

Statics and dynamics of spin and electric dipoles in three, four, and other dimensions

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Properly, spin is an antisymmetric tensor, and therefore in n -dimensional spaces where polar vectors have n components, spin has $n(n-1)/2$ components. Moreover, although a rotation can make an arbitrary polar vector have only one nonzero component, the same is not true for spin (and magnetic field). In particular, for $n=4$ an experimentalist can generate two independent field components (e.g., H_{12} and H_{34}) and, further, systems can develop two types of spontaneous symmetry-breaking internal fields. To illustrate some dynamical implications of the additional field component we have derived the equation of motion for spin in n dimensions, and for $n=4$ we apply it to free Larmor precession, where we find two modes [at $\gamma(H_{12} \pm H_{34})$]. Simple ferromagnets and spin glasses are also discussed for $n=4$. Since no true two-component spin can exist in any dimension, we consider XY spin dynamics for $n=3$ spins \vec{S} subject to strong uniaxial anisotropy. The behavior of electric and magnetic dipoles is contrasted, for the usual ($n=3$) case. It is also shown that normal modes for XY electric dipoles \vec{p} have only an in-plane polarization, contrasting to XY spins, which have a normal mode with an out-of-plane magnetization. The hypothesis, for $n=3$, of dipoles due to magnetic charge and a "gyroelectric effect" ($\vec{p} \propto \vec{S}$) is briefly discussed. It is noted that the usual concept of a scalar magnetic source (magnetic charge) is appropriate only to $n=3$.

I. INTRODUCTION

The study of both static and dynamic behavior of interacting spins \vec{S} has yielded a rich harvest of results. Normally the case of three-component spins is studied, but for some purposes one generalizes to both higher and lower dimensions. In that case, it is conventional to extend the exchange interaction

$$\mathcal{H}_{\text{ex}} = -\frac{1}{2} \sum_{i,j} J_{ij} S_{i\alpha} S_{j\alpha} \quad (\alpha=1,2,3), \quad (1.1)$$

and the Zeeman interaction

$$\mathcal{H}_z = -\gamma H_\alpha \sum_i S_{i\alpha} \quad (\alpha=1,2,3), \quad (1.2)$$

by simply letting the maximum value of the index α take on any integral value. (Here i and j are site indices, J_{ij} is the exchange constant, γ is the gyromagnetic ratio, and H_α is the external field.) This is a well-defined procedure, but it involves treating spin as a polar vector, despite the fact that it is actually an antisymmetric tensor.

To illustrate the point that the conventional procedure may take some liberties with the physics of true spins in higher dimension, consider the following. By a suitable rotation one can always find a coordinate system in which a polar vector (e.g., the electric field \vec{E}) has only one nonzero component. This is true for any dimensionality. It is not so obvious that the same is true for an antisymmetric tensor (e.g., the magnetic field $H_{\mu\nu}$) in any dimensionality. Indeed, it is generally true only for $n=2$ and 3 [note that when polar vectors have n components, antisymmetric tensors have $n(n-1)/2$ components].

Specifically, in Sec. II we show that there can be two independent components of $H_{\mu\nu}$ (e.g., H_{12} and H_{34}) in $n=4$, and in Sec. III we show that this yields two free-spin Larmor precession frequencies, at $\gamma(H_{12} \pm H_{34})$. To do this, we must generalize the equation of motion for spin angular momentum, which for $n=3$ takes the form

$$\frac{\partial S_\alpha}{\partial t} = -\frac{\partial E}{\partial \theta_\alpha}, \quad (1.3)$$

where E is the energy and θ_α represents a rotation about the α th axis.

It is also nontrivial to extend the physics for $n=3$ to $n=2$, where spin has only one component. In that case, true spins are Ising spins. Thus, to model XY spins it is necessary to consider $n=3$ spins confined to a plane by strong anisotropy. Moreover, the dynamics of $n=3$ spins is very different from the dynamics of $n=3$ electric dipoles, even with the same types of interactions, due to the differing relationships of the dipole moments to angular orientation and angular momentum.

Each of the above points is more fully discussed below. In Sec. II we show that the number of independent spin and magnetic field components is either $n/2$ or $(n-1)/2$, according to whether n is even or odd. In Sec. III we derive the equation of motion for spins in arbitrary dimension, and apply it in $n=4$ to Larmor precession by noninteracting spins in a magnetic field. Some care is required here, for spin in $n=4$ has more complicated representations than in $n=3$. In Sec. IV we consider the equations of motion for what we will define to be "simple" ferromagnets in $n=4$, showing that the spin waves satisfy

$\omega = Dk^2$, as for $n=3$. Section IV also considers a spin glass for $n=4$; the equations of motion are derived and applied to spin waves, where $\omega = ck$, as for $n=3$. In Section V we consider XY spins as $n=3$ spins with strong planar anisotropy. The spin waves for XY spins with nonparamagnetic order are determined, for nonuniform rotations about z , of wave vector k , and it is shown that $\omega = ck$, where c^2 is proportional to the geometric mean of the exchange J and the anisotropy D . We find that an out-of-plane magnetization must accompany the spin wave. The normal modes for rotations about x and y are also considered. Finally, in Sec. VI we consider the problem of determining the normal modes of $n=3$ electric dipoles. In the case of strong planar anisotropy and weak exchange (with no ordinary dipolar anisotropy), it is shown that the normal modes satisfy $\omega = ck$, where c^2 is proportional to the exchange J divided by the molecular moment of inertia I . The polarization is purely in-plane for this case, in contrast to what occurs for the magnetic case. Some implications of the possibility of magnetic charge and, associated with it, a "gyroelectric" effect, are also discussed in this section.

