda_T(\phi_T^2)^2/4!$. The new parameters are related to m^2 and λ in the following manner:

$$m_L^2=m^2+\tfrac{1}{2}\lambda M^2,\quad m_T^2=m^2+\tfrac{1}{6}\lambda M^2\,,$$
$$\lambda_1=\lambda_2=\lambda M,\quad \lambda_L=\lambda_{LT}=\lambda_T=\lambda \tag{8}$$

[cf. (6)]. The mass m_T^2 is related to H and M by $m_T^2=H/M$. Moreover, since $\lambda>0$ and since M is also expected to be non-negative for $n\geq 0$, we note that $m_L^2>m_T^2$. As $H\to 0$, m_T^2 vanishes but m_L^2 takes a non-negative finite value, $\lambda M^2/3$. As H vanishes, the vanishing mass of the transverse modes will cause infrared singularities in our theory. This will make diagrams with the maximum number of the transverse propagators the most dominant ones at any given order in perturbation.

Let us now calculate various bare coupling constants in terms of the renormalized ones. First, let us consider the bare parameter λ_{L0}. In the one-loop calculation, the diagrams required are shown in Figs. 2(a)—2(b). Since our interest is in studying the implications of the difference between the two masses, we will keep the ultraviolet cutoff Λ *fixed* throughout the paper. It is easily seen that

$$\lambda_{L0}=\lambda_L+\tfrac{3}{2}\lambda_L^2 I(m_L)+\frac{n-1}{6}\lambda_{LT}^2 I(m_T)\,. \tag{9}$$

Here,

$$I(a)=S\int_0^\Lambda d^dq/(q^2+a^2)^2\,, \tag{10}$$

with

$$S=(1/2\pi)^2[2\pi^{d/2}/\Gamma(d/2)]\,.$$

Before expressing λ_{LT0} and λ_{T0} in terms of the renormalized coupling constants, we note that the diagram shown in Fig. 2(c) contributes only to λ_{LT0}, but is absent for both λ_{L0} and λ_{T0}. This is the first indication that the three quartic coupling constants renormalize *differently*.[19] Indeed, it is easily seen that

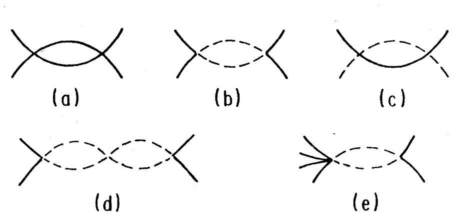

FIG. 2. Solid line represents the longitudinal mode and dashed line represents one of the $n-1$ transverse modes.

$$\lambda_{LT0}=\lambda_{LT}+\tfrac{1}{2}\lambda_L\lambda_{LT}I(m_L)+\tfrac{2}{3}\lambda_{LT}^2 I(m_L,m_T)+\frac{n+1}{6}\lambda_{LT}\lambda_T I(m_T)\,, \tag{11}$$

$$\lambda_{T0}=\lambda_T+\tfrac{1}{6}\lambda_{LT}^2 I(m_L)+\frac{n+7}{6}\lambda_T^2 I(m_T)\,, \tag{12}$$

where

$$I(a,b)=S\int_0^\Lambda d^dq/[(q^2+a^2)(q^2+b^2)]\,.$$

Now it is evident from Eqs. (9), (11), and (12) that the three quartic coupling constants renormalize differently, because the two masses are different. This observation does not appear to have been appreciated before in the literature. For the two cubic vertices, we find, to the one-loop level, that

$$\lambda_{10}=\lambda_1+\tfrac{3}{2}\lambda_1\lambda_L I(m_L)+\frac{n-1}{6}\lambda_2\lambda_{LT}I(m_T)\,, \tag{13}$$

$$\lambda_{20}=\lambda_2+\tfrac{1}{2}\lambda_1\lambda_{LT}I(m_L)+\tfrac{2}{3}\lambda_2\lambda_{LT}I(m_L,m_T)+\frac{n+1}{6}\lambda_2\lambda_T I(m_T)\,. \tag{14}$$

Indeed, we find that the graphs required for λ_{10} (or λ_{20}) can be obtained from the graphs for λ_{L0} (or λ_{LT0}) by identifying one of the external longitudinal legs as a net magnetization M. Therefore if we set $\lambda_1=\lambda_L M$ and $\lambda_2=\lambda_{LT}M$, then (9) and (11) are identical with (13) and (14), respectively. Thus, one must consider only the three quartic coupling constants λ_L, λ_{LT}, and λ_T in describing the low-temperature phase ($T<T_c$). This should be contrasted with only one coupling constant λ at high temperatures ($T>T_c$). The parameter space corresponding to the low-temperature phase is *larger* than the parameter space corresponding to the high-temperature phase.

We are now in a position to study the influence of the Nambu-Goldstone—mode singularities as the magnetic field H vanishes. For a sufficiently small magnetic field, we have

$$I(m_T)\gg I(m_T m_L)\gg I(m_L)\,. \tag{15}$$

In this limit, nothing peculiar happens to λ_{LT0} and λ_{T0} for any $n\geq 0$. The situation is not the same for λ_{L0}, (9), which includes a term proportional to $n-1$ that becomes negative for $n<1$. Because of the above inequality, (15), we observe that λ_{L0} can become *negative* for $n<1$ when H becomes sufficiently small: For fixed but small values of λ_L and λ_{LT}, so that the perturbation expansion is valid, there exists a critical value $H_c(n,\lambda_L,\lambda_{LT},\Lambda)$ of H such that, if $n<1$, we have

$$\text{(i)}\ \lambda_{L0}>0\ \text{ for } H>H_c\,,$$
$$\text{(ii)}\ \lambda_{L0}=0\ \text{ for } H=H_c\,, \tag{16}$$
$$\text{(iii)}\ \lambda_{L0}<0\ \text{ for } H<H_c\,.$$

The integral over ϕ_{T0} is well behaved everywhere. Therefore, this integral causes no problem in the partition function. However, when λ_{L0} becomes negative, the integral over ϕ_{L0} will make the partition function meaningless. The free energy becomes *unbounded*. Therefore, to this order in the perturbation, we find that the analytic continuation of the theory does *not* exist for $H < H_c$ when $n < 1$. This is exhibited in Fig. 1(b) by the area below AC where AC is defined by $H = H_c$. This pathology is caused by the diagram of Fig. 2(b), which involves two transverse components in the loop. This loop becomes dominant for small H and, for $n < 1$ and $H < H_c$, drives λ_{L0} negative. We wish to emphasize that the pathology occurs for $\Lambda < \infty$ and $H > 0$, and is really due to n being less than unity.

