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Structure factor for dilute magnetic systems

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The structure factor of dilute magnetic systems in the ordered phase contains, in addition to the usual q-dependent susceptibility, another term, which results from the fluctuations in the local quenched magnetic moments. This term, first considered by Grinstein, Ma, and Mazenko (GMM), is discussed here using scaling arguments, and calculated explicitly in $d=4-\epsilon$ and $d=2+\tilde{\epsilon}$ dimensions, as well as for an infinite number of spin components, $n\to\infty$. Contrary to the earlier calculation by GMM, this term is shown to have no coexistence singularities when $n\geq 2$. In the Ising case, this term combines with the usual susceptibility term to affect the measured critical amplitude ratios: The measured amplitude of the structure factor below T_c is *not* equal to that calculated for the susceptibility, as previously assumed.

I. INTRODUCTION AND RESULTS

The critical properties of quenched random systems have been the subject of much active research in the last decade.[1] In the present paper we discuss in particular the theory of the structure factor, measured by neutron scattering, of dilute magnets. To be specific, a possible model Hamiltonian describing such systems is

$$H=-\sum_{\langle i,j\rangle} J_{ij}\vec{S}_i\cdot\vec{S}_j=-\sum_{\langle i,j\rangle}([J]+\delta J_{ij})\vec{S}_i\cdot\vec{S}_j\ , \qquad (1.1)$$

where the $\{\vec{S}_i\}$ are n-component spins on the sites i of a d-dimensional lattice, and the J_{ij} are quenched random exchange interactions, with configurational average $[J]$ and with mean-squared fluctuation $\Delta=[\delta J_{ij}^2]$. An alternative formulation starts with the Ginzburg-Landau-Wilson formulation,

$$\bar{H}=\tfrac{1}{2}\int d^dx\{[\vec{\nabla}\vec{S}(x)]^2+r_0|\vec{S}(x)|^2+\tfrac{1}{4}u|\vec{S}(x)|^4+\phi(x)|\vec{S}(x)|^2\}\ , \qquad (1.2)$$

in which r_0 is linear in the temperature, and $\phi(x)$ represents the fluctuation in the local interactions,

$$[\phi]=0,\ \ [\phi(x)\phi(x')]=\Delta\delta(x-x')\ . \qquad (1.3)$$

It was shown heuristically by Harris[2] [for Eq. (1.1)], and confirmed using the renormalization group[3–5] [for Eq. (1.2)], that such randomness does not affect the critical exponents of systems with a negative specific heat exponent, α. This applies, e.g., to spin systems with $n>2$ at $d=3$. Renormalization-group studies in $d=4-\epsilon$ dimensions found a new, "random," fixed point, when the nonrandom ("pure") system has $\alpha>0$.[3–6] In particular, this "random" behavior is expected for the Ising case ($n=1$) at $d=3$.

Since α is typically small, it has been generally believed that this modified critical behavior might be difficult to observe. However, recent experiments on the dilute Ising antiferromagnet $Fe_{1-x}Zn_xF_2$ seem to yield new values for the exponents and the universal amplitude ratios.[7] These amplitudes roughly agree with recent one-loop calculations (to order $\epsilon^{1/2}$) for the "random" Ising behavior.[8]

Much of the experimental effort on critical phenomena relies on neutron-scattering measurements of the spin structure factor $\mathscr{S}_\mu(q)$.[7] This factor is related to the average Fourier transform of the spin-spin correlation function, $[\langle S_\mu(0)S_\mu(x)\rangle]$, where $\langle\cdots\rangle$ denotes a thermal average and $[\cdots]$ denotes a configurational one (over the quenched random variables). In the present paper we discuss the properties of

$$\mathscr{S}_{\parallel}(\vec{q})=[\langle S_{\parallel}(\vec{q})S_{\parallel}(-\vec{q})\rangle]\ ,$$

where $S_\parallel$ is the longitudinal-spin component (parallel to the average magnetization $\vec{M}=[\langle\vec{S}(x)\rangle]$), in the ordered phase. A simple addition and subtraction shows that $\mathscr{S}_\parallel(q)$ can be split into three terms,

$$\mathscr{S}_{\parallel}(q)=\chi_{\parallel}(q)+C^{(s)}(q)+M^2\delta(q)\ . \qquad (1.4)$$