II. ON THE EIGENVALUES OF ANTISYMMETRIC TENSORS IN n DIMENSIONS

The results of this section are probably well known to at least some mathematicians, but we provide demonstrations here, both for completeness and because of the importance of the results for Larmor precession when $n=4$ (Sec. III).

Consider a real, antisymmetric tensor $K_{ij} = -K_{ji}$. With it we can define a Hermitian, pure imaginary tensor J_{ij} :

$$J_{ij} = iK_{ij}, \quad (2.1)$$

$$(J_{ji})^* = (iK_{ji})^* = -iK_{ji} = J_{ij}, \quad (2.2)$$

$$J_{ij}^* = -J_{ij}. \quad (2.3)$$

J can be diagonalized by a unitary transformation. In other words, there is an orthonormal basis set of (possibly complex) eigenvectors of J :

$$J |n\rangle = \lambda_n |n\rangle. \quad (2.4)$$

Taking the complex conjugate of (2.4), and using the fact that J is pure imaginary, we see that

$$J |n\rangle^* = -\lambda_n |n\rangle^*. \quad (2.5)$$

(Here we have used the fact that λ_n is real, since J is Hermitian.) The number of linearly independent eigenvectors with eigenvalue $-\lambda_n$ is therefore the same as for eigenvalue λ_n . Thus we conclude that the eigenvalues of J come in pairs with equal and opposite values, unless $\lambda_n = 0$. In that case, the process of complex conjugation yields that $|n\rangle$ can be chosen to be real.

Applying (2.4) and (2.5) to $K = iJ$, we find that

$$K |n\rangle = -i\lambda_n |n\rangle, \quad K |n\rangle^* = i\lambda_n |n\rangle^*. \quad (2.6)$$

The basis set

$$|n_R\rangle = \frac{1}{\sqrt{2}}(|n\rangle + |n\rangle^*), \quad |n_I\rangle = \frac{-i}{\sqrt{2}}(|n\rangle - |n\rangle^*) \quad (2.7)$$

is real, and from (2.6) it has the properties

$$K |n_R\rangle = \lambda_n |n_I\rangle, \quad K |n_I\rangle = -\lambda_n |n_R\rangle. \quad (2.8)$$

Since the transformation given in (2.7) is unitary, we have a sequence of two unitary transformations from one real basis (i.e., the one in which K_{ij} is defined) to another real basis [i.e., (2.7)]. As a consequence the combined unitary transformation is real, making the transformation an orthogonal one ($U^\dagger = U^{-1}$ and $U^* = U$ implies that $U^T = U^{-1}$). Since rotations are expressed via orthogonal transformations (and vice versa), it has been possible to find a rotation to a new orthogonal basis, whereby K satisfies (2.8) for each pair $|n_R\rangle$ and $|n_I\rangle$. In the general case $\lambda_n \neq 0$, so that (2.8) is nontrivial.

The above discussion has the following consequence for an even-dimensional space: the magnetic field H_{ij} has $n/2$ independent components, since (2.8) implies that $H_{ij} = 0$ if i and j are not pairs as in (2.7), for $H_{ij} = \pm\lambda_n$, if i and j are pairs as in (2.7).

For an odd-dimensional space there must be one component which is unpaired, corresponding to $\lambda_n = 0$. Thus, for odd-dimensional spaces H_{ij} has $(n-1)/2$ independent components.

III. FREE-SPIN DYNAMICS AND LARMOR PRECESSION

In this section we will obtain the equation of motion for an isolated spin, by considering the response of the system's energy to an infinitesimal rotation. This result will then be applied to the example of Larmor precession for a paramagnetic system in $n=4$.

The dynamical variables of our n -dimensional system are taken to be subject to an operation of the n -dimensional rotation group, $SO(n)$. (A system may admit more than one such operation, leading to the distinction among orbital, spin, and total angular momentum—see below.) These rotations are not necessarily dynamical symmetries (i.e., they may change the energy of the system). We will be concerned with infinitesimal rotations, which can be represented by antisymmetric tensors $d\theta_{\mu\nu}$. The first-order rotation of a vector is then described by

$$dA_\mu = d\theta_{\mu\nu} A_\nu. \quad (3.1)$$

Note that for $n=3$ the infinitesimal angle $d\theta_\beta$ satisfying

$$dA_\alpha = \epsilon_{\alpha\beta\gamma} d\theta_\beta A_\gamma \quad (n=3) \quad (3.2)$$

is in accord with (3.1) if we define

$$d\theta_\beta = \frac{1}{2} \epsilon_{\beta\mu\nu} d\theta_{\mu\nu} \quad (n=3). \quad (3.3)$$

To find the transformation properties of an antisymmetric tensor, we first consider the special type

$$C_{\alpha\beta} \equiv [A, B]_{\alpha\beta} = A_\alpha B_\beta - A_\beta B_\alpha. \quad (3.4)$$

Note that for $n=3$ we can set

$$C_\gamma \equiv \frac{1}{2} \epsilon_{\gamma\alpha\beta} C_{\alpha\beta} \quad (n=3), \quad (3.5)$$

and recover the cross product, $C_1 = A_2 B_3 - A_3 B_2$, etc. For general n , (3.4) implies

$$\begin{aligned} dC_{\alpha\beta} &= (dA_\alpha)B_\beta + A_\alpha(dB_\beta) - (dA_\beta)B_\alpha - A_\beta(dB_\alpha) \\ &= d\theta_{\alpha\gamma}C_{\gamma\beta} + d\theta_{\beta\gamma}C_{\alpha\gamma}. \end{aligned} \quad (3.6)$$

