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Microscopic approach to critical behavior in ^{3}He-^{4}He mixtures. II. Thermodynamics of the effective Hamiltonian

K. K. Singh*
International Centre for Theoretical Physics, Trieste, Italy

Partha Goswami
Department of Physics and Astrophysics, University of Delhi, Delhi 110007, India
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The thermodynamics of a weakly interacting fermion-boson mixture has been worked out on the basis of the effective Hamiltonian derived in an earlier paper. Tricritical-point behavior is discussed in terms of the fields (T,μ_3,μ_4). For the degenerate phase of the mixture, the theory reproduces the classical Landau expansion near a tricritical point. For the nondegenerate phase, the theory differs materially from the Landau theory; it predicts tricritical exponents in agreement with those calculated by applying renormalization-group theory to phenomenological models, and a slope for the upper line larger than that of the λ line in the x-T plane.

I. INTRODUCTION

In an earlier paper[1] (hereafter referred to as I) a system of weakly interacting bosons and fermions was used as a model to develop a theory of critical behavior in ^{3}He-^{4}He mixtures. The fermion amplitudes and the short-wavelength boson amplitudes were eliminated from the problem to obtain an effective, low-momentum boson Hamiltonian. It was pointed out that if one completely ignored fluctuations of the order parameter $(b_0/\sqrt{V})$, the effective Hamiltonian assumed the form of the well-known Landau expansion[2] near a tricritical point. For the nondegenerate phase of the mixture, this approximation is obviously inadequate. The known disagreement[3] between predictions of the Landau theory and experimentally observed tricritical behavior in the normal phase of ^{3}He-^{4}He mixtures is, therefore, not surprising from the point of view of the microscopic theory.

The simplest approximation which takes fluctuations of the order parameter into account is the self-consistent Hartree-Fock (HF) approximation. In this paper the thermodynamics of the mixture is worked out in this approximation. For the degenerate phase of the mixture, the theory reproduces, essentially, the Landau expansion near a tricritical point. For the nondegenerate phase, the theory is an improvement over the Landau theory. It gives tricritical exponents in agreement with those obtained by applying a renormalization-group approach to classical phenomenological models,[4] and also a slope for the upper line larger than that of the λ line in the x-T plane. The latter result is in qualitative agreement with experiments.[5]

An outline of the contents of the paper is as follows: The self-consistent HF approximation is introduced in Sec. II and the thermodynamic potential and the equation of state for the mixture are calculated. As (T,μ_3,μ_4) appear as natural variables in the theory, the thermodynamics is discussed in the T-μ_3 plane with μ_4 playing the role of a parameter. The domains of the nondegenerate and degenerate phases and the existence of the tricritical point (TCP) form the content of Sec. III. In Sec. IV we deal with the calculation of tricritical exponents and the slopes of the upper line and the λ line in the x-T plane.

A general discussion of the work reported in I and this paper is given in Sec. V. A derivation of the expression for μ_4 used in I to discuss the stability of the mixture has also been indicated.

II. THERMODYNAMIC POTENTIAL IN HARTREE-FOCK APPROXIMATION

The effective boson Hamiltonian derived in I is [cf. Eqs. (51) and (73) of I]

$$H_e=c_0+\sum_q\left[\frac{q^2}{m_4}-\mu_4'\right]b_q^\dagger b_q+h_4+h_6\ , \tag{1}$$

$$h_4=\frac{u_4'}{V}\sum_{q_1,q_2,q} b_{q_1}^\dagger b_{q_2}^\dagger b_{q_1-q}b_{q_2+q}\ , \tag{2}$$

$$h_6=\frac{u_6}{V^2}\sum_{\substack{q_1,q_2,q_3\\ q_1',q_2',q_3'}} b_{q_1}^\dagger b_{q_1-q_1'}b_{q_2}^\dagger b_{q_2-q_2'}b_{q_3}^\dagger b_{q_3-q_3'} \times\delta(q_1'+q_2'+q_3')\ , \tag{3}$$

where

$$c_0=V[c_2+\tfrac{1}{2}u_3(n_3^F)^2+u_{34}n_3^Fn_4'+2u_4n_4'^2]\ , \tag{4}$$

$$\mu_4'=\mu_4-u_{34}n_3^F-4u_4n_4'+(u_{34}^2n_4'+u_3u_{34}n_3^F)\frac{\partial n_3^F}{\partial\mu_3}+\frac{1}{V}\sum_q\nu_4(q)\ , \tag{5}$$

$$u_4' = u_4 - \tfrac{1}{2} u_{34}^2 \frac{\partial n_3^F}{\partial \mu_3} + O(u_{34}^3) , \tag{6}$$

$$u_6 = \frac{u_{34}^3}{6} \left[\frac{\partial^2 n_3^F}{\partial \mu_3^2} \right] . \tag{7}$$

The expression for c_2 in (4) is given by Eq. (47) of I. It will not be reproduced here. Although u_6 is of order (u_{34}^3), as will become evident it is not necessary to calculate the third-order contributions to u_4', μ_4', and c_0.

