Magnetoresistance of a periodic superconducting network near T_c

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The dc magnetoresistance of a superconducting ladder network is computed using linear-response theory. The calculation employs the time-dependent Ginzburg-Landau equations and is thus expected to be relevant for $T \approx T_c$. The results obtained exhibit a surprising qualitative similarity to experimental data for networks of Josephson junctions.

I. INTRODUCTION

Recent experimental[1-6] and theoretical[7-12] work on the properties of two-dimensional granular superconductors, and arrays of coupled Josephson junctions, has led to renewed interest in the properties of superconducting micronetworks. Following the ideas of DeGennes,[13] these networks are modeled by superconducting wires whose transverse dimension is sufficiently small that the superconducting order parameter varies only along the length of the wire (the energy for transverse variation is sufficiently large to suppress transverse degrees of freedom).

In this limit, networks of thin wires are described by a sum of one-dimensional Ginzburg-Landau free-energy functionals, i.e.,

$$F(\psi,T)=\sum_n \int dl_n \left[a\,|\psi(l_n)|^2+\frac{b}{2}|\psi(l_n)|^4+\delta\left|\left[\frac{1}{i}\frac{\partial}{\partial l_n}+\frac{2eA(l_n)}{\hbar c}\right]\psi(l_n)\right|^2\right], \tag{1}$$

where $a=a_0(T-T_c)/T_c$, $\delta=\hbar^2/2m$, b is a constant, and n indicates the nth element with l_n the position along the nth element. $A(l_n)$ is the component of the vector potential tangent to the displacement l_n. The probability density for finding a particular configuration ψ for a network in thermal equilibrium is then given by

$$\rho=\frac{e^{-F(\psi,T)/k_BT}}{Z}, \tag{2}$$

where $Z=\mathrm{Tr}\,e^{-F(\psi,T)/k_BT}$ and the trace is over all possible configurations of the order parameter. The order parameter ψ also determines the supercurrent density in the nth element through the standard expression

$$j(l_n)=\frac{-2e\delta}{\hbar}\left[\psi^*(l_n)\left[\frac{1}{i}\frac{\partial}{\partial l_n}+\frac{2eA(l_n)}{\hbar c}\right]\psi(l_n)+\text{c.c.}\right]. \tag{3}$$

For T near T_c we expect the equilibrium value of $|\psi|^2$ to be small and, if we ignore terms $O(|\psi|^4)$, we obtain an approximate expression

$$f(\psi,T)=\sum_n \int dl_n \left[a\,|\psi(l_n)|^2+\delta\left|\left[\frac{1}{i}\frac{\partial}{\partial l_n}+\frac{2eA(l_n)}{\hbar c}\right]\psi(l_n)\right|^2\right] \tag{4}$$

for the free-energy functional. The mean-field solution[14-18] for the temperature, magnetic field (T,B) phase diagram of a particular network may be obtained by demanding that

$$\delta f/\delta\psi=0, \tag{5}$$

subject to the constraint that ψ be continuous on the network and that the superconducting current be conserved at every network junction. Equations (4) and (5), together with the junction boundary condition, yield the two equations

$$\left[\frac{1}{i}\frac{\partial}{\partial l_n}+\frac{2eA(l_n)}{\hbar c}\right]^2\psi(l_n)+a\psi(l_n)=0, \tag{6}$$

$$\sum_p\left[\frac{1}{i}\frac{\partial}{\partial l_p}+\frac{2eA(l_p)}{\hbar c}\right]\psi(l_p)\Big|_{l_p=0}=0. \tag{7}$$

Equation (6) is the linearized Ginzburg-Landau equation,

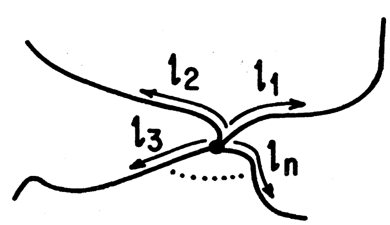

FIG. 1. General node of a superconducting network. The convention for measuring distances l_n from a node is as shown.