This equation is valid for antisymmetric tensors in general, since every antisymmetric tensor is a sum of objects of the form (3.4). We may rewrite (3.6) as

$$\begin{aligned} dC_{\mu\nu} &= \frac{1}{2}d\theta_{\alpha\beta}(\delta_{\alpha\mu}C_{\beta\nu} - \delta_{\beta\mu}C_{\alpha\nu} - \delta_{\alpha\nu}C_{\beta\mu} + \delta_{\beta\nu}C_{\alpha\mu}) \\ &\equiv \frac{1}{2}d\theta_{\alpha\beta} \frac{\partial C_{\mu\nu}}{\partial \theta_{\alpha\beta}}. \end{aligned} \quad (3.7)$$

Similarly, we write

$$dA = \frac{1}{2}d\theta_{\alpha\beta} \frac{\partial A}{\partial \theta_{\alpha\beta}} \quad (3.8)$$

for any quantity A . (The factor $\frac{1}{2}$ compensates for the redundancy $d\theta_{\beta\alpha} = -d\theta_{\alpha\beta}$.) In general, however, the "partial derivatives" in (3.8) cannot be interpreted literally, since there will not exist—even infinitesimally—corresponding configuration-space coordinates $\theta_{\alpha\beta}$. This is quite clear for any system whose configuration is completely described by a single vector (for instance, a point particle, or any axially symmetric rigid body with fixed center of mass): There are $\frac{1}{2}n(n-1)$ independent components $d\theta_{\alpha\beta}$, but the part of configuration space to which the vector can be rotated has only $n-1$ dimensions. On the other hand, if the system is an asymmetric rigid body (in n dimensions), then its configurations are in one-to-one correspondence with the elements of the rotation group, and (3.8) makes literal sense [the $\theta_{\alpha\beta}$ being coordinates for the Lie algebra of $SO(n)$]. In that case the derivatives are then evaluated at $\theta_{\alpha\beta}=0$.

We now consider explicitly the case where the quantity A is an angular momentum $S_{\alpha\beta}$. Note that the orbital angular momentum is the antisymmetric tensor

$$L_{\alpha\beta} = m[r, \dot{r}]_{\alpha\beta} \quad (3.9)$$

and any angular momentum $S_{\alpha\beta}$ is also an antisymmetric tensor. With $\hbar=1$, the Heisenberg equation of motion reads

$$\begin{aligned} \dot{S}_{\alpha\beta} &= i[\mathcal{H}, S_{\alpha\beta}] \\ &= -i[S_{\alpha\beta}, \mathcal{H}] \\ &= \frac{\partial}{\partial \theta_{\alpha\beta}} [e^{-(i/2)\theta_{\alpha\beta}S_{\alpha\beta}} \mathcal{H}(0) e^{-(i/2)\theta_{\alpha\beta}S_{\alpha\beta}}]_{\theta_{\alpha\beta}=0} \\ &= \frac{\partial \mathcal{H}}{\partial \theta_{\alpha\beta}} \\ &= \frac{\partial \mathcal{V}}{\partial \theta_{\alpha\beta}}. \end{aligned} \quad (3.10)$$

In deriving (3.10) we have used the fact that, for $\hbar=1$, the $S_{\alpha\beta}$ are the generators of (the relevant operations of) the rotation group. For $n=3$ we recover (1.3).

For a particle with spin (or a rigid body), $S_{\alpha\beta}$ will be the intrinsic spin if the transformation rotates only the orientation of the object, keeping its position in space fixed; it will be the total angular momentum if the posi-

tion of the particle (or center of mass of the body) is also changed by the rotation. Note that for orbital angular momentum, application to (3.9) of Newton's law for a conservative force,

$$F_\alpha = -\partial_\alpha \mathcal{V} = m\ddot{r}_\alpha, \quad (3.11)$$

can be shown to lead directly to

$$\dot{L}_{\alpha\beta} = \frac{\partial \mathcal{V}}{\partial \theta_{\alpha\beta}}. \quad (3.12)$$

We now apply (3.10) to a spin interacting with a magnetic field $H_{\mu\nu}$, so that the rotationally noninvariant part of the potential is

$$\mathcal{V} = -\frac{1}{2}\gamma S_{\mu\nu} H_{\mu\nu} \quad (3.13)$$

(which becomes $-\gamma \vec{S} \cdot \vec{H}$ for $n=3$). Setting (3.13) into (3.10) yields

$$\dot{S}_{\alpha\beta} = \frac{-\gamma}{2} H_{\mu\nu} \frac{\partial S_{\mu\nu}}{\partial \theta_{\alpha\beta}}. \quad (3.14)$$

From (3.7) we obtain

$$\frac{\partial S_{\mu\nu}}{\partial \theta_{\alpha\beta}} = \delta_{\alpha\mu} S_{\beta\nu} - \delta_{\beta\mu} S_{\alpha\nu} - \delta_{\alpha\nu} S_{\beta\mu} + \delta_{\beta\nu} S_{\alpha\mu}, \quad (3.15)$$

so that (3.15) becomes

$$\begin{aligned} \dot{S}_{\alpha\beta} &= \frac{\gamma}{2} (H_{\beta\nu} S_{\alpha\nu} - H_{\alpha\nu} S_{\beta\nu} - H_{\mu\beta} S_{\alpha\mu} + H_{\mu\alpha} S_{\beta\mu}) \\ &= \gamma (H_{\beta\nu} S_{\alpha\nu} - H_{\alpha\nu} S_{\beta\nu}). \end{aligned} \quad (3.16)$$