Let us now estimate the value of H_c to this order in the perturbation. Setting $\lambda_L = \lambda_{LT} = \lambda$ in (9), we find that H_c is determined by the solution of $\lambda_{L0} = 0$, i.e.,

$$1 + \tfrac{3}{2}\lambda I(m_L) + \frac{n-1}{6}\lambda I(m_T) = 0 \ . \tag{17}$$

For $n < 1$, (17) possesses a nonzero real solution $H = H_c$ determining the curve AC in Fig. 1(b). As $T \to T_c^-$, H_c tends to zero since the two modes become degenerate at T_c: m_L and m_T become identical as $H \to 0$. The same is true for $T > T_c$. Here again one cannot make a distinction between the two masses as $H \to 0$. All three integrals appearing in (9), (11), and (12) become identical in this limit and the one-loop contributions become proportional to $n+8$, as we expect. This means that the three coupling constants renormalize identically, i.e., there is only one coupling constant, λ_0, in the theory. Thus, the new catastrophe alluded to above for $n < 1$ is present only in the low-temperature phase below AC. It does not affect the high-temperature phase. This also means that the critical behavior of the theory is also *not* affected by this catastrophe for $n \geq 0$, if one approaches the critical point along a path [see curve 3 in Fig. 1(b)] that lies above AC, or from the high-temperature side. However, it is conceivable that the critical behavior along a path approaching T_c but which lies below AC is quite different from that along the path [curve 3 in Fig. 1(b)] that lies above it.

Let us now turn our attention to the relation between the bare and the renormalized longitudinal mass. To the one-loop level, the required relation is as follows:

$$m_{L0}^2 \simeq m_L^2 + \frac{n-1}{18}\lambda_2^2 I(m_T) \ , \tag{18}$$

where we have shown only the dominant contribution for sufficiently small H. (Since Λ is kept fixed, we have also neglected the self-energy diagrams.) We again observe that, for $n < 1$, m_{L0}^2 can become negative when H is very small. However, it should be emphasized that the new catastrophe for $n < 1$ arises not because $m_{L0}^2 < 0$ but because $\lambda_{L0} < 0$. If λ_{L0} were positive, the negativity of m_{L0}^2 would not cause any problem for the existence of the theory.

The new catastrophe discussed above does not disappear at higher orders. We can easily extend the calculation of λ_{L0} to the two-loop level. The dominant contributions come from those diagrams that involve two transverse propagators in each loop, such as Fig. 2(d) at the two-loop level. All other diagrams can be neglected at small H due to the inequalities (15). It is easily verified that the dominant contributions to λ_{L0} are given by

$$\lambda_{L0} \simeq \lambda_L + \frac{n-1}{6}\lambda_{LT}^2 I(m_T) + \frac{(n-1)(n+1)}{6^2}\lambda_T \lambda_{LT} I^2(m_T) \ . \tag{19}$$

We again observe that due to the presence of $n-1$ in (19), λ_{L0} can become negative for sufficiently small H when $n < 1$. However, λ_{LT0} and λ_{T0} remain positive to this order for all $H \geq 0$ and all $n \geq 0$. We have done the calculation to the three-loop level and have found that the problem persists. We believe that the problem persists to any given order in the perturbation: There is a *genuine* catastrophe in the analytic continuation below AC for $n < 1$, and the phase below AC must be a *new* phase.

It should be remarked at this stage that the asymmetric renormalization below T_c was already noted by Nelson[19] some time ago, while calculating the equation of state for the $O(n)$ model. However, he did not pursue the implications of it any further. Amit and Goldschmidt[20] have provided a much stronger argument in support of an asymmetric scheme in cases where there are more than one length scale in the problem, as we have here. We now present a reformulation of the arguments due to Amit and Goldschmidt[20] in support of an asymmetric renormalization as we approach the coexistence curve. Let us, for the sake of convenience, focus our attention on some renormalized vertex function and consider its infrared behavior as we approach the coexistence curve ($H = 0$). This requires rescaling of various momenta by a factor of ρ. It is evident from dimensional analysis that m_L, m_T, and H scale as m_L/ρ, m_T/ρ, and $H/\rho^{3-\epsilon/2}$. To study the infrared behavior of the vertex function, we need to consider the limit $\rho \to 0$. In order for the perturbation theory to be valid as we approach the coexistence curve, we need to choose a path along which $H/\rho^{3-\epsilon/2}$ is finite and constant. Thus, $H \sim \rho^{3-\epsilon/2}$. This implies that

$$(m_T/\rho)^2 \sim \rho^{1-\epsilon/2} \to 0, \quad (m_L/\rho)^2 \sim \rho^{(\epsilon/2)(3-\epsilon/2)-2} \to \infty \tag{20}$$

as $\rho \to 0$ ($\epsilon < 2$, i.e., $d > 2$). Since the two masses behave so differently under rescaling, the arguments due to Amit and Goldschmidt indicate that the symmetric renormalization is not correct. One must consider an asymmetric renormalization.[21] This conclusion should not be surprising. What we have discovered is that as H is lowered, we begin to freeze out the longitudinal mode, whereas the transverse modes become critical, giving rise to an $O(n-1)$ theory, and this forms the basis for the calculation of Nelson.[19]

The present catastrophe does not disappear even if we include higher-order terms such as $\lambda_{60}(\phi^2)^3$, $\lambda_{80}(\phi^2)^4$, etc., as is evident from the consideration of diagrams such as the one shown in Fig. 2(e): All corresponding coupling constants λ_{60}, λ_{80}, etc. are driven to negative values for $n < 1$ below T_c as H becomes sufficiently small. Thus, we are forced to conclude that the analytic continuation of the $O(n)$ theory does not exist below AC for $n < 1$. There appears to be an essential singularity for $T < T_c$ on AC in that the theory becomes meaningless when λ_{L0} becomes negative, no matter how small. We *cannot* follow the path P_1P_2 [see Fig. 1(b)] all the way to $H=0$ without crossing this singularity on AC. Thus, the new phase lying below AC cannot be an analytic continuation of the phase above it. This new phase must be of a very different nature than the phase above it, which is identical in nature to the phase at high temperatures ($T>T_c$). The new phase is present only for $n<1$ and disappears completely for any $n\geq 1$.

