Critical exponents for ϕ^3-field models with long-range interactions

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The critical exponents for two ϕ^3-field theories with long-range (LR) interactions decaying as $1/R^{d+\sigma}$, $\sigma>0$, are calculated to two-loop order in renormalized perturbation theory in $d=3\sigma-\epsilon'$ dimensions. One is the continuum version of the p-state Potts model and the other is the scalar field theory with imaginary coupling that describes the Yang-Lee edge-singularity problem. The two crossover exponents for quadratic symmetry breaking discussed by Wallace and Young and in recent work by the present authors are also calculated in the first case. By means of renormalization-group recursion relations to one-loop order, it is shown that the LR fixed point is stable for all $\sigma<2$ whenever η_{SR}, the critical correlation-function exponent for short-range (SR) interactions, is negative, implying a discontinuity of critical exponents at $\sigma=2$. This is the case for the $(p<2)$-state Potts model and for the Yang-Lee edge-singularity problem, and is in agreement with recent results by Chang and Sak for the Ising spin-glass problem. For the $(p>2)$-state Potts model there is an indication of a continuous crossover to SR behavior at $\sigma=2-\eta_{SR}$, with $\eta_{SR}>0$. It is pointed out that a number of exact results [$\beta=(1-\epsilon'/2\sigma)\sigma\nu$, $\hat{\sigma}=(d-\sigma)/(d+\sigma)$, in which $\hat{\sigma}$ is the Yang-Lee edge-singularity exponent, and $\nu^{-1}=(d-\sigma)/2$ for a scalar theory] may apply within the LR expansion depending on $\eta=2-\sigma$ (shown here to hold at least to two-loop order) being exact, to all orders.

I. INTRODUCTION

Renormalization of the continuum ϕ^3-field theory for the p-state Potts model[1,2] with short-range (SR) exchange interaction and the nature of the phase transition have been considered for some time.[3–6] Results for the model with long-range (LR) interaction decaying as $1/R^{d+\sigma}$, $\sigma>0$, were already obtained by Priest and Lubensky to one-loop order.[3] Of particular interest is the crossover to SR behavior, and they argued that this takes place when $\sigma=2-\eta_{SR}$ if $\eta_{SR}>0$ (the case for $p>2$), following earlier work by Sak on the n-vector model with LR interaction.[7] They also pointed out that the SR fixed point is unstable to a LR perturbation for all $\sigma<2$ if $\eta_{SR}<0$ (the case for $p<2$). Although this could suggest a discontinuity of critical exponents at $\sigma=2$, a detailed discussion of how this would take place was not given in their work.

In a recent paper on the n-vector model with LR interaction, we pointed out that the crossover problem to SR behavior may not be completely settled and that critical exponents may be discontinuous[8] at $\sigma=2$. In further work by Chang and Sak,[9] it was also shown that the exponents for the Ising spin-glass model with LR interaction in the ϕ^3 theory of Kotliar *et al.*[10] are discontinuous at $\sigma=2$ because the LR fixed point is stable for all $\sigma<2$. This is a theory where η_{SR} is negative.

One of the purposes of the present paper is to show that the LR fixed point is stable for two other ϕ^3-field theories with LR interaction whenever $\sigma<2$ if $\eta_{SR}<0$. One is the p-state Potts model with $p<2$ and the other is the Yang-Lee edge-singularity problem.[11] A detailed discussion is needed because at the LR fixed point the coefficients of the q^2 and q^σ dependence ($\vec{q}$ being the wave vector) in the renormalization-group (RG) recursion relation for the inverse two-point correlation function[12] cannot both be positive, a point apparently not noted before, and which also arises in the model of Kotliar *et al.*

A second purpose of this work is the renormalization of the p-state Potts model with LR interaction and the calculation of critical exponents, to two-loop order in $d=3\sigma-\epsilon'$ dimensions, including the two crossover exponents for quadratic symmetry breaking (QSB) discussed by Wallace and Young,[13] following our recent work for SR interactions.[14] From our calculation for the Potts model there is a short way to obtain the exponents for the Yang-Lee edge-singularity problem, based on the result that $\eta=2-\sigma$, at least to two-loop order, and this is the third purpose of our work.

The present paper is restricted to calculations with a pure ϕ^3 Landau-Ginzburg-Wilson Hamiltonian without the quartic coupling needed to stabilize the theory. However, as recently shown by Pytte,[5] when the latter is taken into account a second-order transition follows from a RG calculation to two-loop order in $d=6-\epsilon$ dimensions for $p\leq 2$. Owing to the presence of instanton solutions that appear by resummation of the perturbation series to all orders in pure ϕ^3 theory,[15] and which are usually associated with a first-order transition,[16] we warn that the exponents calculated here for the Potts model with $p<2$ may not correspond to a second-order transition, except in the limit where $p\rightarrow 1$.[17] Nevertheless, since the classical argument involving instantons breaks down in this limit, there may be a changeover to different instanton solutions at some low value of $p-1$ which can still allow for a second-order transition, and our results would apply to that case. These difficulties do not arise in the ϕ^3 theory for the Yang-Lee edge singularity, since the coupling is purely imaginary.[11]

The renormalization and calculation of critical exponents are done to two-loop order in renormalized perturbation theory[18–20] in $d=3\sigma-\epsilon'$, while the crossover to SR behavior is studied to one-loop order by means of the simpler RG recursion relations. The paper is organized as follows. The model is introduced in Sec. II, where the bare vertex functions can also be found. The renormalization and calculation of critical exponents for the Potts model and the Yang-Lee edge-singularity problem for LR interactions is done in Sec. III and the stability of the LR fixed point is discussed in Sec. IV. In Sec. V we summarize our results and discuss their relevance for percolation with LR interactions, while technical details are left for the Appendix.

II. MODELS AND VERTEX FUNCTIONS

We start with the effective Landau-Ginzburg-Wilson Hamiltonian for the symmetric theory in momentum space,

$$\mathscr{H}=-\frac{1}{2}\int_{\vec{q}}(m_0^2+sq^2+lq^\sigma)\sum_i \phi_i(\vec{q})\phi_i(-\vec{q}) + \frac{1}{3!}g_{30}\sum_{i,j,k} d_{ijk}\int_{\vec{q}}\int_{\vec{q}\,'}\phi_i(\vec{q})\phi_j(\vec{q}\,')\phi_k(-\vec{q}-\vec{q}\,') , \tag{2.1}$$

where ϕ_i, $i=1,2,\ldots,n$, are the components of a real field, with the tensorial coefficients

$$d_{ijk}=\sum_{\alpha=1}^{p} e_i^\alpha e_j^\alpha e_k^\alpha \tag{2.2}$$

for the $p(=n+1)$-state Potts model in terms of the Potts vectors $\vec{e}^{\,\alpha}$, while

$$d_{ijk}=\begin{cases} i\ (=\sqrt{-1}), & i=j=k=1 \\ 0, & \text{otherwise}\end{cases} \tag{2.3}$$

for the Yang-Lee edge-singularity problem.[11] Furthermore, m_0 is the bare "mass," and the momentum dependence in q^2 and q^σ follows from the LR interaction in the original Hamiltonian decaying as $1/R^{d+\sigma}$, $0<\sigma<2$. The coefficients s and l are fixed by the RG recursion relations in Sec. IV. Anticipating the result that will be obtained there, that the LR fixed point is stable for all $\sigma<2$ when $\eta_{\rm SR}<0$, we set $s=0$ and $l=1$ for the calculation in renormalized perturbation theory. Then, the dimensional coupling $g_{30}=\kappa^{\epsilon'/2}w_0$, in terms of the arbitrary momentum-scale parameter κ, yields the LR expansion in $\epsilon'\equiv 3\sigma-d$ with the trilinear dimensionless coupling w_0. The momentum-space integrals $\int_{\vec{q}}\equiv(2\pi)^{-d}\int d^dq$ are done over all space in renormalized perturbation theory with dimensional regularization,[18,20] and over the shell $b^{-1}<|\vec{q}|<1$, $b>1$, in the RG recursion relations.