Upon taking into account the symmetry-breaking term H_s [Eq. (2) of I], the thermodynamic potential per unit volume can be written as

$$\Omega = -P = -p_F(T,\mu_3) + \Omega_B^0(p > p_c) + \Omega_B(q < p_c) , \tag{8}$$

where P denotes the pressure of the mixture, and

$$p_F(T,\mu_3) = \frac{1}{\beta V} \sum_{K,\sigma} \ln \left\{ 1 + \exp \left[-\beta \left[\frac{K^2}{m_3} - \mu_3 \right] \right] \right\} , \tag{9}$$

$$\Omega_B^0(p > p_c) = \frac{1}{\beta V} \sum_{|p| > p_c} \ln \left\{ 1 - \exp \left[-\beta \left[\frac{p^2}{m_4} - \mu_4 \right] \right] \right\} , \tag{10}$$

$$\Omega_B(q < p_c) = -\frac{1}{\beta V} \ln \operatorname{Tr} \exp[-\beta(H_e + H_s)] . \tag{11}$$

Following Bogoliubov,[6] we replace $b_0/\sqrt{V}$ by a c number, M. The four- and six-operator terms in (1) then take the form

$$h_4 = V u_4' M^4 + 4 M^2 u_4' \sum_q b_q^\dagger b_q + h_4' + h_4'' , \tag{12}$$

$$h_4' = \frac{u_4'}{V} \sum_{q_1, \ldots, q_4} b_{q_1}^\dagger b_{q_2}^\dagger b_{q_3} b_{q_4} \delta(q_1 + q_2 - q_3 - q_4) , \tag{13}$$

$$\begin{aligned} h_6 = {} & V u_6 M^6 + u_6 V f^2 M^2 + 3 V u_6 f M^4 \\ & + (9 u_6 M^4 + 12 u_6 f M^2) \sum_q b_q^\dagger b_q + h_6' + h_6'' , \end{aligned} \tag{14}$$

$$\begin{aligned} h_6' = {} & \frac{9 u_6 M^2}{V} \sum_{q_1, \ldots, q_4} b_{q_1}^\dagger b_{q_2}^\dagger b_{q_3} b_{q_4} \delta(q_1 + q_2 - q_3 - q_4) \\ & + \frac{u_6}{V^2} \sum_{\substack{q_1, q_2, q_3 \\ q_1', q_2', q_3'}} b_{q_1}^\dagger b_{q_1 - q_1'} b_{q_2}^\dagger b_{q_2 - q_2'} b_{q_3}^\dagger b_{q_3 - q_3'} \\ & \times \delta(q_1' + q_2' + q_3') . \end{aligned} \tag{15}$$

Here,

$$f = \frac{1}{V} \sum_{|q| < p_c} 1 , \tag{16}$$

and h_4'' and h_6'' represent terms containing unequal numbers of creation and annihilation operators. Each q summation excludes the point $q = 0$. The unknown order parameter M will be determined by the requirement that Ω be minimum with respect to M.

It was pointed out in the discussion in I that one expects the effective Hamiltonian to yield a Landau expansion for the mixture if fluctuations in the order parameter are completely ignored. Upon ignoring terms containing b_q's $(q \neq 0)$, one obtains

$$\Omega = c' + (-\mu_4' + u_6 f^2) M^2 + (u_4' + 3 u_6 f) M^4 + u_6 M^6 - h M , \tag{17}$$

$$c' = -p_F(T,\mu_3) + \Omega_B^0(p > p_c) + \frac{c_0}{V} . \tag{18}$$

The approximation (17) for Ω is exactly of the form postulated by Landau. It is evidently inadequate for the normal phase where the order parameter vanishes.

The simplest approximation which takes fluctuations into account is the self-consistent Hartree-Fock approximation. It corresponds to replacing the four- and six-operator terms in h_4 and h_6 by their diagonal parts, i.e.,

$$h_4' \simeq \frac{2 u_4'}{V} N''^2 , \tag{19}$$

$$h_6' \simeq (18 u_6 M^2 + 6 u_6 f) \frac{N''^2}{V} + \frac{6 u_6}{V^2} N''^3 + u_6 f^2 N'' , \tag{20}$$

$$N'' = \sum_q b_q^\dagger b_q . \tag{21}$$

Since in thermal equilibrium N'' is expected to be a macroscopic quantity, fluctuations in N'' about its mean value will be small. In calculating the thermodynamic potential, one may, therefore, allow only such quantum states of the system as are characterized by small fluctuations in N'' about its mean value $\langle N'' \rangle$. For such states the terms containing N''^2 and N''^3 in (19) and (20) can be linearized in the fluctuations in the following manner:

$$\frac{2 N''^2}{V} = -2 n''^2 V + 4 n'' N'' , \tag{22}$$

$$\frac{6 N''^3}{V^2} = -12 n''^3 V + 18 n''^2 N'' , \tag{23}$$

where

$$n'' = \langle N'' \rangle / V \tag{24}$$

denotes the mean density of bosons in the range $0 < |q| < p_c$. The Hamiltonian in the HF approximation consequently takes the form

$$H_e + H_s = V C_B(n'', M) - h M V + \sum_{|q| \leq p_c} \left[\frac{q^2}{m_4} + b_4 \right] b_q^\dagger b_q , \tag{25}$$

where

$$\begin{aligned} C_B(n'', M) = {} & -2(u_4' + 3 u_6 f) n''^2 - 12 u_6 n''^3 + c_0 V^{-1} \\ & + (-\mu_4' + u_6 f^2 - 18 u_6 n''^2) M^2 \\ & + (u_4' + 3 u_6 f) M^4 + u_6 M^6 , \end{aligned} \tag{26}$$

and the effective boson chemical potential $(-b_4)$ is given by

$$b_4 = (-\mu_4' + u_6 f^2) + 4(u_4' + 3u_6 f)n'' + 18u_6 n''^2 + 4(u_4' + 3u_6 f + 9u_6 n'')M^2 + 9u_6 M^4 . \tag{27}$$

In C_B as well as b_4, $3u_6 f$ appears as a correction of order u_{34}^3 to u_4', and will be omitted. Similarly, $u_6 f^2$ appears as a small renormalization of μ_4 and will be omitted.