We now specialize to $n=4$, where $n(n-1)/2=6$. In this case, $H_{\mu\nu}$ has two nontrivial components, which we will take to be H_{12} and H_{34} . Rather than consider an individual spin, we will take $S_{\alpha\beta}$ to represent the sum over a large collection of spins, and we will assume that the equilibrium value satisfies $S_{\alpha\beta}^{(0)} = \gamma^{-1} \chi V H_{\alpha\beta}$. Here χ is the susceptibility and V is the volume of the system. (We will consider individual spins shortly.) Linearizing (3.16) about equilibrium yields

$$\dot{S}_{12} = \dot{S}_{34} = 0, \quad (3.17a)$$

$$\dot{S}_{13} = \gamma (H_{34} S_{14} - H_{12} S_{32}) = \gamma (H_{34} S_{14} + H_{12} S_{23}), \quad (3.17b)$$

$$\dot{S}_{14} = \gamma (H_{43} S_{13} - H_{12} S_{42}) = -\gamma (H_{34} S_{13} - H_{12} S_{24}), \quad (3.17c)$$

$$\dot{S}_{23} = \gamma (H_{34} S_{24} - H_{21} S_{31}) = \gamma (H_{34} S_{24} - H_{12} S_{13}), \quad (3.17d)$$

$$\dot{S}_{24} = \gamma (H_{43} S_{23} - H_{21} S_{41}) = -\gamma (H_{34} S_{23} + H_{12} S_{14}). \quad (3.17e)$$

Letting $A_p^+ = S_{p3} + iS_{p4}$ for $p=1,2$, Eqs. (3.17) yield

$$\begin{aligned} \dot{A}_1^+ &= -i\gamma H_{34} A_1^+ + \gamma H_{12} A_2^+, \\ \dot{A}_2^+ &= -i\gamma H_{34} A_2^+ - \gamma H_{12} A_1^+. \end{aligned} \quad (3.18)$$

Assuming that A_1^+ and A_2^+ vary as $e^{-i\omega t}$, (3.18) then yields

$$\begin{aligned} -i(\omega - \gamma H_{34})A_1^\dagger &= \gamma H_{12}A_2^\dagger, \\ -i(\omega - \gamma H_{34})A_2^\dagger &= -\gamma H_{12}A_1^\dagger, \end{aligned} \quad (3.19)$$

so that

$$(\omega - \gamma H_{34})^2 = (\gamma H_{12})^2, \quad (3.20)$$

$$\omega = \gamma(H_{34} \pm H_{12}). \quad (3.21)$$

What happens, then, is that H_{34} and H_{12} produce two Larmor precessions, at their sum and difference frequencies. Each mode involves the four transverse spin components. We have thus shown that both the static and the dynamic properties of paramagnetic spin systems are more complex for $n=4$ than one would expect from a simple generalization of polar vectors from $n=3$ to $n=4$.

The above analysis, however, is oversimplified. The angular momentum eigenstates of $SO(4)$ can be specified in terms of the eigenvalues k and l , associated with the operators¹

$$K_\beta = \frac{1}{2}(\frac{1}{2}\epsilon_{\mu\nu\beta 4}S_{\mu\nu} + S_{4\beta}), \quad (3.22)$$

$$L_\beta = \frac{1}{2}(\frac{1}{2}\epsilon_{\mu\nu\beta 4}S_{\mu\nu} - S_{4\beta}). \quad (3.23)$$

These are each like $SO(3)$ angular momentum operators, and they commute with one another. Thus k and l can take on both integral and half-integral values, where the effect on eigenstates is to yield $K^2 \rightarrow k(k+1)$ and $L^2 \rightarrow l(l+1)$.

For states with both k and l nonzero, if one treats the spins classically one finds no restrictions on the spin components. As a consequence, most of the previous discussion [and, in particular, (3.21)] remains valid. However, if $l=0$ and $k \neq 0$ then L_β must give zero when acting on such a state, and thus the values of $S_{\mu\nu}$ are restricted. In that case, only the $\omega = \gamma(H_{34} - H_{12})$ mode occurs, and $S_{12}^{(0)} = -S_{34}^{(0)} \propto (H_{12} - H_{34})$. On the other hand, for $k=0$ and $l \neq 0$ one finds $\omega = \gamma(H_{34} + H_{12})$, and $S_{12}^{(0)} = S_{34}^{(0)} \propto (H_{12} + H_{34})$.

One complication for both k and l nonzero concerns the statics. Only if $k=l$ does the relationship $S_{\alpha\beta}^{(0)} = \gamma^{-1}\chi V H_{\alpha\beta}$ hold. The reason is that the susceptibility χ , for $n=3$ spins in the paramagnetic regime, varies as $s(s+1)$, where s is the $n=3$ spin.² For $n=4$, the response can be broken into two $SO(3)$ responses: Unless each $SO(3)$ component has the same spin (i.e., $k=l$), the k and l susceptibilities will differ, and thus the response will be a tensor rather than a scalar.

One can also inquire into how $n=4$ spins interact. For $n=3$, exchanging two spins affects the energy in a fashion describable in terms of the angular momentum of the spins.³ We assume, without proof, that the same holds for $n=4$, but that the interaction involves the \vec{K} and \vec{L} $SO(3)$ subspaces individually. Thus we take two spins to interact via

$$V_{ex} = -J_k \vec{K}^{(1)} \cdot \vec{K}^{(2)} - J_l \vec{L}^{(1)} \cdot \vec{L}^{(2)}. \quad (3.24)$$

Such an interaction permits individual ordering in the \vec{K} and \vec{L} degrees of freedom. It is conceivable that J_k and J_l have opposite signs for $k \neq l$, which could lead to coexistent ferromagnetic and antiferromagnetic order. Even if

J_k and J_l have the same signs, but are unequal, one can have the interesting possibility of ordering of the same type, but with two different transition temperatures. Only if exchange terms of the form

$$\mathcal{V}'_{ex} = -J'(\vec{K}^{(1)} \cdot \vec{L}^{(2)} + \vec{K}^{(2)} \cdot \vec{L}^{(1)}) \quad (3.25)$$

occur can the k and l degrees of freedom interact. A more detailed study would be needed to determine if (3.25) can occur.