Two representations for the Potts model can be found in the literature. One is the representation of Wallace and Young[13] (WY) in which $e_i^\alpha=\pm 1$ for any component i, while in the other representation, due to Priest and Lubensky[3] (PL),

$$e_i^\alpha=\left[\frac{p(p-i)}{p-i+1}\right]^{1/2}\times\begin{cases}0 & \text{if } \alpha<i, \\ 1 & \text{if } \alpha=i, \\ -1/(p-i) & \text{if } \alpha>i .\end{cases} \tag{2.4}$$

The main difference between the two is that in the representation of PL there is one Potts vector that lies along a coordinate axis in order-parameter space, namely the vector $\vec{e}^{\,1}$ along the $i=1$ axis, whereas in the WY representation none of the vectors are aligned with a coordinate axis. As far as calculations in the symmetric theory with an $O(n)$-invariant quadratic part in the Hamiltonian of Eq. (2.1) are concerned, the two representations yield identical results, of course, since they only differ in a rotation of coordinates. In the presence of QSB, however, the two representations are no longer equivalent and, as shown explicitly in our recent work,[14] each of them serves to calculate a different crossover exponent.

We implement QSB in the Potts model by adding to Eq. (2.1) the anisotropy term[21]

$$\mathscr{H}_g=\frac{1}{2}g\int_{\vec{q}}B(\vec{q}) \tag{2.5}$$

that favors ordering into m "longitudinal" components if $g>0$, with

$$B(\vec{q})=\frac{1}{n}\left[(n-m)\sum_{i=1}^{m}\phi_i(\vec{q})\phi_i(-\vec{q}) - m\sum_{i=m+1}^{n}\phi_i(\vec{q})\phi_i(-\vec{q})\right] . \tag{2.6}$$

For simplicity, it will be assumed that $m=1$ and the effect of $\mathscr{H}_g$ is then to add a "mass" term to the remaining $n-1$ "transverse" components. In general, QSB generates a break in trilinear symmetry with new fixed-point behavior, as shown by one of us in recent work to one-loop order.[22] Actually, part of that break in trilinear symmetry remains even in the absence of QSB.[23] Indeed, in addition to the usual fixed point of the symmetric theory one finds asymmetric (in the field components) fixed points. Renormalization to two-loop order, with[24] or without QSB,[25] confirm these results. However, to keep the present work simple, we restrict ourselves to trilinear symmetry, with a single trilinear coupling g_{30}. The QSB term (2.5) will then serve to calculate crossover exponents to a quadratic perturbation, with $g\ll 1$, about the symmetric theory.

Here we present vertex functions. The bare one-particle irreducible (1PI) two- and three-point vertex functions $\Gamma_{ij}^{(2)}$ and $\Gamma_{ijk}^{(3)}$, as well as the longitudinal two-point vertex function with a ϕ^2 insertion, $\Gamma_{11}^{(2,1)}$, for the symmetric theory in the Potts model, are calculated in standard way[18,19] to two-loop order, with the result

$$\Gamma_{ij}^{(2)}=q^\sigma\delta_{ij}[1-B_{11}I_1w_0^2-(B_{21}I_2^{(1)}+B_{22}I_2^{(2)})w_0^4] , \tag{2.7}$$

$$\Gamma_{ijk}^{(3)}=\kappa^{\epsilon'/2}d_{ijk}[w_0+A_{11}L_1w_0^3+(A_{21}L_2^{(1)}+A_{22}L_2^{(2)}+A_{23}L_2^{(3)})w_0^5] , \tag{2.8}$$

$$\Gamma_{11}^{(2,1)}=1+C_{11}L_1w_0^2+[(C_{21}+C_{22})L_2^{(1)}+(C_{23}+C_{24})L_2^{(2)}+C_{25}L_2^{(3)}]w_0^4 , \tag{2.9}$$

in which a factor

$$K_d^{1/2}=[2^{-1+d}\pi^{d/2}\Gamma^{-1}(d/2)]^{1/2},$$

in terms of the Γ function $\Gamma(z)$, is absorbed, as usual, in w_0. I_i and $L_i^{(j)}$, $i=1,2$, are one- and two-loop integrals corresponding to the diagrams in Figs. 1–3 and A_{ij}, B_{ij}, and C_{ij} are the multiplicity coefficients of the diagrams. These coefficients, drawn from Amit's paper,[4] are collected in Table I, while for the LR integrals calculated with dimensional regularization at zero mass we obtain the results given in the Appendix. Note that, for fixed $\sigma<2$, some of the integrals are less singular than for the SR Potts model, either with a lower-order pole in ϵ' or without a pole at all. This is responsible for the absence of wave-function renormalization and for the results derived in Sec. III.

The calculation of the crossover exponents for QSB, ϕ and $\bar{\phi}$, requires the 1PI two-point longitudinal vertex function with a B insertion, which can be written as[14]

$$\Gamma_{11\,B}^{(2)}(w_0)=\Gamma_{11,1}^{(2)}(w_0)-\frac{1}{n}\Gamma_{11}^{(2,1)}(w_0), \tag{2.10}$$

in which $\Gamma_{11,1}^{(2)}$ is the 1PI two-point vertex function with the insertion of a longitudinal-field component squared, $\phi_1(\vec{q})\phi_1(-\vec{q})$. With this, $\Gamma_{11\,B}^{(2)}$ becomes precisely the two-point vertex function with insertion of the operator $\{\phi_1^2\}\equiv\phi_1^2-\phi^2/n$, in coordinate space, that belongs to the irreducible representation $(n-1,2)$ in the work of WY.[13] In contrast with the vertex functions for the symmetric theory, $\Gamma_{ij}^{(2)}$, $\Gamma_{ijk}^{(3)}$, and $\Gamma_{11}^{(2,1)}$, which are independent of the representation for the vectors $\vec{e}^{\,\alpha}$, $\Gamma_{11,1}^{(2)}$ is representation dependent,[14] as will become clear below.