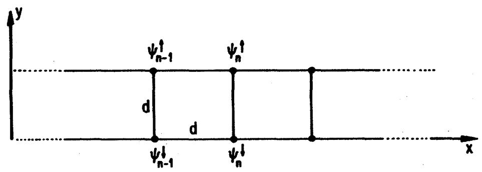

FIG. 2. A superconducting ladder network illustrating the geometry and order parameters labeling convention at each junction.

while (7) represents the superconducting-current conservation requirement, satisfied at each node in the network. The sum over p is taken over all links connected to a given node, and l_p is measured outward from the node in question (Fig. 1). Equations (6) and (7) have been used by a number of authors[13–17] to study the (T,B) phase diagram for a variety of superconducting networks.

However, since much of the experimental work[1–6] motivating the network model involves the study of transport measurements (in particular, magnetoresistance[4–6]), a study of the transport properties of superconducting networks appears to be worthwhile. The initial results of such a study are reported in this paper. In particular, we have computed for $T \gtrsim T_c$ the contribution of superconducting fluctuations to the conductivity of the ladder network shown in Fig. 2. The calculation employs the time-dependent Ginzburg-Landau equation and follows the general procedures developed to treat the case of a single, thin superconducting filament.[19,20]

In Sec. II we describe the eigenstates of the order parameter for the ladder network. These eigenstates permit us to express $f(\psi,T)$ as a sum of independent normal modes. Section III contains an analysis of the linear response of these superconducting normal modes to an applied electric field. In Sec. IV we present our results for the resistance of the ladder network and compare our calculation, in a qualitative way, to recent experimental data.[21]

II. ORDER-PARAMETER EIGENSTATES

Using the expression given by (4) for $f(\psi,T)$, we may integrate by parts on each link of a network. The terms arising from the end-point contributions may be grouped according to the node defined by a given end point. This grouping results in a collection of terms at each node proportional to the expression in (7). The resulting expression for the free-energy functional then assumes the form

$$f(\psi,T)=\sum_n \int dl\, \psi^*(l_n)\left[a+\delta\left[\frac{1}{i}\frac{\partial}{\partial l_n}+\frac{2eA(l_n)}{\hbar c}\right]^2\right]\psi(l_n)+\cdots , \tag{8}$$

where the ellipsis represents end-point contributions. Equation (8) may be diagonalized by introducing the normalized eigenstates $\psi_k(l)$, defined by

$$\sum_n \left[\delta\left[\frac{1}{i}\frac{\partial}{\partial l_n}+\frac{2eA(l_n)}{\hbar c}\right]^2+a\right]\psi_k(l_n)=\epsilon_k\psi_k(l_n) \tag{9}$$

with

$$\sum_p \left[\frac{1}{i}\frac{\partial}{\partial l_n}+\frac{2eA(l_n)}{\hbar c}\right]\psi_k(l_p)\Bigg|_{l_p=0}=0 \tag{10}$$

at each node of the network. The eigenstates defined by (9) and (10) form a complete orthonormal set, a linear combination of which can be used to express a general-order-parameter configuration (we assume periodic boundary conditions over the length of the chain). Thus, we may write

$$\psi(l_n)=\sum_k a_k\psi_k(l_n) ,$$

and (8) becomes

$$f(\psi,T)=\sum_k \epsilon_k a_k^* a_k , \tag{11}$$

with the end-point contributions vanishing by virtue of the fact that each eigenstate satisfies (10). The resulting probability density for the linearized theory is then given by

$$\rho=\frac{e^{-f(\psi,T)/k_BT}}{Z} , \tag{12}$$

and the trace used in obtaining thermal averages is defined by

$$\mathrm{Tr}=\prod \int da_k^* da_k ,$$

where the integral is to be taken over the complex a_k plane.