The presence of the new phase below AC for the $O(n)$ model for $n<1$, and, in particular, $n=0$, also means that there must be a new phase below AC for the corresponding polymer problem. Since the *same* polymer problem is also identical to the $n=0$ limit of the discrete n-vector model,[13] which does *not* have any Nambu-Goldstone modes, we can safely conclude that the appearance of the new phase is *not* due to any Nambu-Goldstone modes, but rather due to the fact that n is less than unity. Therefore, we do not have to worry about the infrared singularities caused by these modes in our $O(n)$ model considered here. The new phase ($n<1$) must persist even if we carefully account for the singularities of the Nambu-Goldstone modes as the coexistence curve is approached. This point is discussed in detail in Sec. V.

IV. FIXED BARE PARAMETERS: ASYMMETRIC RENORMALIZATION

In the preceding section we considered in detail the situation where the renormalized parameters m^2 and λ were kept fixed as H was varied. As mentioned previously, this analysis was simplified considerably due to the fact that, in the limit of small H, each term in the series for λ_{L0}, (19), was non-negative for $n\geq 1$. The series was well behaved and therefore the positivity of λ_{L0} for any $\lambda>0$ was easily inferred by considering only a truncated series such as the one in (19). If it were an oscillating series, no such conclusion could be inferred from the knowledge of the truncated series. Let us express the relation between λ_{L0} and λ_L as follows:

$$\lambda_{L0}=Z_L(n)\lambda_L\ , \tag{21}$$

where $Z_L(n)$ is the multiplicative renormalization constant which is, in general, a function of n, Λ, H, and the three renormalized couplings. For a sufficiently small magnetic field H, where (15) is valid, $Z_L(n)$ can be inferred from (19). It is easily seen that

$$Z_L(n)\begin{cases}\geq 1 & \text{for } n\geq 1\ ,\\ <1 & \text{for } n<1\ .\end{cases} \tag{22}$$

The most important observation is that $Z_L(n)$ can even become negative for $n<1$. The value of H for which $Z_L(n)=0$ determines the curve AC in Fig. 1(b).

We now wish to understand how does this new anomaly appear in the analytic continuation of the theory for $n<1$ when we keep the bare quantities m_0^2 and λ_0 fixed as H is varied. We choose λ_0 to be positive. With this choice of λ_0, we ensure that the theory exists everywhere. Therefore, it is evident that the new catastrophe cannot appear in the form of the nonexistence of the theory for $n<1$. However, it must appear in some other form. We will show below that the new catastrophe now appears as the breakdown of the perturbation expansion, which is what is used to define the analytic continuation of the theory[5,6] as a function of n.

The situation now is much more complicated. For example, consider (9). Here, λ_{L0} is a function of two variables, λ_L and λ_{LT}. This equation defines a two-dimensional surface. Let us consider a cross section of this surface along the plane $\bar{\lambda}=\lambda_L=\lambda_{LT}$. The resulting curves are shown in Fig. 3 for $n\geq 0$. For $n\geq 1$ the curve is a monotonic function and we can express $\bar{\lambda}$ as a function of λ_0: For any given value of $\lambda_{LO}=\lambda_0$, there is a unique value of $\bar{\lambda}=\bar{\lambda}(\lambda_0)$. The situation is different for $n<1$. The curve is no longer monotonic. For $0<\lambda_0<\bar{\lambda}_0$, there are two different values of $\bar{\lambda}$ satisfying (9). The relation between $\bar{\lambda}$ and λ_0 is *no* longer unique. For $\lambda_0=\bar{\lambda}_0$ the two values of $\bar{\lambda}$ become identical. However, for $\lambda_0>\bar{\lambda}_0$ the only solutions for $\bar{\lambda}$ satisfying (9) are, in general, complex: The theory has become unphysical for $\lambda_0>\bar{\lambda}_0$. As $H\to 0$, $\bar{\lambda}_0\to 0$. Therefore, if we fix $\lambda_0>0$, the $\bar{\lambda}$ becomes complex for $n<1$, which is certainly not acceptable.

However, this problem of nonuniqueness may not be of any physical relevance. For example, from what has been said in Sec. III, we expect the renormalized couplings λ_L, λ_{LT}, and λ_T to be very different, even though all the bare couplings are identical: $\lambda_{LO}=\lambda_{LT0}=\lambda_{T0}=\lambda_0$. This indicates that the physics does not lie on the above special surface of equal λ_L and λ_{LT}. Moreover, it is also of interest for its own sake to know what can be said about the relationships between the renormalized and the bare quantities obtained perturbatively. For this purpose, let us consider (19). Using (9), (11), and (12), we find that

$$\lambda_L\simeq\lambda_{L0}-\frac{n-1}{6}\lambda_{LT0}^2I(m_T)+\frac{(n-1)(n+1)}{6^2}\lambda_{LT0}^2\lambda_{T0}I^2(m_T)\ . \tag{23}$$

We immediately observe that the series for λ_L is an oscillatory one. Thus, even for $n>1$ an integer, where we expect λ_L to be positive for $\lambda_0>0$, we *must* know the entire series to infer that λ_L is indeed positive. We *cannot* use the truncated series (23) to infer that λ_L is positive for all

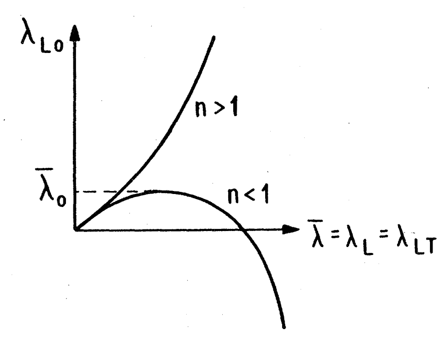

FIG. 3. λ_{L0} as a function of $\bar{\lambda}=\lambda_L=\lambda_{LT}$.