To second order in the loop expansion, we obtain

$$\Gamma_{11,1}^{(2)}=1+D_{11}L_1w_0^2+[(D_{21}+D_{22})L_2^{(1)}+(D_{23}+D_{24})L_2^{(2)}+D_{25}L_2^{(3)}]w_0^4, \tag{2.11}$$

with the same integrals as in Eqs. (2.8) and (2.9), and new coefficients

$$\begin{aligned}D_{11}&=(n+1)[d_{1111}-(n+1)],\\ D_{21}&=2D_{25}=(n+1)^3[(n-3)d_{1111}+2(n+1)],\\ D_{22}&=2(n+1)^3(n-2)[d_{1111}-(n+1)],\\ D_{23}&=2D_{24}=(n+1)^3(n-1)[d_{1111}-(n+1)],\end{aligned} \tag{2.12}$$

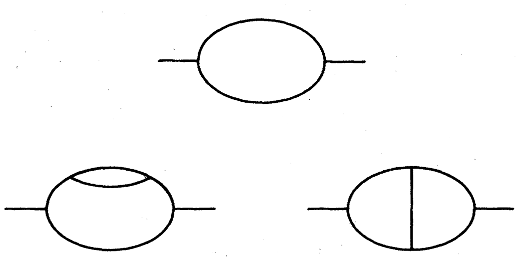

FIG. 1. Diagrams for the two-point 1PI vertex function $\Gamma_{ij}^{(2)}$, to two-loop order, given by Eqs. (A1)–(A3) and the coefficients in Table I.

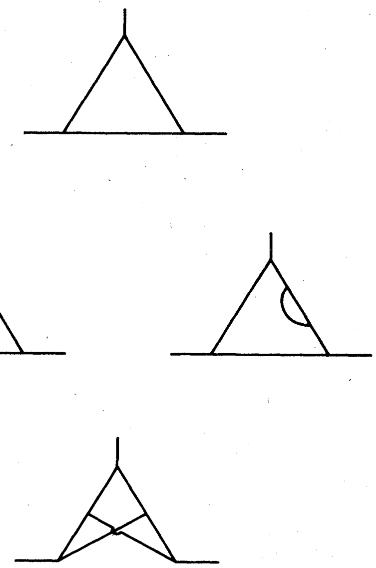

FIG. 2. Diagrams for the three-point 1PI vertex function $\Gamma_{ijk}^{(3)}$, to two-loop order, given by Eqs. (A5)–(A11) and the coefficients in Table I.

where

$$d_{ijkl}=\sum_{\alpha=1}^{p}e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha \tag{2.13}$$

depends explicitly on the representation. Indeed, we find

$$d_{1111}^{\rm PL}=(n^3+1)/n \tag{2.14}$$

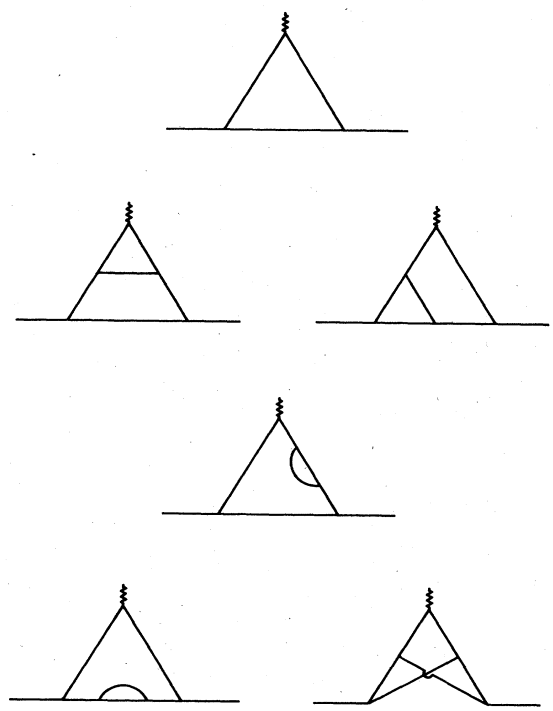

FIG. 3. One- and two-loop diagrams for the vertex functions $\Gamma_{11}^{(2,1)}$ or $\Gamma_{11,1}^{(2)}$ with a ϕ^2 insertion or ϕ_1^2 insertion, respectively.

TABLE I. Numerical coefficients of the diagrams for the symmetric theory in terms of the notation used in Ref. 6, where $\alpha=(n+1)^2(n-1)$, $\beta=(n+1)^2(n-2)$, and $\gamma=(n+1)^4\times(n^2-4n+5)$, for the $p(=n+1)$-state Potts model. For the Yang-Lee edge-singularity problem, $\alpha=\beta=-1$ and $\gamma=1$.

$A_{11}=\beta$
$A_{21}=3\beta^2$, $A_{22}=\frac{3}{2}\alpha\beta$, $A_{23}=\frac{1}{2}\gamma$
$B_{11}=\frac{1}{2}\alpha$
$B_{21}=\frac{1}{2}\alpha^2$, $B_{22}=\frac{1}{2}\alpha\beta$
$C_{11}=\alpha$
$C_{21}=C_{23}=2C_{24}=\alpha^2$, $C_{22}=4C_{25}=2\alpha\beta$

in the PL representation, while

$$d^{\mathrm{WY}}_{1111}=n+1 \tag{2.15}$$

in the WY representation.

Equations (2.9)—(2.11) then yield

$$\Gamma^{(2)}_{11\,B}=\frac{n-1}{n}\{1+F_{11}L_1w_0^2+[(F_{21}+F_{22})L_2^{(1)}+(F_{23}+F_{24})L_2^{(2)}+F_{25}L_2^{(3)}]w_0^4\}\ , \tag{2.16}$$

with

$$F_{ij}=\frac{nD_{ij}-C_{ij}}{n-1}$$

given explicitly for the two representations in Table II.

The vertex functions given so far correspond to the Potts model. For the Yang-Lee edge-singularity problem, represented by a scalar field theory with imaginary coupling, Eq. (2.3), only the symmetric theory needs to be considered and the results follow directly from those for the Potts model with the replacements $\alpha=\beta=-1$ and $\gamma=1$.

TABLE II. Numerical coefficients of the diagrams for $\Gamma^{(2)}_{11\,B}$, Eq. (2.16), for the theory with QSB in the representations of WY and PL, in terms of α, β, and γ defined in Table I.

WY representation
$F_{11}=-p^2$
$F_{21}=F_{25}=p^4$, $F_{22}=-2p^2\beta$, $F_{23}=2F_{24}=-p^2\alpha$
PL representation
$F_{11}=\beta$
$F_{21}=\frac{1}{2}F_{22}=\beta^2$, $F_{23}=2F_{24}=\alpha\beta$, $F_{25}=\frac{1}{2}\gamma$