The last term represents the usual Bragg peak, centered at the wave vector $q=0$ and related to the long-range order (for antiferromagnets, the point $q=0$ is at a corner of the Brillouin zone). $\chi_\parallel(q)$ is the Fourier transform of the usual longitudinal-spin correlation function,

$$\chi_{\parallel}(q)=[\langle S_{\parallel}(q)S_{\parallel}(-q)\rangle-\langle S_{\parallel}(q)\rangle\langle S_{\parallel}(-q)\rangle]\ . \qquad (1.5)$$

The function $C^{(s)}(q)$, first discussed by Grinstein, Ma, and Mazenko (GMM),[9] is specific for random systems. It measures the fluctuations in the local quenched magnetization $m(q)=\langle S_\parallel(q)\rangle$,

$$C^{(s)}(q)=[m(q)m(-q)]-M^2\delta(q)\ . \qquad (1.6)$$

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No such term is expected for the transverse structure factor, $\mathcal{S}_\perp(q)$.

In a nonrandom system, one has $C^{(s)}=0$, and the measurement of $\mathcal{S}_{||}(q)$ at $q\neq 0$ yields the q-dependent longitudinal susceptibility $\chi_{||}(q)$. For the Ising case, $\chi=\chi_{||}(q)$ is often well approximated by a Lorentzian,

$$\chi(q)\simeq 1/[\chi^{-1}(0)+Cq^2] , \qquad (1.7)$$

neglecting effects of the order of the small critical exponent η. For the systems with continuous symmetry, $n\geq 2$, one encounters coexistence-curve singularities; at $T<T_c$ and small q one has $\chi_{||}\propto q^{-(4-d)}$ and $\chi_\perp\propto q^{-2}$.[10] Equation (1.4) shows that a fit like (1.7) to $\mathcal{S}(q)$ is wrong for random systems. In addition to $\chi(q)$, $\mathcal{S}(q)$ contains the nontrivial term $C^{(s)}(q)$. For some purposes it is convenient to write[9]

$$C^{(s)}(q)=\chi_{||}^2(q)D(q) . \qquad (1.8)$$

For $d>4$, where mean-field theory applies, GMM showed that $D(q)=\Delta M^2$. In the Ising case, this becomes

$$C^{(s)}(q)=\frac{(\Delta M)^2}{[\chi^{-1}(0)+Cq^2]^2} , \qquad (1.9)$$

which is very reminiscent of the squared Lorentzian observed in systems with random fields.[1] Indeed, one might argue that a fluctuation in the exchange $J_{ij}S_iS_j$ generates, within mean-field theory, the term $J_{ij}M_iS_j$, which acts as a random field.[1]

In Sec. II we present a general scaling analysis of $C^{(s)}(q)$, and show that Eq. (1.9) must be modified for $d<4$. In the Ising case, near the "random" critical behavior, we find that for small q one has

$$C^{(s)}(0)=A\Delta\chi(0) , \qquad (1.10)$$

where A is a constant. Thus,

$$\mathcal{S}(q)\rightarrow(1+A\Delta)\chi(0) \quad \text{as } q\rightarrow 0 . \qquad (1.11)$$

If one writes $\chi(0)=\Gamma'|t|^{-\gamma}$, with $t=(T-T_c)/T_c$, then one concludes that a measurement of $\mathcal{S}(q)$ will yield the amplitude $(1+A\Delta)\Gamma'$, instead of Γ'. The amplitude ratio measured in Ref. 7 is therefore *not* Γ/Γ', but rather $(\Gamma/\Gamma')/(1+A\Delta)$. For the Hamiltonian (1.2), we show in Sec. V that $A=1/u+O(1)$, so that the measured amplitude ratio is

$$(\Gamma/\Gamma')/\{1+\Delta^*/u^*+O(\epsilon^{1/2})\}=(\Gamma/\Gamma')/\{\tfrac{7}{4}+O(\epsilon^{1/2})\} ,$$

and *not* Γ/Γ'. Of course, the corrections of order $\epsilon^{1/2}$ may be large, and the detailed comparison with the experiment should await their evaluation.