$H \geq 0$. A similar problem arises when we express m_L^2 in terms of bare parameters. We find from (18), or directly, that

$$m_L^2 \cong m_{L0}^2 - \frac{n-1}{18}\lambda_{20}^2 I(m_T) . \tag{24}$$

As H decreases, i.e., as $I(m_T)$ increases, we observe that m_L^2 can become negative even for $n>1$. However, for $n>1$ we certainly expect the longitudinal susceptibility $\chi_L = 1/m_L^2$ to be non-negative, implying that one must consider the entire series for m_L^2 to ensure the positivity of χ_L. We now show that there is a way of rewriting (18) in a form which amounts to resumming the "entire" series for m_L^2. Rewriting (18) as

$$\frac{1}{m_L^2} \simeq \frac{1}{m_{L0}^2}\left[1+\frac{n-1}{18}\frac{\lambda_2^2}{m_L^2}I(m_T)\right] ,$$

and replacing λ_2, m_L, and m_T by λ_{20}, m_{L0}, and m_{T0}, respectively, to this order, we have

$$\chi_L \simeq \frac{1}{m_{L0}^2} + \frac{n-1}{18}\frac{\lambda_{20}^2}{m_{L0}^4} I(m_{T0}) , \tag{25}$$

or, equivalently,

$$m_L^2 \simeq m_{L0}^2 \left[1+\frac{n-1}{18}\frac{\lambda_{20}^2}{m_{L0}^2}I(m_{T0})\right]^{-1} . \tag{26}$$

Expression (25) for χ_L is indeed what we obtain if we evaluate it directly. Moreover, the first two terms in the expansion of (26) are shown in (24). It is clear from (25) and (26) that they are consistent with the positivity of χ_L for $n>1$. Now these can be analytically continued to $n<1$ for all H. We immediately notice that for $n<1$, χ_L can become negative for small H because of the presence of $n-1$.

This simple exercise has paved the way for us to express the renormalized couplings in terms of the bare couplings in a way that ensures the positivity constraints for $n>1$ without knowing the entire series. We rewrite (9), (11), and (12) as follows:

$$\begin{aligned}
\lambda_L^{-1} &= \lambda_{L0}^{-1}\left[1+\tfrac{3}{2}\lambda_{L0}I(m_L)+\frac{n-1}{6}\frac{\lambda_{LT0}^2}{\lambda_{L0}}I(m_T)\right] ,\\
\lambda_{LT}^{-1} &= \lambda_{LT0}^{-1}\left[1+\frac{1}{2}\frac{\lambda_{L0}^2}{\lambda_{LT0}}I(m_L)+\tfrac{2}{3}\lambda_{LT0}I(m_L,m_T)\right.\\
&\qquad\left.+\frac{n+1}{6}\lambda_{T0}I(m_T)\right] ,\\
\lambda_T^{-1} &= \lambda_{T0}^{-1}\left[1+\frac{1}{6}\frac{\lambda_{LT0}^2}{\lambda_{T0}}I(m_L)+\frac{n+7}{6}\lambda_{T0}I(m_T)\right] ,
\end{aligned} \tag{27}$$

where we have set all renormalized couplings equal to their bare values inside the large parentheses, and where we must finally set $\lambda_{L0}=\lambda_{LT0}=\lambda_{T0}=\lambda_0$. For $n<1$ we see the appearance of a pole in λ_L which is a reflection of the catastrophe discussed in Sec. III. By keeping $\lambda_0>0$ and fixed, we find that the renormalized coupling λ_L starts to grow indefinitely for $n<1$ as H is lowered, indicating a breakdown of the perturbation expansion. This result is consistent with the existence of a zero of $Z_L(n)$ on AC for $n<1$ in (21). For nonzero and fixed $\lambda_L>0$, (21) implies that λ_{L0} is zero on AC for $n<1$. If we keep $\lambda_{L0}=\lambda_0>0$ and fixed, we find that λ_L must possess a plot on AC. Let us now determine the equation describing curve AC. This curve is given by the solution of the following equations, at which Z_L possesses a zero:

$$1+\tfrac{3}{2}\lambda_0 I(m_L)+\frac{n-1}{6}\lambda_0 I(m_T)=0 , \tag{28}$$

where we have set $\lambda_{L0}=\lambda_{LT0}=\lambda_0$. Let us consider $d=4-\epsilon$, $\epsilon>0$. For small ϵ, we observe that

$$I(a) \simeq a^{-\epsilon}\ln(\Lambda/a) ,$$

and neglecting 1 compared to other terms in (28), we have, for $n<1$,

$$(1-n)m_T^{-\epsilon}\ln(\Lambda/m_T) \simeq 9m_L^{-\epsilon}\ln(\Lambda/m_L) . \tag{29}$$

For $\epsilon>0$ we can neglect the logarithmic singularities. Thus, we find that

$$m_T^2 \sim m_L^2 .$$

Let us now solve this near the critical point ($T=T_c$, $H_0=0$). We know that, here, $M_0 \sim \tau^{\beta}$, $m_T^2=H_0/M_0$, and $m_L^2=1/\chi_L=H_0^{1-1/\delta}$, where $\tau=(T_c-T)/T_c$. Thus,

$$H_0 \sim M_0^{\delta} , \tag{30}$$

or

$$H_0 \sim \tau^{\beta+\gamma} , \tag{31}$$

where we have used the relations $\Delta=\delta\beta=\gamma+\beta$, describes curve AC near T_c.

V. SYMMETRIC RENORMALIZATION SCHEME

In the preceding two sections we considered an asymmetric renormalization of the $(\phi^2)^2$ theory and have shown explicitly that the analytic continuous of the theory breaks down below AC for $n<1$. Because of the asymmetry due to different masses of the longitudinal and the transverse modes, either λ_{L0} becomes negative or λ_L develops a pole as soon as $n<1$. Thus, one might be inclined to think the pathologies appear only because we have chosen an asymmetric renormalization scheme.