Equations (9) and (10) have been previously analyzed[17,18] in the study of the (T,B) phase diagram of the ladder network using a mean-field approximation. In that case, attention is focused on the minimum eigenvalue ϵ_k of (9), and the values of T and B for which the minimum eigenvalue is equal to zero are taken to be the normal-superconducting phase boundary in the (T,B) plane.

In the study of the transport properties of the network, it is necessary to know all of the eigenvalues ϵ_k and eigenstates ψ_k. Following previous work, we define $\psi_k^{\uparrow\downarrow}(n)$ to be the amplitude of the order parameter at the nth junction on the upper ($\uparrow$) or lower ($\downarrow$) rail of the ladder network (Fig. 2). Choosing the gauge $\mathbf{A}=(0,Bx,0)$ the order parameter on each link of the network may be written as, for $(n-1)d<x<nd$,

$$\begin{aligned}\psi_k^{\uparrow}(x)&=\frac{1}{\sin(kd)}(\psi_{n-1}^{\uparrow}(k)\sin[k(nd-x)]\\&\quad+\psi_n^{\uparrow}(k)\sin\{k[x-(n-1)d]\}) ,\\ \psi_k^{\downarrow}(x)&=\frac{1}{\sin(kd)}(\psi_{n-1}^{\downarrow}(k)\sin[k(nd-x)]\\&\quad+\psi_n^{\downarrow}(k)\sin\{k[x-(n-1)d]\}) ,\end{aligned} \tag{13}$$

while for $x=nd$, $0<y<d$,

$$\psi_k(y)=\frac{1}{\sin(kd)}e^{-i2n\gamma y/d}[\psi_n^{\downarrow}(k)\sin[k(d-y)] + e^{+i2n\gamma}\psi_n^{\uparrow}(k)\sin(ky)] \ .$$

In (13) we have assumed each link of the ladder network to have length d and introduced the notation

$$\gamma=\pi\phi/\phi_0, \quad \phi=Bd^2, \text{ and } \phi_0=2\pi\hbar c/2e \ .$$

Substituting (13) into (9), we obtain

$$\epsilon_k=\delta k^2+a \tag{14}$$

for the energy corresponding to the eigenstate ψ_k. The allowed values of k, for a given ϕ, are determined by satisfying (10) at each node of the network. Substituting (13) into (10) yields the equations

$$\begin{aligned}&-3\cos(kd)\psi_n^{\uparrow}(k)+\psi_{n-1}^{\uparrow}(k)+e^{-i2n\gamma}\psi_n^{\downarrow}(k)+\psi_{n+1}^{\uparrow}(k)=0 \ ,\\ &-3\cos(kd)\psi_n^{\downarrow}(k)+\psi_{n-1}^{\downarrow}(k)+e^{+i2n\gamma}\psi_n^{\uparrow}(k)+\psi_{n+1}^{\downarrow}(k)=0 \ .\end{aligned} \tag{15}$$

The substitution

$$\begin{aligned}&\psi_n^{\uparrow}(k)=e^{-in\gamma}e^{inqd}f_k^{\uparrow}(q) \ ,\\ &\psi_n^{\downarrow}(k)=e^{-in\gamma}e^{inqd}f_k^{\downarrow}(q)\end{aligned} \tag{16}$$

into (15) yields the equations

$$\begin{aligned}&[-3\cos(kd)+2\cos(qd-\gamma)]f_k^{\uparrow}(q)+f_k^{\downarrow}(q)=0 \ ,\\ &f_k^{\uparrow}(q)+[-3\cos(kd)+2\cos(qd+\gamma)]f_k^{\downarrow}(q)=0 \ ,\end{aligned} \tag{17}$$

and the resultant eigenvalue equation for kd is given by[17,18]

$$3\cos(kd)=2\cos(qd)\cos\gamma\pm[4\sin^2\gamma\sin^2(qd)+1]^{1/2} \ . \tag{18}$$