The linearizations (22) and (23) hold for states having a mean density n'' of bosons. The eigenstates of the linearized Hamiltonian, however, can have arbitrary values of N''/V. Consistency demands that, in calculating $\Omega_B(q<p_c)$ defined by (11), the trace should be restricted to only those states that satisfy (24).

As usual in statistical mechanics, the restriction (24) can be taken into account by calculating

$$\Omega_B'(q,\zeta) = -\frac{1}{\beta V} \ln \operatorname{Tr} \exp\left[-\beta\left[H_e + H_s - \zeta \sum_q b_q^\dagger b_q\right]\right] \tag{28}$$

without any restriction and then choosing the parameter ζ such that

$$\frac{1}{V}\left\langle \sum_q b_q^\dagger b_q \right\rangle = -\frac{\partial \Omega_B'}{\partial \zeta} = n'' , \tag{29}$$

with $\langle\ \rangle$ denoting a thermodynamic average calculated with the Hamiltonian $H_e + H_s - \zeta \sum_q b_q^\dagger b_q$. The required $\Omega_B(q)$ is then given by

$$\Omega_B = \Omega_B' + n''\zeta(n'') . \tag{30}$$

The unknown quantity n'' is fixed by the requirement that Ω_B be minimum with respect to n''. It should be noted that the trace on the right-hand side of (28) is meaningful only if

$$b_4 - \zeta \geq 0 . \tag{31}$$

The calculation of Ω_B is trivial. We find that it has a minimum with respect to n'', provided that

$$n'' = I(b_4) \equiv \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\exp[\beta(q^2/m_4 + b_4)] - 1} , \tag{32}$$

$$[4u_4' + 9u_6 I(b_4) + 9u_6 M^2] - \left[\frac{dI}{db_4}\right]^{-1} > 0 . \tag{33}$$

Equations (29) and (32) imply that at the minimum point ζ is zero.

We can regard (27) and (32) as self-consistent equations for n'' or b_4. The stability condition (33) will prove useful later. The thermodynamic potential in the HF approximation can now be written as

$$\Omega = -p_F(T,\mu_3) + \Omega_B^0(p > p_c;\mu_4) + \Omega_B(q,b_4) - hM + C_B , \tag{34}$$

$$\Omega_B(q,b_4) = \frac{1}{\beta} \int \frac{q^2 dq}{2\pi^2} \ln\left\{1 - \exp\left[-\beta\left[\frac{q^2}{m_4} + b_4\right]\right]\right\} . \tag{35}$$

The requirement that Ω be stationary with respect to M gives the equation of state,

$$\frac{h}{2M} = a_2(b_4) + 2a_4(b_4)M^2 + 3u_6 M^4 , \tag{36}$$

$$a_2(b_4) = -\mu_4' + 4u_4' I(b_4) + 18u_6 I^2(b_4) , \tag{37}$$

$$a_4(b_4) = u_4' + 9u_6 I(b_4) . \tag{38}$$

In the notations (37) and (38), Eq. (27) for b_4 takes the form

$$b_4 = a_2(b_4) + 4a_4(b_4)M^2 + 9u_6 M^4 . \tag{39}$$

Equation (36) is not the same as in the Landau theory in as much as the coefficients a_2 and a_4 are implicit functions of M.

Equations (36) and (39) imply that, for small M and small h/M, b_4 is a small quantity. Upon using the expansion

$$I(b_4) = I_0 - a_1(T) b_4''^{1/2} + c_1(T) b_4 + \cdots , \tag{40}$$

where I_0, a_1, and c_1 are regular functions of T, one finds

$$\Omega = A(T,\mu_3,\mu_4) - 2a_1^2 a_4 b_4 - \tfrac{2}{3} a_1 b_4^{3/2} - hM + a_2 M^2 + a_4 M^4 + u_6 M^6 + O(18a_1^2 u_6 M^2 b_4) , \tag{41}$$

$$\frac{h}{2M} = a_2 + 2a_4 M^2 + 3u_6 M^4 - 4a_1 a_4 b_4^{1/2} - 18a_1 u_6 M^2 b_4^{1/2} + 18a_1^2 u_6 b_4 + O(18c_1 u_6 M^2 b_4) , \tag{42}$$

$$b_4 = a_2 + 4a_4 M^2 + 9u_6 M^4 - 4a_1 a_4 b_4^{1/2} - 36a_1 u_6 M^2 b_4^{1/2} + O(c_1 u_6 M^2 b_4) . \tag{43}$$

Here,

$$A(T,\mu_3,\mu_4) = -p_F(T,\mu_3) + c_0(T,\mu_3,\mu_4)V^{-1} + \Omega_B^0(p > p_c;\mu_4) + \Omega_B(q,0) + 2u_4' I_0^2 + 6u_6 I_0^3 - \mu_4' I_0 , \tag{44}$$

and a_2 and a_4 denote, respectively, $a_2(0)$ and $a_4(0)$. In writing the above expansions, the dimensionless quantities $a_1^2 u_6$, $c_1 a_4$, and $c_1 u_6 I_0$ are considered to be small in comparison with unity. The order of the terms ignored is indicated in each of the equations. It is convenient to absorb u_6 by redefining M^2, b_4, h, and a_2 as follows:

$$m^2 = 3u_6 M^2, \quad a_2' = 3u_6 a_2 , \tag{45}$$

$$b_1 = 3u_6 b_4, \quad h_1 = (3u_6)^{3/2} h . \tag{46}$$

Equations (42) and (43) can then be solved to give

$$b_1^{1/2} \simeq -3\alpha m^2 + \left[2m^4 + 2a_4 m^2 + \frac{h_1}{2m}\right]^{1/2} , \tag{47}$$

$$\frac{h_1}{2m} \simeq a_2' + 2a_4 m^2 + m^4 - 2\sqrt{2}\alpha(2a_4 + 3m^2)\left[m^4 + a_4 m^2 + \frac{h_1}{4m}\right]^{1/2} , \tag{48}$$

$$\alpha = (3a_1^2 u_6)^{1/2} . \tag{49}$$

Note that in the degenerate phase ($h_1 \to 0$, $m \neq 0$) the above expressions are meaningful only if

$$m^2 > -a_4(T,\mu_3,\mu_4) \ . \tag{50}$$

In the region $a_4 > 0$, this condition is satisfied for all m, no matter how small; in the region $a_4 < 0$, (50) implies the absence of a critical line.