We now wish to show that the appearance of a new phase below AC is not due to our choice of the asymmetric renormalization. We will exhibit below that such a phase also appears if one carefully analyzes the results obtained from a symmetric renormalization scheme. This should again strengthen our claim that the catastrophe is really due to n being less than unity, and not due to the scheme of renormalizations, the approximate nature of our calculation, etc. Schäfer and Horner[11] and Lawrie,[12] among others, have calculated the equation of state for the $O(n)$ model, by properly taking into account the singular nature of the Nambu-Goldstone modes. In the following we would only consider the form of the equation of state given by Lawrie,[12] even though our conclusion also holds true for that given by Schäfer and Horner. In terms of the scaling variables $x=\tau/M^{1/\beta}$ and $y=H/M^{\delta}$, $\tau=(T-T_c)/T_c$, the equation of state reads:

$$(3+x)^{\epsilon/2}\left[1+x\left[\frac{2}{3+x}\right]\frac{3\epsilon}{(n+8)}\right]=9y\left[1-\frac{n-1}{9}\left[\frac{3+x}{y}\right]^{\epsilon/2}\right]\bigg/(n+8)\,, \qquad (32)$$

and is certainly satisfied at the coexistence curve for all n, where $x=-1$ and $y=0$. Thus, it would appear at first that (32) is valid for all $n\geq 0$ all the way up to the coexistence curve ($H=0$, $\tau\leq 0$). Unfortunately, this is not the case when one looks at (32) carefully as H is reduced toward its zero value.

Let us first remark that the left-hand side (lhs) of (32) is always non-negative for all n, and $H\geq 0$. Since the critical behavior disappears as soon as $H>0$, the magnetization curve must be a smooth function of T. For $n\geq 0$ it can be argued that the magnetization is never negative. This is easily seen for $n=0$ where ϕ_p, the density of polymer chain, is equal to $MH/2$. Since $\phi_p>0$ and $H>0$, we find that M must be non-negative. Thus, at least near T_c, the magnetization in nonzero field ($H>0$) must be larger than the spontaneous magnetization. There may be temperatures away from T_c where $M(H>0)<M_0$ (when $n<1$), but just near T_c there is always a region where $M(H>0)>M_0$. This is consistent with the requirement of a smooth and non-negative $M(H)$ near T_c. Thus, $x=\tau/M^{1/\beta}\geq -1$ ($\tau<0$). Thus, the lhs is never negative. On the other hand, the right-hand side (rhs) of (32) is non-negative only when $n\geq 1$. Thus, for $n\geq 1$, (32) is perfectly sensible and one can take the limit $H\rightarrow 0$ without encountering any problem. This is consistent with our analysis.

The situation changes dramatically for $n<1$. As H is reduced we find that the rhs of (32) starts from a value which is positive, goes through zero, and eventually becomes negative when H is very small. Let us evaluate the value of H when the rhs is zero. There are two such values. The first value is evidently $H=0$, i.e., $y=0$. However, the second value is a nonzero value given by the zero of the expression within large square brackets on the rhs. Since $x\geq -1$, we find that the lhs becomes zero only when $x=-1$. Thus, the second zero of the rhs is given by

$$(2/y^*)^{\epsilon/2}=9/(1-n)\,.$$

Thus, $y^*=$const, at which $x=-1$. Now, in terms of H and τ, this implies $H\sim\tau^{\Delta}$, which is nothing but the equation describing the curve AC in our previous analysis.

It is easily seen that the same problem affects the equation of state proposed by Schäfer and Horner,[11] when n becomes less than unity. Thus, one cannot take their equation of state and set $n=0$ to describe the low-temperature phase of the polymer problem, despite the claim made recently by Schäfer.[22] He argues that any criticism of the $n=0$ limit lacks any foundation because the negative susceptibility of the magnetic system ($n=0$) poses no problem for the polymer system.[14,23] As we remarked in the Introduction, the pathologies we allude to are not due to negative susceptibility: They are more serious. The negative susceptibility is not the cause of the breakdown of analytic continuation. Therefore, the arguments of Refs. 14 and 23 are not relevant for our problem.

VI. POLYMERS AND THE $n\rightarrow 0$ LIMIT

It is a well-known fact[5,8,9] by now that there is a very close connection between the $n\rightarrow 0$ limit of the $O(n)$ theory and self-avoiding walks, which are thought of as an idealized representation of linear polymer chains. However, this formal correspondence is established only at high temperatures[9] ($T>T_c$), where there is no distinction between the longitudinal and the $n-1$ transverse modes as the magnetic field vanishes. It has been implicitly assumed[8] that the formal correspondence also works at low temperatures ($T<T_c$) for all $H\geq 0$, as was discussed in detail in the Introduction. We now briefly describe this connection.[24] The osmotic pressure Π of the polymer solution is equal to the negative of the free energy $W(H_0)$ of the $O(n)$ theory as $n\rightarrow 0$:

$$\Pi=W=M_0H_0-\Gamma(M_0)\,,$$

where M_0 is the magnetization of the system and $\Gamma(M_0)$ is the Legendre transform of $W(H_0)$:

$$M_0=\frac{\partial W(H_0)}{\partial H_0},\quad H_0=\frac{\partial\Gamma(M_0)}{\partial M_0}\,.$$

Let ϕ_m and ϕ_p denote the equilibrium concentrations of monomers and of polymer chains, respectively. According to the analogy between the polymer solution and the field theory, we have

$$\phi_m=\frac{\partial\Gamma(M_0)}{\partial\tau},\quad \phi_p=\tfrac{1}{2}M_0H_0\,, \qquad (33)$$

where $\tau=(T_c-T)/T_c$. The polymerization index N is given by

$$N=\phi_m/\phi_p\,.$$

Assuming that $W(H_0)$ scales the same way as the free energy of the $O(n)$ model for $n\geq 1$ an integer,[8,24] we expect that

$$\phi_m=\tau^{\nu d-1}f_m(x),\quad \phi_p=\tau^{\nu d}f_p(x),\quad x=\tau/H_0^{1/\Delta}\,, \qquad (34)$$