III. RENORMALIZATION AND CRITICAL EXPONENTS FOR LR INTERACTIONS

Finite renormalized vertex functions are obtained, as usual,[18,19] by means of renormalization of the field ϕ, of the ϕ^2 insertion, and now also of the B insertion,[21] through the functions Z_ϕ, Z_{ϕ^2}, and Z_B, with coupling-constant renormalization, such that

$$\Gamma^{(2)}_{ij\ \mathrm{ren}}(w)=Z_\phi(w)\Gamma^{(2)}_{ij}(w_0) \tag{3.1a}$$

$$\Gamma^{(3)}_{ijk\ \mathrm{ren}}(w)=Z_\phi^{3/2}(w)\Gamma^{(3)}_{ijk}(w_0) \tag{3.1b}$$

$$\Gamma^{(2,1)}_{11\ \mathrm{ren}}(w)=Z_\phi(w)Z_{\phi^2}(w)\Gamma^{(2,1)}_{11}(w_0)\equiv\bar{Z}_{\phi^2}(w)\Gamma^{(2,1)}_{11}(w_0) \tag{3.1c}$$

$$\Gamma^{(2)}_{11\,B\ \mathrm{ren}}(w)=Z_\phi(w)Z_B(w)\Gamma^{(2)}_{11\,B}(w_0)\equiv\bar{Z}_B(w)\Gamma^{(2)}_{11\,B}(w_0)\ , \tag{3.1d}$$

in which the expansion coefficients in

$$w_0=w\sum_{n=0}a_{2n}w^{2n}\ , \tag{3.2}$$

where w is the renormalized coupling constant, and

$$Z_\phi(w)=\sum_{n=0}b_{2n}w^{2n}\ , \tag{3.3a}$$

$$\bar{Z}_{\phi^2}(w)=\sum_{n=0}c_{2n}w^{2n}\ , \tag{3.3b}$$

$$\bar{Z}_B(w)=\sum_{n=0}f_{2n}w^{2n}\ , \tag{3.3c}$$

are determined by minimal subtraction of dimensional poles in the bare vertex functions.[18] The expansions start with $a_0=b_0=c_0=f_0=1$, and next we find $b_2=0=b_4$, which means that

$$Z_\phi(w)=1\ , \tag{3.4}$$

as one would expect on the basis of the known renormalization of ϕ^4-field theory with LR interaction.[8,26] Equation (3.4) states that there is no field renormalization. It should be pointed out that $b_2=0$ follows from the finite result in Eq. (A1), whereas the vanishing of b_4 comes from the exact cancellation of dimensional poles involving coupling-constant renormalization to one-loop order, to be considered next. Although the results reported here are to two-loop order, one may expect Eq. (3.4) to hold to all orders.

For the remaining expansion coefficients, we obtain

$$a_2=-\beta/\epsilon' \tag{3.5a}$$

$$a_4=\frac{3\beta^2}{2\epsilon'^2}-\frac{1}{4\epsilon'}[3\beta^2S(\sigma)+\tfrac{3}{2}\alpha\beta F(\sigma)+\tfrac{1}{2}\gamma G(\sigma)]\ , \tag{3.5b}$$

$$c_2=-\alpha/\epsilon'\ , \tag{3.5c}$$

$$c_4=\frac{\alpha^2+2\alpha\beta}{2\epsilon'^2}-\frac{1}{4\epsilon'}[(\alpha^2+2\alpha\beta)S(\sigma)+\tfrac{3}{2}\alpha^2F(\sigma)+\tfrac{1}{2}\alpha\beta G(\sigma)]\ , \tag{3.5d}$$

in the symmetric theory, where α, β, and γ are the standard coefficients for the Potts model given in Table I, while $F(\sigma)$ and $G(\sigma)$ are defined in the Appendix, and

$$S(\sigma)\equiv\psi(\tfrac{3}{2}\sigma)-\psi(\sigma)-\psi(\sigma/2)+\psi(1)\ , \tag{3.6}$$

in which $\psi(z)$ is the logarithmic derivative of the Γ function. The momentum independence of the expansion coefficients follows, as it should,[18] from an exact cancellation of the terms involving the integral $\hat{L}_\sigma(k_1,k_2)$ defined in Eq. (A7).

With QSB, the coefficients, directly in terms of n, are

$$f_2^{\rm WY}=(n+1)^2/\epsilon'\ , \tag{3.7a}$$

$$f_4^{\rm WY}=\frac{(n+1)^4}{2\epsilon'^2}\{(5-2n)-\tfrac{1}{2}[(5-2n)S(\sigma)-\tfrac{3}{2}(n-1)F(\sigma)+G(\sigma)]\epsilon'\} \tag{3.7b}$$

in the WY representation, and

$$f_2^{\rm PL}=-(n+1)^2(n-2)/\epsilon'\ , \tag{3.8a}$$

$$f_4^{\rm PL}=\frac{3(n+1)^4}{2\epsilon'^2}\{(n-2)^2-\tfrac{1}{2}[(n-2)^2S(\sigma)+\tfrac{1}{2}(n-1)(n-2)F(\sigma)+\tfrac{1}{6}(n^2-4n+5)G(\sigma)]\epsilon'\} \tag{3.8b}$$

in the PL representation.

The Wilson functions, defined as[18,19]

$$\beta(w)=-\tfrac{1}{2}\epsilon'\left[\frac{\partial\ \ln w_0}{\partial w}\right]^{-1}\ , \tag{3.9}$$

$$\gamma_\phi(w)=\beta(w)\frac{\partial\ \ln Z_\phi}{\partial w}\ , \tag{3.10}$$

$$\bar{\gamma}_{\phi^2}(w)=\beta(w)\frac{\partial\ \ln\bar{Z}_\phi}{\partial w}\ , \tag{3.11}$$

$$\bar{\gamma}_B(w)=-\beta(w)\frac{\partial\ \ln\bar{Z}_B}{\partial w}\ , \tag{3.12}$$

then become, with Eqs. (3.3),

$$\beta(w)=-\tfrac{1}{2}\epsilon' w[1-2a_2w^2-(4a_4-6a_2^2)w^4]\ , \tag{3.13}$$

$$\gamma_\phi(w)=0\ , \tag{3.14}$$

$$\bar{\gamma}_{\phi^2}(w)=-\epsilon' w^2[c_2+(2c_4-c_2^2-2a_2c_2)w^2]\ , \tag{3.15}$$

$$\bar{\gamma}_B(w)=\epsilon' w^2[f_2+(2f_4-f_2^2-2a_2f_2)w^2]\ , \tag{3.16}$$

which yield the critical exponents for the symmetric theory in terms of the fixed-point value w^*, where $\beta(w^*)=0$, as

$$\eta=2-\sigma+\gamma_\phi(w^*)=2-\sigma\ , \tag{3.17}$$

$$\nu^{-1}=2-\eta+\bar{\gamma}_{\phi^2}(w^*)\ , \tag{3.18}$$

while the crossover exponents for QSB follow in the standard way from[14,21]

$$\phi=\nu[2-\eta-\bar{\gamma}_B^{\rm WY}(w^*)] \tag{3.19}$$

in the WY representation, and

$$\bar{\phi}=\nu[2-\eta-\bar{\gamma}_B^{\rm PL}(w^*)] \tag{3.20}$$

in the PL representation, both calculated at the symmetric fixed point.

For the case of the four-state Potts model, where there is a clear distinction between the two representations, the two crossover exponents correspond to different ways of implementing QSB, as we pointed out before.[14] Indeed, ϕ corresponds to a break in quadratic symmetry that maintains the permutation symmetry within a pair of Potts vectors, but breaks the equivalence between pairs, whereas $\bar{\phi}$ is related to QSB that favors a single Potts vector against the others. Both are exponents that characterize the response of the symmetric theory to QSB perturbations, in the form of a crossover to a lower-symmetry state. In the percolation limit $n\to 0$, ϕ yields the crossover to a random Ising magnet, whereas $\bar{\phi}$ is the exponent β of the symmetric theory, according to the group-theoretical arguments of WY.[13] This will be checked here for a LR interaction.