In (16) and (18) the range of q is given by $-\pi/d\leq q\leq\pi/d$ with the allowed values of q given by $q=2\pi m/L$ $(m=0, \pm1, \ldots)$. It is convenient to think of (18) as defining the band structure (in qd and γ) of both kd and by (14), the eigenenergy ϵ_k. Clearly, for any solution (kd) to (18), there exist other solutions given by $kd+2\pi n$, $n=1, 2, \ldots$, which define a series of energy bands. It is sufficient, therefore, to consider only those solutions of (18) in the range $0\leq kd\leq 2\pi$; all other solutions then follow from the addition of $2\pi n$. Further simplification arises from the fact that if kd is a solution, then $2\pi-kd$ is also a solution. Thus, the entire band structure may be obtained from the smallest values of $kd(0\leq kd\leq\pi)$ which satisfy (18). These solutions to (18)

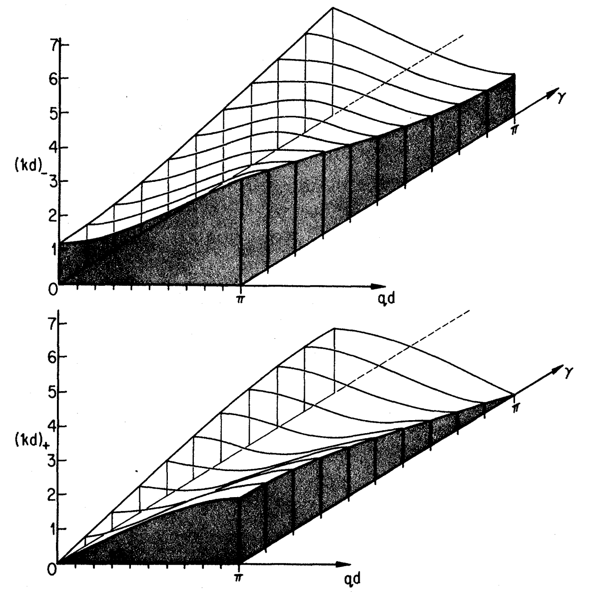

FIG. 3. Eigenvalues $kd_{\pm}$ vs qd and γ for the order-parameter normal modes. Only the two smallest solutions to Eq. (18) are shown. Other solutions in the range $0\leq kd\pm\leq 2\pi$ are given by $2\pi-kd_{\pm}$.

are shown as a function of qd and γ in Fig. 3. We have used $(kd)\pm$ to indicate the sign chosen in (18) and have taken $0\leq qd, \gamma\leq\pi$. Solutions for qd and γ outside of this range may be obtained from the symmetry properties of (18). The remaining energy bands are generated using the properties of (18) discussed above.

The calculation of the electrical response of the network requires the eigenstates ψ_k to be normalized, such that

$$\int_{\text{network}}\psi_k^*\psi_k dl=1 \ .$$

Equivalently,

$$\int[\psi_k^{\uparrow *}(x)\psi_k^{\uparrow}(x)+\psi_k^{\downarrow *}(x)\psi_k^{\downarrow}(x)]dx + \int\psi_k^*(y)\psi_k(y)dy=\frac{L}{d}=N \ , \tag{19}$$

where the integrals are taken over one unit cell of the network. Using (13), (16), and (17), together with (19), yields the normalization

$$\begin{aligned}&\psi_n^{\downarrow}(k)=\beta(k,q)\psi_n^{\uparrow}(k) \ ,\\ &\beta(k,q)=3\cos(kd)+2\cos(qd-\gamma)=\beta \ ,\end{aligned} \tag{20}$$

with

$$\frac{\sqrt{N/k}\,\psi_n^{\uparrow}(k)}{\sin(kd)}=e^{in\gamma}e^{inqd}\left[3(1+\beta^2)\left[\frac{kd}{2}-\frac{1}{4}\sin(2kd)\right] + [\sin(kd)-kd\cos(kd)][\beta^2\cos(qd+\gamma)+\beta+\cos(qd-\gamma)]\right]^{-1/2} .$$