where f_m and f_p are some homogeneous functions of the variable x. In the following we will focus our attention on $\tau>0$, i.e., $T<T_c$. (The variables x and τ defined above should not be confused with those in Sec. V.) We will see below that the semidilute regime of the polymer solution corresponds to $x\rightarrow\infty$, while the dilute limit corresponds to $x\rightarrow 0$. As $x\rightarrow\infty$ we expect $f_m(x)$ to approach a finite constant, so that ϕ_m can take a finite nonzero value for $\tau>0$. In the same limit, ϕ_p must have a linear dependence on H_0, so that

$$f_p(x)\sim x^{-\Delta}\quad \text{as } x\rightarrow\infty\,.$$

Therefore, the polymerization index N in this limit is given by

$$N\tau=f_m(x)/f_p(x)\sim x^{\Delta}\quad \text{as } x\rightarrow\infty\,.$$

Introducing the overlap concentration ϕ_m^* by

$$\phi_m^* = 1/N^{\nu d-1} ,$$

which is the monomer concentration of a chain with N monomers confined in a volume $N^{\nu d}$ (N is also the length of the polymer chain in a suitable unit of length), we have

$$\psi_m = \phi_m/\phi_m^* = (N\tau)^{\nu d-1} f_m(x) .$$

The semidilute limit is characterized by the limit $\phi_m \gg \phi_m^*$, i.e., $\psi_m \to \infty$. This can be achieved if $N\tau \to \infty$, i.e., $x \to \infty$. On the contrary, the dilute limit corresponds to $\phi_m \ll \phi_m^*$, i.e., $x \to 0$. Consider curve 1 in Fig. 1(b). Evidently, $x \to \infty$ on this curve as we approach $H=0$. Thus, this curve should describe the scaling limit of the semidilute regime as we approach $H=0$. It can be shown[8] that the osmotic pressure Π in this limit is a function only of ϕ_m and not of ϕ_p:

$$\Pi \sim \phi_n^{\nu d/(\nu d-1)} ,$$

thus showing that, below T_c, the free energy $W(H_0)$ of the $O(0)$ theory, which is equal to $-\Pi$, tends to a *nonzero* value as $H_0 \to 0$. However, for any finite system, the free energy is identically *zero* at $H_0=0$. This is easily seen[24] by considering a finite system of n-component spins $\mathbf{S}=(S_1,S_2,\ldots,S_n)$ of length $\sqrt{n}$ in a magnetic field H_0 and calculating the partition function $Z(H_0)$ as $n \to 0$. It is found that $Z(0) \equiv 1$ for any finite system, indicating that the corresponding free energy at $H_0=0$ must be identically zero. Parisi and Sourlas[25] claim that it remains zero even for an infinite system (N or $V \to \infty$). This means that, for $n=0$, the thermodynamic limit $V \to \infty$ and the limit $H_0 \to 0$ *cannot* be interchanged,[25] a result which is certainly very different from that for positive integer n:

$$\lim_{H_0 \to 0} \lim_{V \to \infty} W(H_0) \neq \lim_{V \to \infty} \lim_{H_0 \to 0} W(H_0) . \qquad (35)$$

However, we now show that there is no basis for the validity of (35) and, in particular, the right-hand side is *not* identically zero.

Let us consider a one-dimensional discrete n-vector model[13] and first look at the right-hand side of (35). The transfer matrix at $H_0=0$ for the model corresponding to the polymer problem is $T(K)=e^{K\mathbf{S}\cdot\mathbf{S}'}/2n$, where $\mathbf{S}$ and $\mathbf{S}'$ are n-component axis spins, each with the following $2n$ possibilities:

$$(\pm\sqrt{n},0,0,\ldots,0) , \quad (0,\pm\sqrt{n},0,0,\ldots,0) ,\ldots, \quad (0,0,\ldots,\pm\sqrt{n}) .$$

It is easily seen that the two dominant eigenvalues of T are the following:

$$\lambda_1 = [X + 1/X + 2(n-1)]/2n$$

and

$$\lambda_2 = (X - 1/X)/2n ,$$

where $X=\exp(Kn)$. For $n=0$ and $N \to \infty$ it is easily seen that $K_c=1$ is the critical point and that $W(K)=0$ for $K<K_c$ and $W(k)=\ln K$ for $K>K_c$. Thus, $W(K)\neq 0$ for $H_0=0$ as $N\to\infty$. Moreover, it can also be seen easily that, if one calculates $W(K,H_0)$ in this case as $N\to\infty$ and then lets $H_0\to 0$, one recovers the above values of $W(K)$ obtained for $H_0=0$.

Thus, we have argued above that the claim of Parisi and Sourlas that polymer problem gives rise to a zeroth-order phase transition because of (35) is incorrect.

However, the pathologies that appear in the $O(n)$ model for $n<1$ mean that curve 1 in Fig. 1(b) must necessarily cross curve AC, below which we enter a new phase. Thus, one cannot take the limit $H_0 \to 0$ on the lhs of (35) without encountering the new phase where the behavior of $W(H_0)$ may be quite different from what has been proposed in Ref. 8. For the same reason, one cannot use curve 1 of Fig. 1(b) to describe the scaling limit of the semidilute regime. One may argue that one can describe this regime along some other curve which never crosses curve AC. We now show that it is impossible to do this for any $d<4$ and, in particular, for $d=3$. From the definition of x, we find that $x\to\infty$, provided that $H_0 \sim \tau^p$, with $p>\Delta$ and $\tau\to 0$, i.e., $T\to T_c^-$. Since $\Delta=\gamma+\beta$, where γ and β are the susceptibility and the magnetization exponents, respectively, we find that any curve defined by $H_0\sim\tau^p$, on which $x\to\infty$ as $\tau\to 0$, must necessarily lie below curve AC, on which $H_0\sim\tau^{\gamma+\beta}$ [see (31), valid for $d<4$]. This is exhibited in Fig. 1(b) by curve 2, on which $x\to\infty$. From what has been said above, this curve must cross AC. On the other hand, on curve 3, which lies above AC, $x\to 0$ as $\tau\to 0$ and can be used to describe the dilute limit of the polymer solution.

The form of AC calculated in (31) is valid only to two-loop order in the perturbation, and we believe this form to remain valid to any finite order in the perturbation. Thus we expect that the problem with the description of the semidilute regime will persist to any finite order in the perturbation.