III. NONDEGENERATE AND DEGENERATE PHASES

It is convenient to discuss the nondegenerate and degenerate phases of the mixture separately.

A. Nondegenerate phase

The nondegenerate phase is defined by

$$h_1 \to 0,\ m \to 0, \text{ and } h_1/m \neq 0 \ . \tag{51}$$

The equations for h_1/m and b_1 for this phase are

$$h_1/m = b_1, \quad b_1^{1/2} = -2\alpha a_4 + (4\alpha^2 a_4^2 + a_2')^{1/2} \ . \tag{52}$$

A positive sign for the square root in (52) is required by the stability condition (33).

If we hold μ_4 fixed, $a_2 = 0$ and $a_4 = 0$ define two curves in the μ_3-T plane. It is not difficult to check that in the limit of a degenerate Fermi gas ($\mu_3/kT \gg 1$), the curve $a_2 = 0$ is concave downward, while the curve $a_4 = 0$ is concave upward (cf. Fig. 1). The two curves intersect provided that

$$\mu_4 > \frac{64\pi^4}{3} \frac{u_4^3}{m_3^3 u_{34}^5} \ . \tag{53}$$

We shall refer to the intersection of the curves as the *tricritical point* (TCP) and shall show that thermodynamic behavior in its neighborhood corresponds to tricritical behavior.

Equation (52) implies that, in the region $a_4 > 0$, $b_4^{1/2}$ does not exist at points $a_2' < 0$, whereas in the region $a_4 < 0$, $b_4^{1/2}$ does not exist in the domain $a_2' < -4\alpha^2 a_4^2$. The nondegenerate phase is thus possible only in the hatched area in Fig. 1. It will, however, be seen below that, in the region $a_4 < 0$, the degenerate phase is more stable than the nondegenerate phase below the line $a_2' = \frac{3}{4} a_4^2$. In the region $a_4 > 0$, the boundary $a_2 = 0$ of the nondegenerate phase will be defined as the λ line.

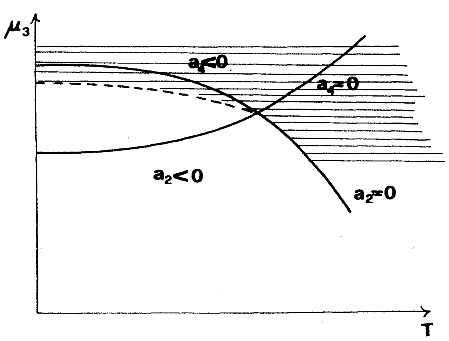

FIG. 1. Qualitative plots of the curves $a_2 = 0$ and $a_4 = 0$. A nondegenerate phase is possible in the hatched region only. The dashed line corresponds to $a_2' = -4a_1^2 a_4^2$.

B. Degenerate phase

The degenerate phase is defined by $h_1 \to 0$, $m \neq 0$. The equation determining m in this case is (48).

Consider first the region $a_4 < 0$ in the T-μ_3 plane. The square-root term in (48) is meaningful provided that $m^2 > |a_4|$. It is easy to see that, for $m^2 > |a_4|$, the last term in (48) is only a correction term. We consequently obtain the Landau-theory solution for m^2,

$$m^2 = -a_4 + (a_4^2 - a_2')^{1/2} \ . \tag{54}$$

Substitution for m^2 in (47) gives

$$b_1 \simeq 2\{ |a_4|^2 - a_2' + |a_4| [(a_4^2 - a_2')^{1/2}] \} \ . \tag{55}$$

The solutions (54) and (55) exist provided that $a_2' < a_4^2$, but not too close to a_4^2. Together with the results obtained above for the nondegenerate phase, this condition implies that in the region $a_4 < 0$ the degenerate and nondegenerate phases overlap in the domain

$$-4\alpha^2 a_4^2 < a_2' < a_4^2 \ . \tag{56}$$

In order to determine the relative stability of the two phases in this domain, we compare the values of the thermodynamic potential in the two phases. Using (41), (52), (54), and (55), we obtain

$$\Omega_D - \Omega_{\rm ND} = \frac{|a_4|^3}{27 u_6^2} \{ -2 + 3x - 2(1-x)^{3/2} + 2\alpha x^{3/2} - 4\sqrt{2}\alpha [1 - x + (1-x)^{1/2}]^{3/2} \} \ , \tag{57}$$

where

$$x = a_2'/a_4^2 \ , \tag{58}$$

and the subscripts D and ND refer, respectively, to the degenerate phase and the nondegenerate phase. Examination of this expression shows that

$$\Omega_D < \Omega_{\rm ND}, \quad x < \tfrac{3}{4} \tag{59}$$

$$\Omega_D > \Omega_{\rm ND}, \quad x > \tfrac{3}{4} \tag{60}$$

if one ignores the correction term α in (57). More exactly, $\frac{3}{4}$ in (59) and (60) should be replaced by x_0, where x_0 is solution of