III. LINEAR-RESPONSE THEORY

The general linear response of the network to an applied electric field $E(x,t)$ may be written as

$$j(x,t)=\int dx' \int_{-\infty}^{t} dt' K(x,x',t-t')E(x',t') , \qquad (21)$$

where $j(x,t)$ is the supercurrent at point x time t, and x is any point on the ladder network. For convenience $E(x',t')$ has been assumed to be parallel to the rails of the network and to be a function only of position along the rails. To obtain the response to a spatially constant electric field, we may apply a perturbing field of the form

$$E=E_0\delta(t-t_0) \qquad (22)$$

and compute the resulting current to first order in E_0. The coefficient of E_0 is then the desired response function.

Our calculation will concern itself only with the spatially uniform dc response of the network to an electric field E_0. In that case we have

$$\begin{aligned} j_{\text{dc}} &= \frac{1}{L}\int j(xt)dx \\ &= \frac{1}{L}\int dx \int dx' \int_{-\infty}^{t} dt' K(x,x',t-t')E_0 \\ &= \sigma E_0 . \end{aligned} \qquad (23)$$

If the superconducting fluctuations dominate the conductivity, the resistance of the network is then proportional to $1/\sigma$.

Since the order-parameter normal modes ψ_k are statistically independent, it suffices to study the response of a single mode to the external field (22). The total response of the system is then obtained by summing the response of each normal mode. We assume that the time evolution of a perturbed normal mode is governed by the time-dependent Ginzburg-Landau equation[19,20]

$$\frac{\delta}{\alpha}\left[\frac{1}{i}\frac{\partial}{\partial l_n}+\frac{2eA(l_n)}{\hbar c}\right]^2\psi+\frac{a}{\alpha}\psi-2ieV\psi=-\hbar\frac{\partial}{\partial t}\psi , \qquad (24)$$

where $\alpha=\pi a_0/8k_BT_c$ and $V(x,t)$ is the electrochemical potential. Strictly speaking, a fluctuating Langevin noise source[19] should also be included on the left-hand side of (24). The procedure we use yields identical results to those obtained by including such a source term, and we therefore omit it for simplicity.

To obtain the linear response of a normal mode to a perturbation of the form (22), we assume that at time $t=t_0$, a normal mode $\psi(l,t_0)=a_k\psi_k(l)$ with amplitude a_k is present in the system (this mode is presumed to be excited by the noise source). At this time (t_0) we apply the perturbing potential

$$V(xt)=-E_0x\delta(t-t_0)=f(x)\delta(t-t_0) \qquad (25)$$

to the system. Equation (24) may be integrated over time for a small interval centered about t_0, and keeping terms to first order, we find

$$\psi_k'=a_k\psi_k+\frac{2ief}{\hbar}a_k\psi_k \qquad (26)$$

for the perturbed normal mode as $t\to t_0^+$. The second term in (26) may now be expanded using the complete set of states defined by $\{\psi_k\}$. Each of these states then evolves independently via an equation of the form (24), but with V set equal to zero.

The result is that for $t>t_0$ we may write

$$\psi_k'(l,t)=e^{-\omega_k(t-t_0)}a_k\psi_k + \sum_{k'}{}' e^{-\omega k'(t-t_0)}\left\langle\psi_{k'}\left|\frac{2ief}{\hbar}\right|\psi_k\right\rangle a_{k'}\psi_{k'} , \qquad (27)$$

where $\omega_k=\epsilon_k/\alpha\hbar$ and Dirac notation has been used to define the "matrix element"

$$\left\langle\psi_{k'}\left|\frac{2ief}{\hbar}\right|\psi_k\right\rangle=\frac{2ie}{\hbar}\int \psi_{k'}^{*}f\psi_k dl , \qquad (28)$$

where the integral is understood to be over the ladder network.