VII. DISCUSSIONS AND CONCLUSIONS

In the present work we have carefully looked at the question of analytic continuation, in n, $n\geq 0$, and real, of $(\boldsymbol{\phi}^2)^2$ field theory, where $\boldsymbol{\phi}$ is a classical n-component real field. The theory is mathematically well defined only for integer $n=1,2,3,\ldots$. The present study was prompted by the observation, made originally by de Gennes,[5] that the $n=0$ limit of the above theory is intimately related with the problem of polymers, and by the hope that such a study would eventually shed some light on the nature of analytic continuation in other physical models, viz., the replica system, the Potts model, etc. To study the $n=0$ limit of the present theory, we must consider the analytic continuation of the theory to all real $n \geq 0$. (The method can be easily extended to include negative values of n.) As is customary,[4–7] the analytic continuation is defined perturbatively.

We first consider a noncanonical renormalization of the $(\boldsymbol{\phi}^2)^2$ field theory in $d = 4-\epsilon$ dimensions ($\epsilon \geq 0$, $d > 2$). This renormalization scheme is a nonminimal one: We perform all subtractions appropriate for $\epsilon = 0$, even though, for $\epsilon > 0$, the theory does not require any renormalization of the coupling constants. The renormalization of the theory in the symmetric phase ($T > T_c$) is well understood. Our interest here is to study the renormalizibility of the theory below the critical temperature ($T < T_c$), where there is a nonzero spontaneous magnetization for $n = 1,2,3,\ldots$ for any $d > 2$. (For $d = 2$, there is no spontaneous magnetization for any integer $n \geq 2$, which is the reason we have excluded $d = 2$ from the range for d.) Since the symmetry is broken, we make a distinction between the longitudinal and the $n-1$ transverse modes. Owing to the presence of Nambu-Goldstone modes, the renormalization of the theory below T_c is shown to be very *different* from the renormalization of the theory above T_c. It is not difficult to understand why our result is in apparent contradiction with that due to Lee.[3] It has been shown by Lee that the counterterms required to renormalize the theory above T_c are also sufficient to renormalize the theory when the symmetry is broken either spontaneously or by an external source. The key idea in the proof is the observation that when one uses the counterterms of the symmetric theory in the broken-symmetry phase, it merely introduces terms that are finite as far as the ultraviolet behavior ($\Lambda \to \infty$) of the theory is concerned, and, therefore, are not relevant for the renormalization of the theory. Since the infrared behavior is usually of no interest in field theory when making the theory finite as $\Lambda \to \infty$, no attention is paid to the relevance of such terms in the infrared limit, which is the limit relevant for studying the effect of the Nambu-Goldstone modes. When we consider the behavior of the theory in the infrared limit, we find that the theory looks very different in the symmetric and the nonsymmetric phase. For example, it is shown that below T_c there are three different quartic couplings in the theory, while above T_c there is only one quartic coupling in the theory. The three different quartic couplings at $T < T_c$ become identical as $T \to T_c$ and remain identical for all $T \geq T_c$ ($H = 0$). The observation that the infrared behavior of the renormalized theory is *different* above and below T_c does not seem to have been appreciated properly in the literature. On the other hand, the result due to Lee is about the ultraviolet behavior of the renormalized theory and says nothing about the infrared behavior of the theory. As has been shown here, the different infrared behavior of the theory above and below T_c is due to the Nambu-Goldstone modes of the symmetry. This also means that the counterterms we have introduced below T_c in our scheme do not have the full $O(n)$ symmetry, as opposed to the counterterms in the symmetric phase that have full $O(n)$ symmetry.

The analysis carried out in Sec. III in terms of renormalized parameters was greatly simplified because the perturbation expansions were well behaved. This meant that various conclusions about the existence of the theory could be drawn even when only a few terms in the expansions [see (9), (11), and (12)] were known. We find that the infrared behavior of the theory as analytically continued here is devoid of any anomaly for $n \geq 1$, but the situation is drastically *different* for $n < 1$. We find that while the two coupling constants $\lambda_{\rm LT0}$ and $\lambda_{\rm T0}$ do not show any anomaly, the coupling constant $\lambda_{\rm L0}$ develops a zero at some *finite* nonzero magnetic field denoted by H_c and becomes negative for $H < H_c$, see (16), as soon as n becomes less than one. A negative $\lambda_{\rm L0}$ means that the analytic continuation of the theory *cannot* exist below curve AC, Fig. 1(b), where AC is determined by the zero of $\lambda_{\rm L0}$. It is our belief that the zero of $\lambda_{\rm L0}$, i.e., that of $Z_{\rm L}(n)$ in (21), at AC, persist to *all* orders in the perturbation.

The analysis carried out in Sec. IV in terms of bare quantities is, however, complicated by the fact that the series we obtain here are not well behaved at all, as is evident from (23) and (24). Because of this, the truncated forms of the perturbation expansions are not valid for all values of H, even for integer $n \geq 1$, where we do not except any anomalies. We present a method of expressing various quantities in terms of bare ones that remain sensible for all values of H when $n \geq 1$. This method amounts to a tricky resummation of the perturbation expansion as explained in detail in Sec. IV and expressed in (27). In this situation, the catastrophe for $n < 1$ appears in the form of a *pole* in $\lambda_{\rm L}$ on AC. The presence of this pole is due to the zero of $Z_{\rm L}(n)$, and presumably persists to all orders in the perturbation.

In Sec. V we consider the equation of state obtained in a symmetric renormalization scheme to establish that the appearance of a new phase below AC is not due to our choice of an asymmetric renormalization scheme. We show that the pathology appears in the form of a complex solution below AC: For real H, M_0 must be complex. Since the density of a polymer chain, ϕ_p, is related to M_0H, in the analogy $n = 0$, we find that the analytic continuation must break down since ϕ_p must be real. The result of this section also proves that the pathology persists even if one considers Nambu-Goldstone modes carefully, as was done by Schäfer and Horner[11] and by Lawrie.[12]

Thus, we find that the appearance of a new phase below AC is not due to our asymmetric renormalization scheme, or to an oversimplified treatment of the Nambu-Goldstone modes. The catastrophe below AC is genuine and is caused by n as soon as it becomes less than unity.