IV. SPIN DYNAMICS FOR $n=4$ FERROMAGNETS AND SPIN GLASSES

In the preceding section we treated a spatially uniform paramagnetic system. To consider nonuniform ferromagnets and spin glasses, it will be necessary to generalize (3.10). We will consider a collection of spins $S_{\alpha\beta}(j)$ subject to the Hamiltonian \mathcal{H} , so that we begin with (3.10) in the form

$$S_{\alpha\beta}(j) = \frac{\partial \mathcal{H}}{\partial \theta_{\alpha\beta}(j)}. \quad (4.1)$$

To go to the continuum limit we define the magnetization in a small region R via

$$m_{\alpha\beta}(x) \equiv \frac{\gamma}{V_R} \sum_{j \in R} S_{\alpha\beta}(j), \quad (4.2)$$

where x is at the center of R , which is of volume V_R and contains N_R spins. Likewise we define a rotation field

$$d\theta_{\mu\nu}(x) = \frac{1}{N_R} \sum_{j \in R} d\theta_{\mu\nu}(j). \quad (4.3)$$

The continuum version of \mathcal{H} takes the form

$$\mathcal{H} = \int \mathbf{H} d^n x', \quad (4.4)$$

where \mathbf{H} changes under both $d\theta_{\mu\nu}$ and $d(\partial_i \theta_{\mu\nu})$. Then

$$d\mathcal{H} = \frac{1}{2} \int d^n x' d\theta_{\mu\nu}(x') \frac{\delta \mathbf{H} \mathbf{H}}{\delta \theta_{\mu\nu}}, \quad (4.5)$$

where we have performed the usual variation of $d\theta_{\mu\nu}$ and $d(\partial_i \theta_{\mu\nu})$ in \mathcal{H} (and integrated by parts, dropping the surface terms) and

$$\frac{\delta \mathbf{H}}{\delta \theta_{\mu\nu}} \equiv \frac{\partial \mathbf{H}}{\partial \theta_{\mu\nu}} - \partial_i \frac{\partial \mathbf{H}}{\partial (\partial_i \theta_{\mu\nu})}. \quad (4.6)$$

Note that \mathbf{H} is an explicit function of $m_{\mu\nu}$ and $\partial_i m_{\mu\nu}$, so that the effect of $d\theta_{\mu\nu}$ and $\partial_i(d\theta_{\mu\nu})$ is to cause changes in $m_{\mu\nu}$ and $\partial_i m_{\mu\nu}$, via (3.6). This will be seen explicitly when we consider the ferromagnet for $n=4$.

With these preliminaries taken care of, the time derivative of (4.2) becomes

$$\begin{aligned}
\dot{m}_{\alpha\beta}(x) &= \frac{\gamma}{V_R} \sum_{j \in R} \dot{S}_{\alpha\beta}(j) \\
&= \frac{\gamma}{V_R} \sum_{j \in R} \frac{\partial \mathcal{H}}{\partial \theta_{\alpha\beta}(j)} \\
&= \frac{\gamma}{2V_R} \sum_{j \in R} \int d^n x' \frac{\delta \mathbf{H}}{\delta \theta_{\mu\nu}} \frac{\partial \theta_{\mu\nu}(x')}{\partial \theta_{\alpha\beta}(j)} \\
&= \frac{\gamma}{2V_R} \sum_{j \in R} \int d^n x' \frac{\delta \mathbf{H}}{\delta \theta_{\mu\alpha}} \frac{1}{N} (\delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\mu\beta} \delta_{\nu\alpha}) \\
&= \gamma \frac{\delta \mathbf{H}}{\delta \theta_{\alpha\beta}}, \tag{4.7}
\end{aligned}$$

where we have employed, in succession, (4.2), (4.1), (4.5), (4.3), and $\sum_{i \in R} (1) = N_R$, $\int_R d^n x' = V_R$. In practice, we will consider systems for which thermal averages have been performed, so that the microscopic Hamiltonian density \mathbf{H} will be replaced by the macroscopic energy density ϵ .

An alternative demonstration of (4.7) depends upon the fact that $\delta \mathbf{H} / \delta \theta_{\mu\nu}$ is zero in equilibrium. If we assume that $\dot{m}_{\mu\nu}$ is driven by some analytic function of $\delta \mathbf{H} / \delta \theta_{\mu\nu}$, which is zero when $\delta \mathbf{H} / \delta \theta_{\mu\nu} = 0$, then the functional form can be obtained from the uniform case, where $\dot{m}_{\mu\nu} = \gamma \partial \mathbf{H} / \partial \theta_{\mu\nu}$. We are then led to (4.7).

Consider now a ferromagnet. From our discussion of the k and l subspaces in the preceding section, we anticipate that the energy density takes the form

$$\epsilon = \epsilon_0 + \frac{1}{4} \rho_{s\alpha\beta,\mu\nu} \vec{\nabla} m_{\alpha\beta} \cdot \vec{\nabla} m_{\mu\nu},$$

where ϵ_0 is the energy density for uniform magnetization. However, for $k = l$, we expect that ρ_s can be replaced by a scalar, so that

$$\epsilon = \epsilon_0 + \frac{1}{4} \rho_s (\vec{\nabla} m_{\mu\nu})^2. \tag{4.8}$$

Correspondingly, the spontaneous magnetizations associated with k and l are expected to be the same. This means that we may take only $m_{34}^{(0)} = -m_{43}^{(0)} = m$ to be nonzero. (One can also take only $m_{12}^{(0)} = -m_{21}^{(0)} = -m$ to be nonzero. Both choices give equal magnitudes to the \vec{K} and \vec{L} components of the magnetization. Indeed, the \vec{K} and \vec{L} spontaneous magnetizations can be rotated individually without changing the energy. This would lead to more complex forms, which we will not consider here.)