To calculate the critical exponents, we need the nontrivial fixed point w^* in Eq. (3.13),

$$w^{*2}=\frac{1}{2a_2}-\frac{4a_4-6a_2^2}{(2a_2)^3}\ , \tag{3.21}$$

which becomes, with Eqs. (3.5a)—(3.5b),

$$w^{*2}=-\frac{\epsilon'}{2\beta}\left[1+\frac{3\alpha\beta F(\sigma)+\gamma G(\sigma)+6\beta^2S(\sigma)}{8\beta^2}\epsilon'\right]\ , \tag{3.22}$$

to two-loop order. Since $\beta=p^2(p-3)$, there will be a real LR fixed point only when $p<3$, with a runaway at $p=3$, in contrast with the SR case that has a real fixed point for $p<\frac{10}{3}$. Note also that the coefficient of the term of $O(\epsilon'^2)$ involves $F(\sigma)$, defined in Eq. (A4). This contains the Γ function $\Gamma(-\sigma/2)$ which diverges in the SR limit $\sigma\to 2$ as a pole in $\epsilon=6-d$. The last term in w^{*2} then becomes of $O(\epsilon)$ and one may expect further terms of the same order coming from higher-order terms in ϵ' when $\sigma\to 2$. This signals the breakdown of the LR expansion in ϵ' when $\sigma\to 2$. We return to this point in the next section, where the stability of the LR fixed point is analyzed and the crossover to SR behavior is discussed.

Making use of the result that $\eta=2-\sigma$, Eq. (3.17), Eq. (3.18) yields, with Eqs. (3.3), (3.11), and (3.13),

$$\nu^{-1}=\sigma-\frac{c_2}{2a_2}\epsilon'-\frac{2a_2(2c_4-c_2^2-2a_2c_2)-c_2(4a_4-6a_2^2)}{(2a_2)^3}\epsilon'\ , \tag{3.23}$$

in which the last term is actually of $O(\epsilon'^2)$. Indeed, when written in terms of the tensorial coefficients, we find

$$\nu^{-1}=\sigma-\frac{\alpha}{2\beta}\epsilon'+\frac{\alpha\beta(\alpha-\beta)S(\sigma)+\frac{1}{2}\alpha(\beta^2-\gamma)G(\sigma)}{8\beta^3}\epsilon'^2, \tag{3.24}$$

which, as the fixed point, is valid for $p<3$. With the explicit dependence of the coefficients on n, given in Table I, we thus have ν for all $n<2$. In the percolation limit $n\to 0$,

$$\nu^{-1}=\sigma-\tfrac{1}{4}\epsilon'-\frac{4S(\sigma)+G(\sigma)}{128}\epsilon'^2, \tag{3.25}$$

and $\eta=2-\sigma$, the result of Eq. (3.17).

The fixed-point values of the Wilson function $\bar{\gamma}_B(w)$ for the B insertion, defined by Eq. (3.12), are given by

$$\bar{\gamma}_B^{\rm WY}(w^*)=-\frac{1}{2(n-2)}\epsilon' -\frac{(n-1)[2(n-2)S(\sigma)+(n-1)G(\sigma)]}{2^4(n-2)^3}\epsilon'^2 \tag{3.26}$$

in the WY representation, and

$$\bar{\gamma}_B^{\rm PL}(w^*)=\epsilon'/2 \tag{3.27}$$

in the PL representation, with no correction, at least to $O(\epsilon'^2)$. Indeed, we checked that the contributions of terms of this order *cancel identically,* and presumably there is a cancellation to all orders. The reason for the simplicity of this result can be seen as follows, based on the expectation that $\bar{\phi}$, defined in Eq. (3.20), should coincide with the critical exponent β. Using the scaling relations[27]

$$\beta=\tfrac{1}{2}(d\nu-\gamma),\quad \gamma=(2-\eta)\nu, \tag{3.28}$$

with the result $\eta=2-\sigma$, Eq. (3.17), and $d=3\sigma-\epsilon'$, we have

$$\beta=(1-\epsilon'/2\sigma)\sigma\nu. \tag{3.29}$$

Equation (3.20) then requires that $\bar{\gamma}_B^{\rm PL}(w^*)$ take the value given by Eq. (3.27). Note that this, as Eq. (3.29), would hold to all orders if that is the case with $\eta=2-\sigma$, a result not yet confirmed.

Explicit values for the crossover exponents now follow from Eqs. (3.19), (3.20), and (3.24) as

$$\phi=1+\frac{n}{2\sigma(n-2)}\epsilon'+\frac{n(n-1)[\sigma G(\sigma)+4(n-2)]}{2^4\sigma^2(n-2)^3}\epsilon'^2 \tag{3.30}$$

in the WY representation, and

$$\bar{\phi}=1+\frac{1}{2\sigma(n-2)}\epsilon' -\frac{(n-1)[2(n-2)\sigma S(\sigma)-\sigma G(\sigma)-4(n-2)]}{2^4\sigma^2(n-2)^3}\epsilon'^2 \tag{3.31}$$

in the PL representation. Notice that $\phi\to 1$ in the percolation limit $n\to 0$, and that $\bar{\phi}=\beta$, which follows from Eqs. (3.24) and (3.29), in accordance with the results of WY.[13] Also, $\phi=\bar{\phi}=1-(1/2\sigma)\epsilon'$, the result of the Gaussian model[28] for β when $n=1$, due to the vanishing of the trilinear tensorial coefficients. These properties are the same as for the Potts model with SR interaction, discussed in our previous work,[14] showing that they are only symmetry dependent, regardless of the interaction range.

It is interesting to compare the results obtained so far for LR interaction with those for the SR case to $O(\epsilon^2)$, $\epsilon=6-d$. It turns out that the SR exponents are obtained replacing σ by $2-\eta_{\rm SR}$ and, consequently, $\epsilon'=\epsilon-3\eta_{\rm SR}$. When taken together with the stability analysis of the next section, this indicates a continuous crossover to SR behavior at $\sigma=2-\eta_{\rm SR}$ only when $\eta_{\rm SR}>0$, i.e., for $p>2$. The apparent crossover to SR behavior when $n_{\rm SR}<0$ (the case for $p<2$), leading to $\sigma>2$, seems to be unphysical because there is a breakdown of the LR expansion before reaching that point.

Here we discuss the Yang-Lee edge singularity. We can now obtain, as a particular case, the fixed point and the critical exponents for a scalar ϕ^3 theory with imaginary coupling, and LR interaction, making use of the replacements $\alpha=\beta=-1$ and $\gamma=1$. According to Fisher, the critical exponent η for this theory yields the exponent for the Yang-Lee edge singularity, defined here as $\hat{\sigma}$, by means of the hyperscaling relation[11]

$$\hat{\sigma}=\frac{1}{\delta}=\frac{d-2+\eta}{d+2-\eta}. \tag{3.32}$$

The fixed point for the theory with imaginary coupling follows from Eq (3.22) as

$$w^{*2}=\frac{\epsilon'}{2}\left[1+\frac{3F(\sigma)+G(\sigma)+6S(\sigma)}{8}\epsilon'\right]. \tag{3.33}$$

As it should, with the imaginary d_{ijk} in Eqs. (2.1) and (2.3), there is a real fixed-point value for w^* in a LR expansion for all $\sigma<2$. Note that, as in the case of Eq. (3.22), this expansion breaks down as $\sigma\to 2$ for the reason given there.