$$-2 + 3x - 2(1-x)^{3/2} = 2.375\alpha \ . \tag{61}$$

The conclusion is that, below the line

$$a_2' = \tfrac{3}{4} a_4^2 \ , \tag{62}$$

the degenerate phase is the more stable one, while above it the nondegenerate phase is more stable. On the line (62) the two phases have equal thermodynamic potential and can coexist. The order parameter on the coexistence (ce) line has a nonzero value

$$m_{\text{ce}}^2 = \tfrac{3}{2}\,|a_4|\ . \tag{63}$$

In the region $a_4 > 0$, the last term on the right-hand side of Eq. (48) behaves like a correction term if m^2 is much larger than a_4, or is of the order of a_4. The solution for m^2 in this case is given by the Landau-theory result (54) provided that $a_2' < 0$. The conditions for its validity become

$$|a_2'| \gg a_4^2\ , \tag{64}$$

$$|a_2'| \sim a_4^2\ . \tag{65}$$

The first of these is satisfied if a_2' and a_4 are quantities of the same order of smallness; the second is satisfied if a_2' is of the same order of smallness as a_4^2. In the calculation of tricritical exponents (cf. Sec. IV) these are the only cases which arise.

IV. CALCULATION OF TRICRITICAL EXPONENTS

As seen above in the degenerate phase, the HF theory reduces to the classical Landau description. It will, therefore, give the same critical behavior near the TCP as predicted by the Landau theory. In the nondegenerate phase, however, the HF theory differs from the classical theory through the presence of b_4 terms [cf. (41)]. The critical exponents associated with the normal phase will consequently be different. The disagreement between the Landau-theory and experimental results has been noticed by several authors,[3,7] and proposals have been made at a phenomenological level to improve Landau's theory. No attempt, however, appears to have been made to explain the experimental results in terms of a microscopic theory.

In the notation proposed by Griffiths,[8] the tricritical exponents may be divided into two classes: sub-t exponents and sub-u exponents. The sub-t exponents describe critical behavior along a line parallel to the T axis and passing through the TCP. The sub-u exponents describe the thermodynamic behavior associated with the coexistence line near the TCP. For the definitions of the exponents we refer the reader to Griffiths's paper[8] and the report by Kincaid and Cohen.[9] Below, calculations of a few typical exponents are presented.

As pointed out in Sec. I of I, and as may also be seen explicitly from Eqs. (41) and (43), the potential $\Omega + hM$ is a function of the variables (T,μ_3,μ_4,M). It is therefore convenient to discuss critical behavior in the T-μ_3 plane treating μ_4 as a parameter. Some remarks on the use of the variables (T,Δ,P) will be found in the discussion in Sec. V.

An examination of the expressions for a_2 and a_4 shows that both of them are regular functions of T and μ_3 at the TCP (T_t,μ_{3t}). For small deviations from TCP, one may, consequently, write

$$a_2 = d_2(T - T_t) + e_2(\mu_3 - \mu_{3t})\ , \tag{66}$$

$$a_4 = d_4(T - T_t) + e_4(\mu_3 - \mu_{3t})\ . \tag{67}$$

The coefficients d_2, e_2, and d_4 are positive, while e_4 is negative. They depend upon μ_4 through T_t and μ_{3t}.

We first calculate the order-parameter exponent β_t defined as

$$m \sim (T_t - T)^{\beta t},\quad T < T_t,\quad \mu_3 = \mu_3 t\ . \tag{68}$$

The segment $T < T_t$ of the line $\mu_3 = \mu_{3t}$ lies in the region $a_4 < 0$ (cf. Fig. 1). On this line, m is determined by Eq. (54), which yields, for small $T - T_t$,

$$m^2 \simeq (3u_6 d_2)^{1/2}(T - T_t)^{1/2}\ . \tag{69}$$

It follows that $\beta_t = \frac{1}{4}$. Using this result, one finds, for the susceptibility exponent γ_t, the value 1.

In Landau theory the specific-heat exponent α_t is zero in the normal phase. In HF theory the entropy per unit volume for the normal phase obtained from (41) is

$$S = -\frac{\partial A}{\partial T} + (a_1 b_4^{1/2} + 2a_1^2 a_4)\frac{\partial b_4}{\partial T} + 2a_1^2 d_4 b_4\ . \tag{70}$$

At $\mu_3 = \mu_{3t}$, (52) gives

$$b_4 \simeq d_2(T - T_t)\ . \tag{71}$$

The entropy thus contains a term proportional to $(T - T_t)^{1/2}$, which implies

$$\alpha_t = \tfrac{1}{2}\ . \tag{72}$$

It is evident that this result arises from the part of the thermodynamic potential associated with the long-wavelength fluctuations b_q. The Landau theory corresponds to ignoring these fluctuations, and, hence, a zero value for α_t.

The values of other sub-t exponents are given in Table I. For comparison, the values obtained by Riedel and Wegner,[4] by applying renormalization-group theory to a phenomenological one-component spin model, have been listed, along with those calculated by Kincaid and Cohen.[9]

We next calculate two sub-u exponents, namely γ_u and δ_{u^+}. Their values in the Landau theory are, respectively, 0 and 1. The role played by the long-wavelength fluctuations becomes manifest again.

At constant μ_4, γ_u may be defined by

$$\frac{\partial x}{\partial \mu_3} \sim (T - T_t)^{-\gamma_u},\quad x = x_t,\quad T > T_t \tag{73}$$

where x_t denotes the values of the fermion concentration x at the TCP. Alternatively, since the quantity conjugate to μ_3 is n_3, we can set

$$\frac{\partial n_3}{\partial \mu_3} \sim (T - T_t)^{-\gamma_u},\quad x = x_t,\quad T > T_t\ . \tag{74}$$

Both definitions give the same result for γ_u.