The remainder of the calculation now follows standard quantum-mechanical methods,[22] i.e., $\psi_k'(l,t)$ is inserted in (3) to obtain the supercurrent $j_k(l,t)$ generated by the perturbation. The total supercurrent response is then obtained by summing over k and averaging over all the fluctuation amplitudes a_k using the probability density given by (12). The averaging procedure causes the zeroth-order terms to vanish, while the first-order terms yield the desired current response

$$\langle\langle j(x,t)\rangle\rangle=\frac{ie^2\delta}{\hbar^2}\sum_{kk'}e^{-(\omega_k+\omega_{k'})(t-t_0)}(\lambda_k-\lambda_{k'})\left[\langle\psi_{k'}|f|\psi_k\rangle\left[\psi_k^{*}\frac{\partial}{\partial x}\psi_{k'}\right]+\text{c.c.}\right]E_0 , \qquad (29)$$

where $\lambda_m=k_BT/(\alpha\hbar\omega_m)$, and $\partial/\partial x$ is understood to be a derivative parallel to the rails of the network at the location x. The current response to a uniform static field may be obtained by integrating the above expression over space and time according to (28). The resulting expression for the conductivity is given by

$$\sigma=\frac{16e^2\delta^2k_BT}{\hbar\alpha^2L}\sum_{kk'}\frac{|M_{k'k}|^2}{E_kE_{k'}(E_k+E_{k'})} , \qquad (30)$$

where

$$M_{k'k}=\left\langle\psi_{k'}\left|\frac{1}{i}\frac{\partial}{\partial x}\right|\psi_k\right\rangle , \quad E_k=\epsilon_k/\alpha , \qquad (31)$$

and we have used the identity valid for our choice of gauge,

$$\langle \psi_{k'} | x | \psi_k \rangle = \frac{2\delta}{i\alpha} \frac{\left\langle \psi_{k'} \left| \frac{1}{i}\frac{\partial}{\partial x} \right| \psi_k \right\rangle}{E_{k'} - E_k} \tag{32}$$

to evaluate $\langle \psi_{k'} | f | \psi_k \rangle$.

Equation (30) correctly reproduces previous results obtained for a single superconducting filament. In this case, $\psi_k = (1/\sqrt{L})e^{ikx}$ and $M_{k'k} = k\delta_{k'k}$ from which we obtain

$$\sigma_{\text{wire}} = \frac{16e^2\delta^2 k_B T}{\hbar\alpha^2 L} \sum_k \frac{k^2}{2E_k^3} ,$$

in agreement with previous calculations.[23–25]

IV. RESULTS AND DISCUSSION

The matrix element occurring in the expression for the conductivity is given by

$$M_{k'k} = \frac{N\delta_{q'q}}{\sin(kd)\sin(kd)} \left[\frac{2k'k}{k'^2 - k^2} \right] [\cos(kd) - \cos(k'd)][f_{k'}^{\uparrow}(q')f_k^{\uparrow}(q)\sin(qd-\gamma) + f_{k'}^{\downarrow}(q')f_k^{\downarrow}(q)\sin(qd+\gamma)] , \tag{33}$$

where we have chosen the arbitrary phase of the normal modes such that $f_k^{\uparrow}(q)$ and $f_k^{\downarrow}(q)$ are real. Further analysis of (30) is facilitated by separating σ into an intraband $\sigma_1(k=k')$ and interband $\sigma_2(k \neq k')$ contributions. In both cases, the $\delta_{q'q}$ term in (33) ensures that only $q = q'$ states need be considered.