The arguments presented here clearly establish that the analytic continuation of the $(\boldsymbol{\phi}^2)^2$ field theory along P_1P_2, Fig. 1(b), *cannot* be extended below AC for $n < 1$. It should be noted that the catastrophe responsible for this does not appear in a mean-field approximation,[14,25] where the effect of fluctuations are completely neglected. However, it is these fluctuations that are responsible for the new catastrophe.

From what has been said above, it should be clear that the new phase below AC must be very *different* from the paramagnetic phase above AC. However, we have not been able to say anything decisive about whether the transition from the phase above to the phase below AC is sharp or not. Because the new catastrophe appears just below AC and because the phase above AC must remain paramagnetic arbitrarily close to AC, this transition may be sharp. We believe that this transition cannot be a first-order transition because, with a first-order transition, one usually associates the notion of the analytic continuation across the transition to describe a metastable extension of the phase, something that has been shown here to be impossible across AC.

The nature of the new phase below AC is quite unclear. By analogy with $n \geq 1$ an integer, we say that the $H=0$. $T<T_c$ describes a "ferromagnetic" phase even when n is not an integer and, in particular, when $n<1$. It has been known for some time[14] that this ferromagnetic phase has some anomalous behavior for $n<1$. For example, the spontaneous magnetization M_0, which is supposed to be a monotonic function of the temperature for at least integer $n \geq 1$, is a nonmonotonic function of the temperature[14] for $n<1$. Since the new phase, which is an intermediate phase between the ferromagnetic phase and the paramagnetic phase, does not exist for integer $n \geq 1$, it cannot be obtained as an analytic continuation in n of some phase. The analytic continuation along P_1P_2 only indicates the appearance of this phase for $n<1$, but provides no information about the actual nature of this phase. Our intuition at this stage is not sufficiently deep to provide any insight into the nature of this phase. However, it is expected that our analysis would be a first step towards this goal.

We should again emphasize that our analysis regarding the appearance of the new phase for $n<1$ has been restricted to $d>2$, which ensures that there is a nonzero spontaneous magnetization for arbitrary n. Thus, it is not clear if such a phase would exist for $d \leq 2$. However, it appears[26] that there are certain anomalies, for example, a negative value of the exponent η, at low temperatures for $n<1$ even when $d=2$, which might indicate that such a phase also exists for $d=2$. The arguments leading to such anomalies for $d=2$ are quite independent of the arguments presented here. Therefore, if the arguments for $d=2$ are correct[26] and indicate the existence of the new phase below T_c, it is indeed an important result and lends support to our conclusion about the appearance of the new phase for $n<1$. Unfortunately, here again ($d=2$), no information has been obtained regarding this phase. For $d=1$ it appears that such a phase does not exist.[27]

As discussed in Sec. VI, the appearance of this new phase for $n=0$ and $d<4$ poses serious problems for the description of the semidilute regime of the polymer solution by considering $H \rightarrow 0^+$ and $T<T_c$, as was originally proposed by des Cloizeaux.[8] However, as far as the description of the dilute limit of the polymer solution is concerned, there does not appear to be any problem at all. We have excluded $d=4$ from our consideration here because the extensive overlap necessary to define the semidilute regime is impossible for $d=4$.[24]

Before we close, we wish to discuss the relevance of our result about the $O(n)$ model considered here for two other models that have also been used to describe the polymer problem. The first one is the axis model,[13] in which each spin $\mathbf{S}$ of length $\sqrt{n}$, which has n components, points along one of the axes of the n-dimensional spin space: $\mathbf{S}=(\pm\sqrt{n},0,0,\ldots,0)$, $(0,\pm\sqrt{n},0,\ldots,0)$, etc. As $n \rightarrow 0$, this model at high temperatures becomes[13] identical to the $n \rightarrow 0$ limit of the $O(n)$ model at high temperatures: they both describe a polymer solution. The second one is the supersymmetric formulation of the polymer problem using commuting and anticommuting quantities in a symmetric way.[25,28] Here, the loops in the Feynman diagrams are made to disappear by a proper choice of the numbers of commuting and anticommuting fields: Each commuting field contributes a $+1$ to a loop, while each anticommuting field contributes a -1. Thus, choosing $2m$ commuting real fields

$$\phi_1,\phi_2,\ldots,\phi_{2m}$$

and $2m$ anticommuting fields

$$\psi_1,\psi_2,\ldots,\psi_m,\psi_1^\dagger,\psi_2^\dagger,\ldots,\psi_m^\dagger ,$$

we form the superfields

$$\mathbf{\Phi}=(\Phi_1,\Phi_2,\ldots,\Phi_m,\psi_1,\psi_2,\ldots,\psi_m)$$

and

$$\mathbf{\Phi}^\dagger=(\Phi_1^\dagger,\Phi_2^\dagger,\ldots,\Phi_m^\dagger,\psi_1^\dagger,\psi_2^\dagger,\ldots,\psi_m^\dagger) ,$$

where

$$\Phi_k=(1/\sqrt{2})(\phi_{2k-1}+i\phi_{2k}),\quad \Phi_k^\dagger=(1/\sqrt{2})(\phi_{2k-1}-i\phi_{2k}) ,\quad k=1,\ldots,m .$$

Now, the theory in terms of the superfields $\mathbf{\Phi}$ and $\mathbf{\Phi}^\dagger$ is effectively a theory of zero component field, and describes the polymer solution.[25,28] The analogy between the polymer problem and the supersymmetric theory is again shown in the case in which the supersymmetry is not broken.[25] The supersymmetry is spontaneously broken at the phase-transition point. The point we wish to stress is that this supersymmetry model and the axis model as $n \rightarrow 0$ are identical to the $O(n)$ model considered here as $n \rightarrow 0$ only at $T>T_c$: They all have the *same* partition function. Since our study of the $O(n)$ model clearly exhibits a genuine instability in this partition function for $n=0$ below AC, Fig. 1(b), we should observe a certain *instability* in the other two models. We wish to emphasize that this instability in the above supersymmetry model and the $n \rightarrow 0$ limit of the axis model must be present since their respective partition functions are identical to that of the $O(n)$ model for $n=0$. However, there is no reason to assume that the phases of these three models below AC are identical: they may indeed be very different.

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