Upon differentiating the equation

$$1/x - 1 = n_4/n_3 \tag{75}$$

at fixed (T,μ_4), we obtain

$$\frac{n_3(T,x_t)}{x_t^2}\left[\frac{\partial x}{\partial \mu_3}\right]_{x=x_t} = \left[\frac{1}{x_t} - 1\right]\left[\frac{\partial n_3}{\partial \mu_3}\right] - \left[\frac{\partial n_4}{\partial \mu_3}\right], \tag{76}$$

the derivatives on the right-hand side being evaluated at $x = x_t$. Expressions for n_3 and n_4 follow from the ther-

TABLE I. List of tricritical exponents of helium mixtures. For definitions, the reader is referred to Refs. 8 and 9.

	Exponent	Experiment	Kincaid-Cohen theory (Ref. 9)	Present theory[a]	Renormalization-group theory (Ref. 4)
sub t	α_t	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
	α_t'	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	β_t		$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
	γ_t		2	1	1
	γ_t'		undefined	1	1
	δ_t		9	5	5
sub u	α_u	0	0	0	0
	α_u'	0	0	0	0
	β_{u^+}	1	1	1	1
	β_{u^-}	1	1	1	1
	γ_u	1	0	1	1
	γ_{u^+}	1	0	1	1
	γ_{u^-}	1	1	1	1
	δ_{u^+}	2	1	2	2
	δ_{u^-}	2	2	2	2

[a]Although the exponents are calculated in this paper treating μ_4 as a parameter, they are expected to be the same at constant pressure (cf. Sec. V). In renormalization-group theory (Ref. 4) they are defined with respect to "scaling fields" whose relationship to the elementary (experimental) fields cannot be unambiguously specified.

modynamic potential. We find

$$n_3=n_3^F-[u_3n_3^F+u_{34}n_4'(p,\mu_4)+u_{34}I_0]\frac{\partial n_3^F}{\partial\mu_3}+u_{34}(-M^2+a_1b_4^{1/2})\frac{\partial n_3^F}{\partial\mu_3}\,, \tag{77}$$

$$n_4=n_4'(p,\mu_4)+I_0-u_{34}n_3^F\frac{\partial n_4'}{\partial\mu_4}+M^2-a_1b_4^{1/2}\,. \tag{78}$$

In the nondegenerate phase, the terms in (77) and (78) which can lead to singular behavior of the derivatives are those containing $b_4^{1/2}$. We therefore restrict our attention to the derivative of $b_4^{1/2}$, which, in view of (52), is

$$\frac{\partial b_4^{1/2}}{\partial\mu_3}=-2a_1e_4+\frac{1}{2}\frac{e_2+8a_1^2a_4e_4}{b_4^{1/2}+2a_1a_4}\,. \tag{79}$$

We need to calculate the right-hand side of (79) on the line $x=x_t$, defined by replacing x by x_t in (75). For small deviations from the TCP, the equation of this line becomes

$$d_3(T-T_t)-e_3(\mu_3-\mu_{3t})-a_1b_4^{1/2}=0\,, \tag{80}$$

where

$$d_3=\left[\frac{\partial n_4^r}{\partial T}\right]_t-\frac{1-x_t}{x_t}\left[\frac{\partial n_3^r}{\partial T}\right]_t\,, \tag{81}$$

$$e_3=\frac{1-x_t}{x_t}\left[\frac{\partial n_3^r}{\partial\mu_3}\right]_t-\left[\frac{\partial n_4^r}{\partial\mu_3}\right]_t\,. \tag{82}$$

n_3^r and n_4^r denote, respectively, the regular parts of n_3 and n_4, and the subscript t means that the derivatives are evaluated at the TCP. Since the dependence of n_3 on T (degenerate fermion gas) and n_4 on μ_3 are both small, d_3 and e_3 are positive coefficients. Equations (80) and (52) lead to the conclusion that the line $x=x_t$ is given by

$$\mu_3-\mu_{3t}=-\frac{d_2}{e_2}(T-T_t)+O(T-T_t)^2\,. \tag{83}$$

The value of $b_4^{1/2}$ on this line is

$$(a_1b_4^{1/2})_{x=x_t}=\left[\frac{d_2}{e_2}e_3+d_3\right](T-T_t)\,. \tag{84}$$

Equations (79), (83), and (84) allow us to infer that

$$\gamma_u=1\,. \tag{85}$$

The exponent δ_{u^+} is defined along the line $T=T_t$ by writing

$$\mu_3-\mu_{3t}\sim(x-x_t)^{\delta_{u^+}},\quad x>x_t\,. \tag{86}$$

For small deviations from TCP along $T=T_t$, Eqs. (75), (77), (78), and (52) give

$$\frac{n_{3t}}{x_t^2}(x-x_t)=e_3(\mu_3-\mu_{3t})+a_1 b_4^{1/2} , \tag{87}$$

$$b_4^{1/2}\simeq[e_2(\mu_3-\mu_{3t})]^{1/2}, \ \ \mu_3>\mu_{3t} \tag{88}$$

where e_3 is defined in (82). We infer that, for small $\mu_3-\mu_{3t}$, (86) is satisfied with

$$\delta_{u^+}=2 . \tag{89}$$

The values of all the sub-u exponents obtained in the HF approximation are listed in Table I together with the experimental results. The values obtained in other treatments[4,9] are also tabulated. Perusal of the table shows that the tricritical exponents derived in this paper are in agreement with the experimental values as well as with the results of scaling theories.