From the thermal average of (4.6) and (4.7) we have

$$\begin{aligned}
TR &= T\dot{S} + T\partial_i j_i^s \\
&= T\partial_i j_i^s + \dot{\epsilon} - \frac{1}{2} h_{\mu\nu} \dot{m}_{\mu\nu} - \frac{1}{2} \Gamma_{i,\mu\nu} \partial_i \dot{\theta}_{\mu\nu} \\
&= \partial_i (Tj_i^s - j_i^\epsilon) - j_i^s \partial_i T + \frac{\gamma}{2} h_{\mu\nu} \partial_i \Gamma_{i,\mu\nu} - \frac{1}{2} h_{\mu\nu} D_{\mu\nu} - \frac{1}{2} \partial_i [\Gamma_{i,\mu\nu} \omega_{\mu\nu}] - \frac{1}{2} \omega_{\mu\nu} \partial_i \Gamma_{i,\mu\nu} \\
&= \partial_i [Tj_i^s - j_i^\epsilon - \frac{1}{2} \omega_{\mu\nu} \Gamma_{i,\mu\nu}] - j_i^s \partial_i T - \frac{1}{2} h_{\mu\nu} D_{\mu\nu} + \frac{1}{2} \partial_i \Gamma_{i,\mu\nu} (\gamma h_{\mu\nu} + \omega_{\mu\nu}). \tag{4.17}
\end{aligned}$$

$$\dot{m}_{\alpha\beta} = \gamma \frac{\delta \epsilon}{\delta \theta_{\alpha\beta}} = \gamma \left[\frac{\partial \epsilon}{\partial \theta_{\alpha\beta}} - \partial_i \frac{\partial \epsilon}{\partial (\partial_i \theta_{\alpha\beta})} \right]. \tag{4.9}$$

Use of (4.8) in (4.9), and the relationship

$$\begin{aligned}
\partial_i m_{\mu\nu} &= \partial_i \delta m_{\mu\nu} = \partial_i (\delta \theta_{\mu\gamma} m_{\gamma\nu}^{(0)} - \delta \theta_{\gamma\gamma} m_{\gamma\mu}^{(0)}) \\
&= m_{\gamma\nu}^{(0)} \partial_i \theta_{\mu\gamma} - m_{\gamma\mu}^{(0)} \partial_i \theta_{\gamma\gamma} \tag{4.10}
\end{aligned}$$

[which follows from (3.6)], yields

$$\dot{m}_{\alpha\beta} = -\gamma \rho_s (m_{\mu\beta}^{(0)} \nabla^2 m_{\mu\alpha} - m_{\mu\alpha}^{(0)} \nabla^2 m_{\mu\beta}). \tag{4.11}$$

Employing the assumption that only $m_{34}^{(0)} = -m_{43}^{(0)} = m$ is nonzero, (4.11) becomes

$$\dot{m}_{13} = -\gamma \rho_s [(-m) \nabla^2 m_{41}] = -\gamma \rho_s m \nabla^2 m_{14}, \tag{4.12a}$$

$$\dot{m}_{14} = -\gamma \rho_s [(m) \nabla^2 m_{31}] = \gamma \rho_s m \nabla^2 m_{13}, \tag{4.12b}$$

$$\dot{m}_{23} = -\gamma \rho_s [(-m) \nabla^2 m_{42}] = -\gamma \rho_s m \nabla^2 m_{24}, \tag{4.13a}$$

$$\dot{m}_{24} = -\gamma \rho_s [(m) \nabla^2 m_{32}] = \gamma \rho_s m \nabla^2 m_{23}. \tag{4.13b}$$

There are two pairs of independent, but degenerate, solutions of (4.12) and (4.13), where we assume a space-time dependence of $\exp[i(\vec{k} \cdot \vec{r} - \omega t)]$. Then

$$\omega = \pm (\gamma \rho_s m) k^2 \tag{4.14}$$

for each mode, and each mode is circularly polarized.

For a spin glass (a nearly rigid structure of spins pointing in all "directions"), the magnetization $m_{\mu\nu}$ and the orientation $\theta_{\mu\nu}$ are independent variables. Thus we must determine the equation of motion for $\theta_{\mu\nu}$. This is obtained using the methods of generalized hydrodynamics.⁴ We assume that

$$d\epsilon = T dS + \frac{1}{2} h_{\mu\nu} dm_{\mu\nu} + \frac{1}{2} \Gamma_{i,\mu\nu} d(\partial_i \theta_{\mu\nu}) \tag{4.15}$$

expresses the thermodynamics of the system, in terms of temperature T , entropy density S , internal field $h_{\mu\nu}$, and "bending field" $\Gamma_{i,\mu\nu}$. Next, we write the equations of motion

$$\begin{aligned}
\dot{\epsilon} + \partial_i j_i^\epsilon &= 0, \quad \dot{S} + \partial_i j_i^s = R > 0, \\
\dot{m}_{\mu\nu} &= \gamma \frac{\delta \epsilon}{\delta \theta_{\mu\nu}} + D_{\mu\nu} = -\gamma \partial_i \Gamma_{i,\mu\nu} + D_{\mu\nu}, \tag{4.16}
\end{aligned}$$

$$\dot{\theta}_{\mu\nu} = \omega_{\mu\nu},$$

where j_i^ϵ , j_i^s , R , $D_{\mu\nu}$, and $\omega_{\mu\nu}$ are to be determined by the requirement that (4.15) be consistent with (4.16) at all times. Thus

Hence, for the uniform state in the absence of dissipation ($R=0$, $D_{\mu\nu}=0$, $\partial_i T=0$) we have $\omega_{\mu\nu}=-\gamma h_{\mu\nu}$. If we now take the internal field and the bending field to be given by

$$h_{\mu\nu} = \frac{m_{\mu\nu}}{\chi}, \quad \Gamma_{i,\mu\nu} = \rho_s \partial_i \theta_{\mu\nu} \quad (4.18)$$