From the independence of Eq. (3.4) on α, β, and γ, it follows that the result of Eq. (3.17), to two-loop order, still applies here. Consequently,

$$\hat{\sigma}=\frac{d-\sigma}{d+\sigma}, \tag{3.34}$$

to that order. Moreover, if Eq. (3.32) is an exact relationship, as Fisher's arguments and discussion based on available results seem to suggest, it is quite possible that Eq. (3.34) holds to all orders in perturbation theory. Making use of $\epsilon'=3\sigma-d$, we can also write

$$\hat{\sigma}=\frac{1}{2}-\frac{1}{8\sigma}\epsilon'+\frac{1}{32\sigma^2}\epsilon'^2+O(\epsilon'^3) \tag{3.35}$$

in the neighborhood of $d^*=3\sigma$, the crossover dimensionality to mean-field behavior. On the other hand, precisely at $d=d^*$, Eq. (3.34) yields $\hat{\sigma}=\frac{1}{2}$, in agreement with the mean-field result of Fisher. Indeed, his argument for a SR interaction (leading to $\hat{\sigma}=\frac{1}{2}$) can be taken over with no change for the present case.

A further check on our results follows from Eq. (3.24) and a general field-theoretic relationship between ν and η for a scalar ϕ^3-field theory valid to all orders in perturbation theory. According to this,

$$\nu^{-1}=\tfrac{1}{2}(d-2+\eta) , \tag{3.36}$$

the derivation of which is given below after discussing the consequences. With $\eta=2-\sigma$, at least to two-loop order, and possibly to all orders,

$$\nu^{-1}=\tfrac{1}{2}(d-\sigma) , \tag{3.37}$$

to the same order. When written in terms of $\epsilon'=3\sigma-d$, this becomes $\nu^{-1}=\sigma-\frac{1}{2}\epsilon'$, to be compared with the result of Eq. (3.24). The last term there is identically zero and what remains is

$$\nu^{-1}=\sigma-\tfrac{1}{2}\epsilon' , \tag{3.38}$$

with no correction, at least to $O(\epsilon'^2)$, in agreement with the above result. If all higher-order terms in Eq. (3.24) vanish, which one may expect together with the vanishing corrections to $\eta=2-\sigma$, then Eq. (3.37) becomes an exact relationship. Note, incidentally, that although this may just be a coincidence, Eq. (3.37) for a scalar ϕ^3-field theory with imaginary coupling yields a ν value which is twice that of the spherical model[28] (the $n\rightarrow\infty$ limit of ϕ^4-field theory for the n-vector model), and perhaps there is a deeper reason why this is so.

We now give a brief derivation of Eq. (3.36) for a scalar ϕ^3-field theory with arbitrary interaction.[29] This is based on the exact relationship

$$w_0\Gamma^{(2,1)}=\Gamma^{(3)} \tag{3.39}$$

between the bare two-point vertex function with a ϕ^2 insertion, $\Gamma^{(2,1)}$, and the three-point vertex $\Gamma^{(3)}$. Since this depends only on the geometrical structure of the diagrams for the vertex functions, a similar exact relationship also holds between renormalized quantities,

$$w\Gamma^{(2,1)}_{\rm ren}=\Gamma^{(3)}_{\rm ren} . \tag{3.40}$$

Using Eqs. (3.1b), (3.1c), and (3.2), written as $w_0=Z_w^{-1}w$, this means that $Z_w\bar{Z}_{\phi^2}=Z_\phi^{3/2}$. Equations (3.9)—(3.11) then yield

$$\bar{\gamma}_{\phi^2}=\tfrac{3}{2}\gamma_\phi-\frac{\epsilon'}{2}-\frac{\beta(w)}{w} , \tag{3.41}$$

which becomes, at the fixed point w^* where $\beta(w^*)=0$,

$$\bar{\gamma}_{\phi^2}(w^*)=\tfrac{3}{2}[\gamma_\phi(w^*)-\epsilon'/3] . \tag{3.42}$$

Equation (3.36) then follows, making use of Eqs. (3.17) and (3.18).

Finally, when Eq. (3.35) is compared with the SR result $\hat{\sigma}_{\rm SR}=\frac{1}{2}-\epsilon/12$ obtained by Fisher to one-loop order,[11] it can be seen that there is a discontinuity at $\sigma=2$, the point where the LR expansion breaks down. One may note, however, that if the LR $\hat{\sigma}$ is continued beyond this point, it goes over into $\hat{\sigma}_{\rm SR}$ at $\sigma=2-\eta_{\rm SR}$ (>2), but this is again unphysical.

IV. STABILITY OF THE LR FIXED POINT

To justify the calculations performed here so far, we analyze the stability of the LR fixed point by means of the Wilson-Fisher RG recursion relations,[12] to one-loop order, which is sufficient for our purpose, since $\eta_{\rm SR}=O(\epsilon)$. These may be written as

$$w'=\zeta^3 b^{-2d}\left[w+\beta w^3\int_{\vec q}^{>}\frac{1}{(sq^2+lq^\sigma+m_0^2)^3}\right] , \tag{4.1}$$

$$s'=\zeta^2 b^{-d-2}\left[s-\tfrac{1}{2}\alpha w^2\frac{\partial}{\partial k^2}\left(\int_{\vec q}^{>}\frac{1}{[s(\vec q+\vec k)^2+l\,|\,\vec q+\vec k\,|^\sigma+m_0^2](sq^2+lq^\sigma+m_0^2)}\right)_{k^2=0}\right] , \tag{4.2}$$

$$l'=\zeta^2 b^{-d-\sigma}l , \tag{4.3}$$

for the Potts model, with the replacement $\alpha=\beta=-1$ for the Yang-Lee edge-singularity problem. The integrations are over $b^{-1}<|\vec q|<1$, while ζ is the spin-rescaling factor and b gives the change in momentum scale, whereas w stands for $g_{30}/3!$ in Eq. (2.1). The recursion relation for m_0^2 (the usual r) will not be needed here. In a calculation to one-loop order one can set $m_0^2=0$ in Eqs. (4.1) and (4.2). Except for the coefficients in front of the integrals, the recursion relations are formally the same as for the Ising—spin-glass (ISG) problem considered recently be Change and Sak[9] (CS). We show next that our conclusions agree with theirs, wherever $\eta_{\rm SR}<0$, and we also indicate a stronger argument for the breakdown of the LR expansion at $\sigma=2$ than they do.