A. Intraband contribution

This contribution to σ is by far the largest for $T \approx T_c$. Taking the limit $k' \to k$ in (33) yields the intraband matrix element

$$M_{kk} = \frac{-N\delta_{qq'}}{\sin(kd)}(kd)[f_k^{\uparrow 2}(q)\sin(qd-\gamma) + f_k^{\downarrow 2}(q)\sin(qd+\gamma)] . \tag{34}$$

This matrix element, together with the normalization defined by (20), may be inserted in (30) to yield the intraband contribution

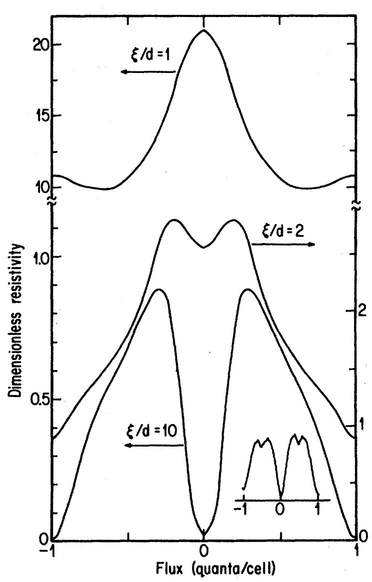

FIG. 4. Dimensionless resistivity of the superconducting chain vs flux density. One flux quantum/cell corresponds to $\gamma = \pi$. Inset illustrates resistance data for a two-dimensional array of Josephson junctions (Ref. 21).

$$\sigma_1 = \frac{16e^2 k_B T d^4 \alpha}{2\hbar L \delta} \sum_q \sum_{\text{bands}} \frac{(\xi/d)^6 |dM_{kk}|^2}{[(\xi/d)^2(kd)^2 + 1]^3}$$
$$= \frac{8e^2 k_B T d^4 \alpha}{\hbar L \delta} S_1 , \tag{35}$$

where we have introduced the coherence length $\xi^2 = \delta/a$ such that $\xi^2 < 0$ for $T < T_c$. The factor S_1 in (35) exhibits the explicit dependence of σ_1 in the ratio (ξ/d), since the factor dM_{kk} is a function only of kd, qd, and γ. Using (35), S_1 may be evaluated numerically. In Fig. 4 we show $1/S_1$ as a function of γ for selected values of ξ/d. Since $\sigma_2 \ll \sigma_1$ (see below), the results shown in Fig. 4 reflect the magnetic-field dependence of the network resistance in cases where the normal conductivity of the network is small compared to σ_1.

A physical interpretation of the intraband contribution may be obtained by referring to (30). For the case $k' = k$ we obtain

$$\sigma_1 = (2e)^2 \sum_k \left[\frac{4\delta^2 |M_{kk}|^2}{\hbar^2 \alpha E_k} \right] \left[\frac{k_B T}{\alpha E_k L} \right] \left[\frac{\hbar}{2E_k} \right]$$
$$= \sum_k \frac{(2e)^2 n_k \tau_k}{m_k^*} , \tag{36}$$

where the following identifications have been made:

(i) From (24) we obtain the lifetime τ_n of the normal-mode density $\psi_k^* \psi_k$ to be

$$\tau_k = \hbar/2E_k . \tag{37}$$

(ii) From (11) and (12) we obtain the average magnitude of the normal mode ψ_k to be

$$\langle a_k^* a_k \rangle = \mathrm{Tr} a_k^* a_k \rho = \frac{k_B T}{\alpha E_k}$$

and thus, the density of this normal-mode excitation is given by

$$n_k = k_B T / \alpha E_k L . \tag{38}$$

(iii) From (14) and (31) we see that dimensionally we may define the effective mass of the fluctuation by

$$m_k^* = \frac{\hbar^2 \alpha E_k}{4\delta^2 |M_{kk}|^2} . \tag{39}$$

Using the above identification, the intraband contribution to σ assumes the standard form for a gas of charged particles. The somewhat surprising result, however, is that the dynamical mass in (36) and (39) is proportional to the *reciprocal* of the band mass defined by the dispersion relation for ϵ_k. This unusual behavior has its origin in the fact that, despite the fact that ϵ_k exhibits a band-structure spectrum, the underlying time-dependent equation (24) for ψ_k is diffusive rather than the usual wavelike Schrödinger equation. Thus, an order parameter wave packet composed of eigenstates centered about k does not propagate ballistically as a function of time, and the standard acceleration theorems familiar to quantum-mechanical behavior do not apply.