In the x-T plane, the coexistence line (62) degenerates into two lines, $x_u(T)$ and $x_l(T)$. The line x_u, called the upper line, represents fermion concentration in the normal phase, while x_l, called the lower line, represents the same quantity in the coexisting degenerate phase. In the classical theory, the upper line has the same slope at the TCP as the λ line. As will be shown below, this is no longer true in the HF approximation.

The equation of the upper line is

$$\frac{1}{x_u}-1=\frac{n_4[T,\mu_3(T),\mu_4]}{n_3[T,\mu_3(T),\mu_4]} , \tag{90}$$

where $\mu_3(T)$ denotes the coexistence line in the μ_3-T plane and n_4 and n_3 refer to the densities in the normal phase. For a point on $x_u(T)$ close to the TCP, one can write

$$-\frac{n_{3t}}{x_t^2}(x_u-x_t)=[d_3(T-T_t)-e_3(\mu_3-\mu_{3t})-a_1 b_4^{1/2}] , \tag{91}$$

where d_3 and e_3 are given by (81) and (82) and the quantity on the right-hand side is to be evaluated on the coexistence line (62). To first order in $T-T_t$, the coexistence line is the same as $a_2=0$, i.e.,

$$\mu_3-\mu_{3t}=-\frac{d_2}{e_2}(T-T_t) . \tag{92}$$

To the next approximation, it is given by

$$a_2=\left[d_4+\frac{d_2}{e_2}\,|\,e_4\,|\right]^2\frac{(T-T_t)^2}{4u_6} . \tag{93}$$

Using these results, (91) becomes

$$x_u-x_t=-\frac{x_t^2}{n_{3t}}\left[d_3+\frac{d_2}{e_2}e_3+\frac{a_1}{(4u_6)^{1/2}}\left[d_4+\frac{d_2}{e_2}\,|\,e_4\,|\right]\right] \times(T-T_t) . \tag{94}$$

The λ line in the x-T plane is given by

$$\frac{1}{x_\lambda}-1=\frac{n_4[T,\mu_{3\lambda}(T),\mu_4]}{n_3[T,\mu_{3\lambda}(T),\mu_4]} , \tag{95}$$

where $\mu_{3\lambda}(T)$ represents the line $a_2=0$ in the μ_3-T plane. Replacing x_u in (91) by x_λ and using the fact that b_4 is zero on the λ line [cf. Eq. (52)], we obtain

$$x_\lambda-x_t=-\frac{x_t^2}{n_{3t}}\left[d_3+\frac{d_2}{e_2}e_3\right](T-T_t) . \tag{96}$$

Equations (94) and (96) imply

$$\left|\frac{dx_u}{dT}\right|-\left|\frac{dx_\lambda}{dT}\right|=\frac{x_t^2}{n_{3t}}\frac{a_1}{(4u_6)^{1/2}}\left[d_4+\frac{d_2}{e_2}\,|\,e_4\,|\right] , \tag{97}$$

i.e., the slope of the upper line is larger in magnitude than that of the λ line. This result is in qualitative agreement with experimental facts.[5] It is easily seen to be a consequence of the $b_4^{1/2}$ term in (91). The role of long-wavelength fluctuations of the order parameter is thus reemphasized.

V. DISCUSSION

The work reported in I and this paper was motivated by the desire to provide a microscopic quantum-mechanical foundation for phenomenological theories of critical behavior in helium mixtures. As the earlier attempts[9] were not successful in obtaining a Landau expansion for the mixture, a primary objective was to understand how such an expansion in powers of the order parameter could arise in a microscopic theory and to discover its limitations. While we have not been able to achieve this objective using realistic interaction potentials between helium atoms, investigations of a model fermion-boson mixture have provided satisfactory, qualitative answers to the above questions.

The quantity of central importance in the investigation turns out to be the effective, low-momentum boson-boson Hamiltonian H_e derived in I, particularly the structure of the coefficients of the four- and six-operator terms of H_e. This structure is similar to that assumed in the Landau theory of a tricritical point. As pointed out in Sec. II, the Landau expansion results on completely ignoring fluctuations of the order parameter in the effective Hamiltonian. This derivation of the Landau theory also makes evident its inadequacy in explaining the tricritical behavior of the normal phase. When the fluctuations are taken into account in an approximate manner, one finds a tricritical behavior in the normal phase, in accord with experiment.

The thermodynamic potential calculated in Sec. II conforms to the scaling hypothesis[7,8] for the TCP. We may write (41) as

$$\Omega+hM=\Omega_r+\Omega_s'(a_2,a_4,M) , \tag{98}$$

where Ω_r denotes the regular part of $\Omega+hM$, and

$$\Omega_s'=-2a_1^2a_4b_4-\tfrac{2}{3}a_1b_4^{3/2}+a_2M^2+a_4M^4+u_6M^6 . \tag{99}$$

It is easily verified that Ω_s' scales as

$$\Omega_s'(la_2,l^{\phi_t}a_4,l^{\beta_t}M)=l^{2-\alpha_t}\Omega_s(a_2,a_4,M) , \tag{100}$$

with $\phi_t=\frac{1}{2}$, $\beta_t=\frac{1}{4}$, and $\alpha_t=\frac{1}{2}$, provided that b_4 scales as

$$b_4(la_2, l^{\phi_t}a_4, l^{\beta_t}M) = lb_4 \ . \tag{101}$$

Equation (43) for b_4 implies that (101) holds. The equation of state [Eq. (42)] enables one to conclude that h scales as l^{Δ_t} with Δ_t equal to $\frac{5}{4}$.