Assuming that the LR interaction determines the critical behavior, and taking $2-\sigma$ of $O(\epsilon')$ or smaller as the range in which crossover to SR behavior can take place, we find, following CS,

$$w'=b^{\epsilon'/2}\left[w+\beta\frac{w^3}{(s+l)^3}\ln b\right] , \tag{4.4}$$

$$s'=b^{-\eta}\left[s+\tfrac{1}{6}\alpha\frac{w^2}{(s+l)^2}\ln b\right] , \tag{4.5}$$

$$l'=l , \tag{4.6}$$

in which l is kept fixed by appropriate choice of ζ, while $\eta=2-\sigma$ and a factor of $K_d^{1/2}$ is again absorbed in w. The LR fixed point is then given by

$$\frac{w^{*2}}{(s^*+l^*)^3}=-\frac{\epsilon'}{2\beta} , \tag{4.7}$$

$$\frac{s^*}{l^*}=\frac{X}{2-\sigma-X}\ , \tag{4.8a}$$

$$X\equiv-\tfrac{1}{12}\epsilon'\frac{\alpha}{\beta}=\begin{cases}\tfrac{1}{12}\epsilon'\dfrac{p-2}{3-p} & \text{(Potts)},\\ -\tfrac{1}{12}\epsilon' & \text{(Yang-Lee)}\ .\end{cases} \tag{4.8b}$$

Note first that Eq. (4.7) coincides with the leading term in Eq. (3.22) for any fixed $\sigma<2$. Indeed, if the LR expansion is justified, s should be an irrelevant variable that can be set to zero, while l is a constant that can be chosen to be 1. Note also that, for both the Potts model with $p<2$ and for the Yang-Lee edge singularity, $X<0$, implying that $s^*/l^*<0$ for any fixed $\sigma<2$. In these two cases, $\eta_{SR}<0$. If l $(=l^*)$ is chosen to be fixed and positive, s^* will be negative, but there is nothing wrong with this as long as $s^*/l^*>-1$. Indeed, at any step of the RG iteration process, $s+l$ may be thought of as an effective l. However, when $s^*/l^*=-1$, $s^*+l^*=0$, and the expansion in terms of the propagator $1/(s+l)q^2$ becomes meaningless at the critical point where $s+l$ takes its fixed-point value. Equation (4.8a) shows that this occurs precisely *at* $\sigma=2$, and this is our criterion for the breakdown of the LR expansion when $X<0$. The case where $X>0$ will be considered separately below. In the ISG problem, where[30] $\eta_{SR}=-(6-d)/3<0$, $s^*/l^*=(3\sigma-d)/(3\sigma+d-12)<0$ for any fixed $\sigma<2$ becomes $s^*/l^*=-1$ at $\sigma=2$, again implying a meaningless critical propagator.

To justify the LR expansion one needs to show that s is an irrelevant variable and also establish the range of validity of the expansion. We have just shown for the latter that this is $\sigma\leq 2$, and that the breakdown of the expansion takes place when the propagator becomes meaningless. In contrast to this, CS argued that one should cease to consider the LR expansion when $\sigma\geq 2$ since the interaction then looks short ranged, but they did not show that the LR expansion actually breaks down at $\sigma=2$. We believe that this is what makes the apparent continuity of critical exponents at $\sigma=2-\eta_{SR}$ unphysical, wherever $\eta_{SR}<0$, as pointed out in the preceding section. The crossover problem with $\eta_{SR}>0$ and $\eta_{SR}<0$ is not symmetric about $\sigma=2$, and it will be shown below that there is a continuous crossover of exponents at $\sigma=2-\eta_{SR}$ when $\eta_{SR}>0$.

To show next the irrelevance of s, and that the LR fixed point is stable when $\sigma<2$, we follow CS and write Eq. (4.5) as

$$s'\cong s-(2-\sigma)s\ln b+X(s+l)\ln b\ . \tag{4.9}$$

This yields

$$\delta s'\equiv s'-s^*\cong b^{-(2-\sigma)+X}\delta s\ , \tag{4.10}$$

which implies that s^* is irrelevant for $X<0$ wherever $\sigma<2$, since these equations follow by assuming that the propagator has the form $1/(s+l)q^2$ up to the fixed point and becomes meaningless at $\sigma=2$. If one merely looks at Eq. (4.10), one may conclude that s^* ceases to be irrelevant only at $2-\sigma=X$. It can easily be checked, for the Potts model with $p>2$, for the Yang-Lee edge-singularity problem, and for the ISG problem, that this is the value of $\sigma=2-\eta_{SR}$ $(\eta_{SR}<0)$ where the LR exponents take the values for the SR expansion, and we agree with CS that this may be just a coincidence. We wish to emphasize that for $\eta_{SR}<0$ the LR expansion becomes meaningless beyond $\sigma=2$ with a negative s^*+l^*, and that the critical exponents are discontinuous at $\sigma=2$ for the three problems, in agreement with CS.

The stability of the LR fixed point for the Potts model with $p>2$ follows from Eq. (4.8) when $\sigma<2-X$, and the LR expansion breaks down for

$$\sigma=2-X\ . \tag{4.11}$$

To see what this means, let $\epsilon'=\epsilon-3\eta$ with $\eta=2-\sigma$. Equation (4.11) then states that

$$\sigma=2-\eta_{SR}\ , \tag{4.12}$$

in which $\eta_{SR}=(p-2)\epsilon/3(10-3p)$ is the exponent for SR interaction.[3–5] At the same time, the fixed-point coupling in Eq. (4.7) becomes

$$\frac{w^{*2}}{(s^*+l^*)^3}=\frac{2\epsilon}{p^2(10-3p)}\ , \tag{4.13}$$

the scaled value for the SR expansion. Equation (4.10) may now be used to show that s is irrelevant as long as $\sigma<2-X$, which is the range of validity of the LR expansion. When the expansion breaks down at $\sigma=2-X$, the calculation has to be reorganized, as shown by Sak for the n-vector model, assuming that the SR expansion applies beyond this point.[7] At $\sigma=2-\eta_{SR}$ the exponents cross over continuously from the LR to the SR expansion, as shown in Sec. III.

V. DISCUSSION AND CONCLUDING REMARKS

We have calculated the critical exponents for two ϕ^3-field theories with a LR interaction: (a) the continuum p-state Potts model and (b) the Yang-Lee edge-singularity problem, to two-loop order. With the usual reservation about ϕ^3-field theories, based on the ground-state instability, our results are meaningful at least for the percolation limit, eventually in a finite domain (in p) around this limit, and also for the Yang-Lee edge singularity. In the region where the LR expansion in $\epsilon'=3\sigma-d$ is valid, i.e., $\sigma<2-\eta_{SR}$ for $\eta_{SR}>0$, and any fixed $\sigma<2$ for $\eta_{SR}<0$, there are presumably a number of exact results that would appear to hold to all orders in renormalized perturbation theory, and these are based on the relation $\eta=2-\sigma$. Within the limitations of our work, we can only assert that the correction to this relation is of $O(\epsilon'^3)$ or smaller, and further work has to be done to confirm if it holds to all orders.[31] If it does hold, however, then $\beta=(1-\epsilon'/2\sigma)\sigma\nu$ given by Eq. (3.29) is an exact relationship, and also $\bar{\gamma}_B^{PL}(w^*)=\epsilon'/2$, Eq. (3.27), is an exact result. The latter follows from the identification of $\bar{\phi}$ with β, which according to the group-theoretical arguments of WY should be an exact result. It also follows that the Yang-Lee edge-singularity exponent, $\hat{\sigma}=(d-\sigma)/(d+\sigma)$, Eq. (3.24), and the relation $\nu^{-1}=\frac{1}{2}(d-\sigma)$, Eq. (3.37), for the Yang-Lee edge-singularity theory, are exact if $\eta=2-\sigma$ holds to all orders.