B. Interband contribution

As previously mentioned, contributions from interband terms are negligible for $\xi/d \gtrsim 2$. The matrix element (33) is nonzero only for the case where $\cos(kd)$ and $\cos(k'd)$ correspond to opposite-sign solutions of (18). In that case we may write (33) in the form

$$M_{k'k} = \pm \frac{N\delta_{q'q} f_{k'}^{\dagger}(q) f_k^{\dagger}(q)}{\sin(k'd)\sin(kd)} \frac{2k'k}{k'^2-k^2} \times \tfrac{4}{3}[4\sin^2\gamma\,\sin^2(qd)+1]^{1/2}\cos(qd)\sin\gamma \; . \quad (40)$$

Equation (40) was inserted in (30), and interband contributions involving the lowest four bands of (kd) were computed. In general, this contribution is $\lesssim (10^{-2})\sigma_1$ for the values $\xi/d \gtrsim 2$ and is thus negligible on the scale of variation exhibited in Fig. 4. For smaller ξ/d, these interband terms increase in relative importance (they amount to $\approx 10\%$ correction for $\xi/d = 1$), but do not qualitatively alter the magnetic-field dependence shown in Fig. 4.

As a final comment, it is worth noting that a two-dimensional grid of superconducting wires belongs to the same universality class as the two-dimensional array of Josephson junctions.[12] The resistive behavior of our ladder network should, therefore, qualitatively reflect the resistive behavior of a similar array of Josephson junctions. The inset to Fig. 4 shows the data of Webb *et al.*[21] on an $\approx 140\times 140$ array of Josephson junctions. The data were obtained well below T_c for Nb. Nevertheless, the general functional variation of resistance with magnetic flux is remarkably similar to that obtained for our network model for $T \approx T_c$. This similarity provides some encouragement for the use of superconducting networks as models for Josephson-junction arrays.

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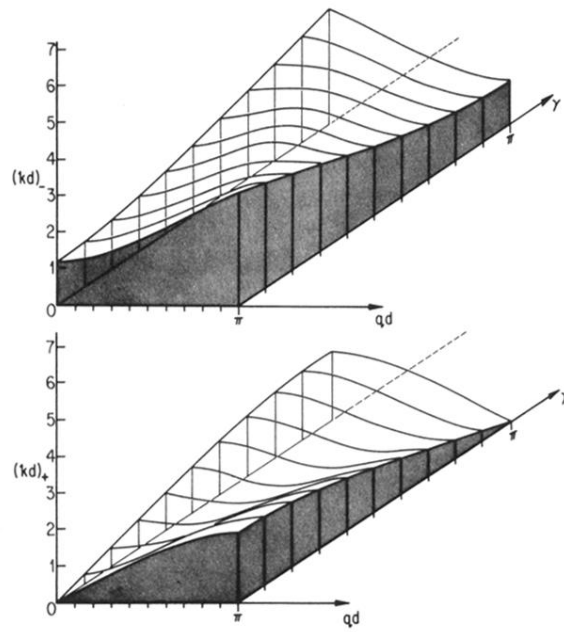

FIG. 3. Eigenvalues $kd_{\pm}$ vs qd and γ for the order-parameter normal modes. Only the two smallest solutions to Eq. (18) are shown. Other solutions in the range $0 \leq kd\pm \leq 2\pi$ are given by $2\pi - kd_{\pm}$.