The agreement between the results for tricritical exponents obtained in this paper and in the renormalization-group approach applied to a phenomenological Hamiltonian[4] can be traced to the scaling property (100) of the thermodynamic potential in both treatments. It should, however, be pointed out that whereas we have defined exponents in terms of the elementary fields $T - T_t$ and $\mu_3 - \mu_{3t}$, in the phenomenological theory they are defined with respect to certain "scaling fields" whose relationship with the elementary fields can only be postulated. The importance of microscopic theories derives from the necessity to illuminate the connection between the scaling fields and the elementary fields which enter the physical description of the system. The expressions for $a_2(0)$ and $a_4(0)$ [cf. Eqs. (37) and (38)] provide an example. It should also be pointed out that values of exponents other than α_t, β_t, and ϕ_t are deduced in the scaling theories from scaling laws[8] which usually require assumptions about the regular behavior of certain multiplying functions.

The HF approximation gives a correct description of tricritical behavior in the degenerate as well as the nondegenerate phase. However, it gives a reasonable description of ordinary critical behavior in the degenerate phase only as long as the inequality (64) is satisfied. In the opposite case ($m^2 \ll a_4$), Eq. (48) gives a solution for M^2, which, instead of approaching zero on the λ line, assumes a finite value $(8a_1^2a_4)$. In the region $a_2 < 0$, the degenerate phase is, therefore, meaningful in an asymptotic sense only, namely if one first fixes $|a_2|$, or the deviation $|T - T_\lambda|$, and then chooses a suitably small a_4 to satisfy (64). This situation, however, is not peculiar to the HF approximation only. More sophisticated approaches such as the Green's function method[10] and the renormalization-group approach[11,12] give equation of state near a critical point in the above asymptotic sense.

An expression for μ_4 up to second order in u_{34} was used in I to discuss the conditions of thermodynamic stability of the mixture. We indicate briefly the derivation of that expression. Elimination of μ_4' from (5) and (36) gives

$$\mu_4 = -\frac{h}{2M} + 4u_4'I(b) + 4u_4n_4' + 2u_4'M^2 + u_{34}n_3^F - u_3u_{34}\frac{\partial n_3^F}{\partial \mu_3} - u_{34}^2 n_4' \frac{\partial n_3^F}{\partial \mu_3} \ . \tag{102}$$

Writing u_4' as $u_4 + u_4''$ and using Eqs. (77) and (78), (102) reduces to Eq. (79) of I with the difference that $-h/2M$ replaces $\mu_B^0(n_4)$. These two quantities, however, are easily seen to be equal. In the degenerate phase both are zero. In the nondegenerate phase, $h/2M$ equals b_4, which, according to (27), is given by $-\mu_4 - u_{34}n_3^F$ up to first order in u_{34}. Equation (75) for n_4 now implies that in the nondegenerate phase, $-b_4$ is equal to the chemical potential of an ideal Bose gas of density n_4.

Finally, we comment on the fact that we have derived tricritical behavior treating μ_4 as a parameter, whereas experimental observations usually refer to a fixed pressure P. Following Bogoliubov's work[6] on symmetry breaking, the correct order parameter for a system of bosons is considered to be $\langle b_0/\sqrt{V}\rangle$ with the b_q's playing the role of fluctuations. It was pointed out in I that with $\langle b_0/\sqrt{V}\rangle$ as the order parameter, just one thermodynamic potential of the fields exists, namely $\Omega'(T,\mu_3,\mu_4,M)$. From a theoretical point of view, therefore, a Landau expansion in powers of the order parameter is possible only with (T,μ_3,μ_4) as field variables; the quantities (T,Δ,P) cannot be used as fields for such an expansion. The experimental situation, however, corresponds to the limit $h \to 0$. In this limit potential function of the variables (T,Δ,P) exists and is simply μ_4. To express the theoretical results for $h \to 0$ in terms of the variables (T,Δ,P), all that is necessary is to replace μ_4 everywhere by $\mu_4(T,\Delta,P)$. Since μ_4 is a regular function of its variables at TCP (its first derivatives being the entropy per particle, the volume per particle, and the fermion concentration x), the critical behavior at constant P may be expected to be the same as at constant μ_4. The nonuniversal aspects of tricritical behavior emphasized by Fisher and Sarbach[13] are being studied in the context of the model introduced in this paper and will be reported later.

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*Permanent address: Department of Physics and Astrophysics, University of Delhi, Delhi 110007, India.

[1]K. K. Singh and Partha Goswami, Phys. Rev. B **29**, 2558 (1984).

[2]L. D. Landau, Phys. Z. Sowjetunion **8**, 113 (1935); in *Collected Papers of L. D. Landau,* edited by D. ter Haar (Pergamon, New York, 1965), p. 96.

[3]R. B. Griffiths, Phys. Rev. Lett. **24**, 715 (1970).

[4]E. K. Riedel and F. J. Wegner, Phys. Rev. Lett. **29**, 349 (1972); Phys. Rev. B **7**, 248 (1973).

[5]E. H. Graf, D. M. Lee, and J. D. Reppy, Phys. Rev. Lett. **19**, 417 (1967).

[6]N. N. Bogoliubov, Physica (Utrecht) Suppl. **26**, 1 (1960).

[7]E. K. Riedel, Phys. Rev. Lett. **28**, 675 (1972).

[8]R. B. Griffiths, Phys. Rev. B **7**, 545 (1973).

[9]J. M. Kincaid and E. G. D. Cohen, Phys. Rep. **22C**, 58 (1975).

[10]K. K. Singh, Physica (Utrecht) **34**, 285 (1967).
[11]E. Brezin, D. J. Wallace, and K. G. Wilson, Phys. Rev. B **7**, 232 (1973).
[12]K. K. Singh, Phys. Rev. B **17**, 324 (1978).
[13]M. E. Fisher and S. Sarbach, Phys. Rev. Lett. **41**, 1127 (1978); Phys. Rev. B **20**, 2797 (1979).