(which are generalizations of the $n=3$ case), and if we neglect the dissipative terms, we obtain

$$\dot{m}_{\mu\nu} = -\gamma \rho_s \nabla^2 \theta_{\mu\nu}, \quad \dot{\theta}_{\mu\nu} = \frac{\gamma m_{\mu\nu}}{\chi}, \quad (4.19)$$

whose solution is $\delta \theta_{\mu\nu} \propto e^{i(\vec{k} \cdot \vec{r} - \omega t)}$ with $\omega^2 = \gamma^2 (\rho_s / \chi) k^2$, just as for $n=3$ spin glasses. However, now there are six degenerate modes rather than three. Including dissipation, we find that

$$j_i^s = -\kappa \partial_i T, \quad D_{\mu\nu} = -D_1 h_{\mu\nu}, \quad (4.20)$$

$$\omega_{\mu\nu} = \gamma h_{\mu\nu} + D_2 \partial_i \Gamma_{i,\mu\nu},$$

where κ , D_1 , and D_2 are all positive, and represent dissipative processes (in the present case, diffusion of heat, magnetization, and $\delta \theta_{\mu\nu}$).

Our discussion of spin in $n=4$ has ended. For completeness, however, Sec. V considers the dynamics of XY spins (i.e., $n=3$ spins with strong uniaxial anisotropy) and Sec. VI treats the dynamics of electric dipoles, with an emphasis on the XY model. From these sections, the behavior of XY spins and XY electric dipoles can be contrasted.

V. XY SPIN DYNAMICS

True spins can only have $n(n-1)/2$ components, or $0, 1, 3, 6, \dots$ according to $n=1, 2, 3, 4, \dots$. Thus a two-component spin is a contradiction. In practice, when the XY model is treated, it is usually the statics which are under consideration, and this is no different from the statics of $n=2$ polar vectors. However, to deal with the dynamics of the XY model one must consider $n=3$ spins which are subject to a large uniaxial anisotropy. Both the ferromagnetic and antiferromagnetic XY models have been treated, from a hydrodynamic viewpoint, by Halperin and Hohenberg,⁵ who find spin waves with linear dispersion. At the microscopic level Villain⁶ has studied the ferromagnetic XY model, again finding that the spin waves have a linear dispersion, $\omega = ck$. The velocity c is found to be proportional to the square root of the product of the anisotropy D and the exchange J .

It turns out that XY spins with *any* type of nontrivial (i.e., nonparamagnetic) spin order have the same linear dispersion, with $c \propto (DJ)^{1/2}$. This is a result most directly and effectively seen by considering the macroscopic (or hydrodynamic) viewpoint. The appropriate variables in this case are θ_z , the macroscopic orientation about the z axis, and the magnetization along the z axis. The energy density ϵ , written in terms of θ_z and m_z , takes the form

$$\epsilon = \frac{m_z^2}{2\chi_z} + \frac{1}{2} \rho_s (\vec{\nabla} \theta_z)^2 \quad (5.1)$$

just as for $n=3$ spin glasses,⁴ except that now $\chi_z \propto D^{-1}$ rather than $\chi \propto J^{-1}$ ($\rho_s \propto J$ in both cases). The equations of motion for m_z and θ_z are⁴

$$\frac{\partial m_z}{\partial t} = -\gamma \frac{\delta \epsilon}{\delta \theta_z} = \gamma \rho_s \nabla^2 \theta_z, \quad (5.2)$$

$$\frac{\partial \theta_z}{\partial t} = \gamma \frac{\delta \epsilon}{\delta m_z} = \frac{\gamma}{\chi_z} m_z. \quad (5.3)$$

Taking $\theta_z, m_z \propto e^{i(\vec{k} \cdot \vec{r} - \omega t)}$, (5.2) and (5.3) yield

$$\omega^2 = c^2 k^2, \quad c = \gamma (\rho_s / \chi_z)^{1/2}. \quad (5.4)$$

It is important to observe that the third component of spin really takes part in the motion, and thus it cannot be eliminated. Indeed, one way to observe spin waves in an XY spin system would be to monitor the out-of-plane magnetization m_z . In Sec. VI we will contrast this to the behavior of XY electric dipoles, which have only an in-plane polarization p_z .

Actually, XY spins do have a response in the plane. However, this is not associated with θ_z . Considering the case with no remanence and no magnetic field, the energy associated with m_x and θ_x is given (in the long-wavelength limit) by

$$\epsilon = \frac{m_x^2}{2\chi_1} + \frac{1}{2} \kappa \theta_x^2, \quad (5.5)$$

where $\chi_1 \propto J^{-1}$ and $\kappa \propto D$. The equations of motion for m_x and θ_x yield [analogously to (5.2) and (5.3)]

$$\frac{\partial m_x}{\partial t} = \gamma \lambda \theta_x, \quad \frac{\partial \theta_x}{\partial t} = \frac{\gamma}{\chi_1} m_x, \quad (5.6)$$

whose solution gives the finite frequency at zero wave vector

$$\omega = \gamma (\kappa / \chi_1)^{1/2}. \quad (5.7)$$

This mode represents the stability of the system against tipping out of the XY plane. Analogous results hold for m_y and θ_y .

VI. THE DYNAMICS OF ELECTRIC DIPOLES

Electric and magnetic dipoles have the same number of components in $n=3$, where their statics is identical, given identical interactions. However, due to their different relationships to angular orientation and angular momentum, electric and magnetic dipoles have very different equations of motion. Magnetic dipoles in $n=3$ satisfy (1.3). To derive the equation of motion for electric dipoles, it will be necessary for us to consider a specific model.