In the limit $p \to 1$ our results apply to percolation critical behavior for a random Ising ferromagnet with a LR interaction decaying as $1/R^{d+\sigma}$. Whether percolation with LR interaction has classical or nonclassical behavior is a matter of some controversy.[32,33] Mean-field arguments plus qualitative fluctuation corrections seem to indicate that the width of the critical region vanishes as the range of the interaction becomes infinite. Short of a more precise treatment of the dependence of the critical region on interaction range, one cannot rule out a small but finite critical region for the interaction used here. The specific exponents calculated in this work for $p=1$ apply to this region.

In studying the stability of the LR fixed point, we restricted the calculation to one-loop order because $\eta_{SR}=O(\epsilon)$, and for $\eta_{SR}<0$, which is the case for the $(p<2)$-state Potts model and the Yang-Lee edge-singularity problem, we find a discontinuity of critical exponents at $\sigma=2$, the site where the LR exponents cross over to SR behavior. This is in agreement with the result of CS for the effective ϕ^3 theory that describes the ISG problem. Although our stability analysis is done with the RG recursion relations, the same results are obtained in renormalized perturbation theory, following our earlier work on the n-vector model.[8] With the latter it can be seen that the expansion breaks down only at $\sigma=2$, as discussed in connection with Eq. (3.22). The situation is different for $\eta_{SR}>0$, the case of the Potts model for $p>2$. This is because in the way we do renormalized perturbation theory there is no sign of the breakdown of the LR expansion other than $\sigma=2$, regardless of whether η_{SR} is positive or not, whereas the RG recursion relations manifest a breakdown of the LR expansion for $\sigma=2-\eta_{SR}$, already to one-loop order. Further work remains to be done in order to bring the results of renormalized perturbation theory into agreement with those obtained by recursion relations, when $\eta_{SR}>0$. Presumably both of them yield a continuity of critical exponents at $\sigma=2-\eta_{SR}$.

An interesting extension of the work presented here is a calculation to one higher order and the resummation of the perturbation series to draw conclusions for $d=3\sigma-\epsilon'$, with finite ϵ', up to $d=3$. This and other extensions will be considered in future work.

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APPENDIX: ONE- AND TWO-LOOP INTEGRALS WITH DIMENSIONAL REGULARIZATION

We give here the results for the one- and two-loop integrals with dimensional regularization[20] for LR interaction, calculated at zero mass.[18] Starting with the integrals in $\Gamma_{ij}^{(2)}$, we find, in the notation of Eq. (2.7),

$$I_1 \equiv k^{-\sigma} I_1(k) \equiv k^{-\sigma} \int_{\vec{q}} \frac{1}{q^\sigma |\vec{q}+\vec{k}|^\sigma} = \tfrac{1}{2} F(\sigma) + O(\epsilon') , \tag{A1}$$

$$I_2^{(1)} \equiv k^{-\sigma} I_2^{(1)}(k) \equiv k^{-\sigma} \int_{\vec{q}} \frac{I_1(\vec{q})}{q^\sigma |\vec{q}+\vec{k}|^\sigma} = \tfrac{1}{4} F^2(\sigma) + O(\epsilon') , \tag{A2}$$

$$I_2^{(2)} \equiv k^{-\sigma} \int_{\vec{q}} \frac{L_1(\vec{k},\vec{q})}{q^\sigma |\vec{q}+\vec{k}|^\sigma} = \frac{1}{\epsilon'} F(\sigma) + O(1) , \tag{A3}$$

in which

$$F(\sigma) \equiv \frac{\Gamma(-\sigma/2)\Gamma(3\sigma/2)\Gamma^2(\sigma)}{\Gamma^2(\sigma/2)\Gamma(2\sigma)} , \tag{A4}$$

and

$$L_1(k_1,k_2) \equiv \int_{\vec{q}} \frac{1}{q^\sigma |\vec{k}_1+\vec{q}|^\sigma |\vec{k}_2-\vec{q}|^\sigma} \tag{A5}$$

is the integral for the one-loop graph in Eq. (2.8). The leading finite results for I_1 and $I_2^{(1)}$, of course, do not need renormalization, but they are kept since they enter the calculation to next-highest order. For the remaining graphs we find

$$L_1(k_1,k_2) = \frac{1}{\epsilon'} \left[1 - \tfrac{1}{2}[\psi(\tfrac{3}{2}\sigma) - \psi(1)]\epsilon' - \frac{1}{2} \frac{\Gamma(\frac{3}{2}\sigma)}{\Gamma^3(\sigma/2)} \epsilon' \hat{L}_\sigma(k_1,k_2) \right] , \tag{A6}$$

in which $\psi(z)$ is the logarithmic derivative of the Γ function $\Gamma(z)$, while

$$\hat{L}_\sigma(k_1,k_2) = \int_0^1 dx \int_0^1 dy\, \Theta(1-x-y)(xy)^{\sigma/2-1}(1-x-y)^{\sigma/2-1} \ln[x(1-x)k_1^2 + y(1-y)k_2^2 + 2xyk_1k_2] , \tag{A7}$$

where $\Theta(x)=1$ if $x \ge 0$, and zero otherwise. It is easy to check that when $\sigma=2$, $L_1(k_1,k_2)$ becomes the SR result given by Eq. (A11) of Amit's paper.[4] Next,

$$L_2^{(1)}(k_1,k_2) = \int_{\vec{q}} \frac{L_1(q,k_2)}{q^\sigma |\vec{k}_1-\vec{q}|^\sigma |\vec{k}_2+\vec{q}|^\sigma} \tag{A8}$$

yields the result

$$L_2^{(1)}(k_1,k_2) = \frac{1}{2\epsilon'^2} \left[1 - \tfrac{1}{2}[\psi(\tfrac{3}{2}\sigma) + \psi(\sigma) + \psi(\sigma/2) - 3\psi(1)]\epsilon' - \frac{\Gamma(\frac{3}{2}\sigma)}{\Gamma^3(\sigma/2)} \epsilon' \hat{L}_\sigma(k_1,k_2) \right] + O(1) , \tag{A9}$$

which can be checked to yield Amit's Eq. (A12), when $\sigma=2$. Then,

$$L_2^{(2)}(k_1,k_2)\equiv \int_{\vec{q}} \frac{I_1(q)}{|\vec{k}_1+\vec{q}|^{\sigma}\,|\vec{k}_2-\vec{q}|^{\sigma}}=\frac{1}{4\epsilon'}F(\sigma)+O(1)\,, \tag{A10}$$

which is less singular than the SR expression and therefore cannot be compared with that case when $\sigma=2$. However, we checked that the SR result, given by Amit's Eq. (A13), is obtained setting $\sigma=2$ at an intermediate step in the calculation. Finally,

$$L_2^{(3)}(k_1,k_2)\equiv \int_{\vec{q}} \frac{1}{q^{\sigma}|\vec{k}_1-\vec{q}|^{\sigma}} \int_{\vec{p}} \frac{1}{p^{\sigma}|\vec{k}_2-\vec{p}|^{\sigma}\,|\vec{k}_1-\vec{q}+\vec{p}|^{\sigma}\,|\vec{k}_2+\vec{q}-\vec{p}|^{\sigma}}=\frac{1}{4\epsilon'}G(\sigma)+O(1)\,, \tag{A11}$$

where

$$G(\sigma)\equiv\frac{\Gamma(\frac{3}{2}\sigma)\Gamma^3(\sigma/2)}{\Gamma^3(\sigma)}\,. \tag{A12}$$

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