Water is a polar molecular with a permanent dipole moment and many complex crystalline phases. Let us abstract this molecule by taking it to be a small rigid body to whose orientation an electric dipole is rigidly attached. In that case, the dipole moment reorients with the rigid body. If we construct a solid out of such rigid bodies, the changes in the dipole moment of the system can be obtained by studying the normal modes of vibration. Be-

sides the acoustic modes, there will also be optical modes (even if there is only one molecule per unit cell) due to the individual molecular vibrations.

The dipole moment \vec{p}_i on the i th molecule changes according to the rotation $\delta\theta_i$ of the i th molecule:

$$\delta\vec{p}_i = \delta\vec{\theta}_i \times \vec{p}_i. \quad (6.1)$$

The angular momentum of the i th molecule,

$$\vec{L}_i = \vec{I}_i \cdot \frac{d\vec{\theta}_i}{dt} \quad (6.2)$$

(where \vec{I}_i is its moment of inertia tensor), satisfies

$$\frac{d\vec{L}_i}{dt} = \vec{\Gamma}_i = -\frac{\delta E}{\delta\vec{\theta}_i}, \quad (6.3)$$

where $\vec{\Gamma}_i$ is the torque and E is the energy. For small $\delta\vec{\theta}_i$ we have

$$\vec{\Gamma}_i \approx -\vec{K}_i \cdot \delta\vec{\theta}_i, \quad (6.4)$$

where \vec{K}_i is a spring constant tensor representing the interaction of the dipole with the internal electric field (we here neglect the possibility of molecular torques due to displacements, and molecular forces due to rotations). Thus (6.2)–(6.4) give

$$\vec{I}_i \cdot \frac{d^2(\delta\vec{\theta}_i)}{dt^2} = -\vec{K}_i \cdot \delta\vec{\theta}_i = -\frac{\delta E}{\delta\vec{\theta}_i} \quad (6.5)$$

as the equation describing the reorientation $\delta\vec{\theta}_i$ of the electric dipole. Contrast this to (3.10), rewritten for $n=3$ as

$$\frac{d\vec{S}_i}{dt} = \frac{d\vec{\theta}_i}{dt} \times \vec{S}_i = -\frac{\delta E}{\delta\vec{\theta}_i}, \quad (6.6)$$

where \vec{S}_i is the spin of the i th molecule. Equation (6.5) is a second-order differential equation, whereas Eq. (6.6) is a first-order differential equation. Since $\vec{S}_i \rightarrow -\vec{S}_i$ under $t \rightarrow -t$, the time-reversal properties of (6.5) and (6.6) match.

In the continuum limit, where $\delta\theta_i \propto e^{i(\vec{k} \cdot \vec{r}_i - \omega t)}$, we expect that \vec{K}_i is replaced by a power series in k^2 , beginning with k^0 (since the interaction of real electric dipoles is anisotropic). Thus the normal modes corresponding to (6.5) are expected to satisfy

$$\omega^2 = \omega_0^2 + c^2 k^2 \quad (\text{dipolar}), \quad (6.7)$$

where ω_0^2 and c^2 are proportional to the dipole energy. On the other hand, if we assume an exchange interaction, then $\omega_0^2 = 0$ and c^2 is proportional to the exchange energy:

$$\omega^2 = c^2 k^2 \quad (\text{exchange}). \quad (6.8)$$

This result is even expected to hold when there is a strong anisotropy keeping the dipoles in the XY plane, so that $\delta\theta_x$ and $\delta\theta_y$ are suppressed. Superficially, then, the normal modes of XY electric and magnetic dipoles, interacting through exchange and planar anisotropy, are very similar, for they both yield $\omega = ck$. However, the source of c is very different for the electric dipole case, with $c \propto (J/I)^{1/2}$, where J is exchange and I a moment of inertia. Moreover, in the electric dipole case there is no tendency for a δp_z to develop, since $\delta\theta_x = \delta\theta_y = 0$. Thus “electric dipole waves” in such a system cannot be observed by monitoring the out-of-the-plane polarization p_z .

If magnetic charge exists (monopoles), a magnetically charged particle with spin \vec{S} might be expected to possess an electric dipole moment with $\vec{p} = \gamma^* \vec{S}$, γ^* being a “gyroelectric” ratio which transforms as $\gamma^* \rightarrow -\gamma^*$ under time reversal. In that case one would expect, by analogy to (5.2) and (5.3), that

$$\frac{\partial p_z}{\partial t} = -\gamma^* \frac{\partial \epsilon}{\delta p_z}, \quad (6.9)$$

$$\frac{\partial \theta_z}{\partial t} = \gamma^* \frac{\partial \epsilon}{\delta p_z}. \quad (6.10)$$

Thus a nonzero p_z would be generated by waves in an XY system of electric dipoles, if the dipoles are due to magnetic charge and a gyroelectric effect. It must be noted, however, that the above discussion very much depends upon the system being $n=3$; otherwise the expression $\vec{p} = \gamma^* \vec{S}$ would not be meaningful. Thus, for $n \neq 3$ one cannot have a “gyroelectric effect,” whereby an antisymmetric angular momentum tensor generates a polar electric dipole vector. This is related to the fact that magnetic charge only has meaning for $n=3$. Specifically, for $n=3$ the magnetic induction \vec{B} satisfies

$$\vec{\nabla} \cdot \vec{B} = \partial_i \frac{1}{2} \epsilon_{ijk} B_{jk} = 4\pi \rho^* \quad (n=3) \quad (6.11)$$

if a free magnetic charge density ρ^* exists. The key element in (6.11) is that i, j , and k differ. Thus there is no $n=2$ analog, and for $n=4$ one would have

$$\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \partial_\nu B_{\alpha\beta} = 4\pi \rho_\mu^* \quad (n=4), \quad (6.12)$$

where the source term has become a vector (rather than scalar). Clearly, only for $n=3$ can the source term be a magnetic charge.

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