For finite small q, $C^{(s)}$ may also have corrections of order q^2. These will modify the measured coefficient of q^2 in $1/\mathcal{S}(q)$, and thus the amplitude ratio of the correlation length. For $n\geq 2$ we combine scaling arguments with new calculations in the limit $n\rightarrow\infty$ (Sec. III) and at $d=2+\tilde{\epsilon}$ dimensions (Sec. IV) to derive the behavior of $C^{(s)}(q)$. In the limit of small q we show that the leading q dependence of $D(q)$ has the form $1/(a+bq^{d-4})^2$, so that (contrary to GMM) $C^{(s)}(q)$ is *not* singular as $q\rightarrow 0$. This result also arises from a corrected exponentiation of the expansion in $\epsilon=4-d$ used by GMM (Sec. V). The leading temperature dependence of $C^{(s)}$ in this limit is $|t|^{-\gamma}$ for the "random" (unstable) behavior and $|t|^{-\gamma-\alpha}$ for the "pure" (stable) behavior. The behavior of $C^{(s)}(q)$ at large q is more complicated, and one needs more information before one can uniquely predict it. However, we expect that, in any case, $C^{(s)}(q)\rightarrow 0$ as $|t|\rightarrow 0$, so that

$$C^{(s)}(q)\ll\chi_{||}(q)\sim q^{-(2-\eta)} \quad (\text{large } q) .$$

Heuristic arguments, consistent with appropriate exponentiations of the ϵ expansion (Sec. V), are given to show that $C^{(s)}$ vanishes as M^2 for the "pure" behavior and as $-t\propto M^{1/\beta}$ for the "random" Ising behavior. In general, $C^{(s)}(q)$ is a mixture of M^2 and $M^{1/\beta}$ for large q.

In addition to the scaling behavior near T_c, we also discuss the behavior at low temperatures, using both scaling (Sec. II) and renormalization-group (Sec. IV) techniques.

II. SCALING

We start with a summary of the scaling properties of $\chi_{||}(q)$. Writing $\chi_{||}$ as a function of the three small variables q, t, and the (dimensionless) magnetic field h, and rescaling lengths by a factor b, one expects[11]

$$\chi_{||}(q,t,h)=\zeta^2 b^{-d}\chi_{||}(bq,b^{1/\nu}t,b^{y_h}h) , \qquad (2.1)$$

where $\zeta=b^{(d-2+\eta)/2}$ is the rescaling factor of the spins, ν is the correlation length exponent, and $y_h=(d+2-\eta)/2$. The exponent η is defined via $\chi_{||}(q,0,0)\propto 1/q^{2-\eta}$. In general, $\chi_{||}$ may also depend on the momentum cutoff Λ, and the right-hand side of Eq. (2.1) may thus depend on $b\Lambda$. We shall ignore this dependence except for special cases, emphasized below. Choosing now $b^{1/\nu}|t|$ to be of order unity, and remembering $hM^{-\delta}$ is a function of $M|t|^{-\beta}$, we conclude that one has

$$\chi_{||}(q,t,M)=|t|^{-\gamma}X(q|t|^{-\nu},M|t|^{-\beta}) , \qquad (2.2)$$

with $\gamma=(2-\eta)\nu$. At $h=0$ and $t<0$ one has $M=B|t|^{\beta}$, so that $M|t|^{-\beta}=B$ is a constant. We then expect that

$$\chi_{||}(q,t)=|t|^{-\gamma}X_0(q|t|^{-\nu}), \quad h=0 . \qquad (2.3)$$

The correct limit $|t|\rightarrow 0$ is now recovered if $X_0(x)\propto x^{-(2-\eta)}$ for $x\rightarrow\infty$. For the Ising case $n=1$ one expects no coexistence-curve singularities, and thus

$$X_0(x)\approx 1/[X_0^{-1}(0)+Cx^2] \quad \text{for } x\ll 1 ,$$

justifying the approximate form (1.7).

For $n\geq 2$, we must have[10] $X_0(x)\propto x^{-(4-d)}$ for $x\ll 1$, so that

$$\chi_{||}\propto|t|^{(4-d)\nu-\gamma}q^{-(4-d)}, \quad q\ll|t|^{-\nu} . \qquad (2.4)$$

When the randomness is "switched on," we must add the variable Δ. Near the "pure" behavior, there exist various arguments[2,3,5] showing that Δ scales as $\Delta\rightarrow b^{\alpha/\nu}\Delta$. Within our scaling approach, this introduces an additional scaling variable, $\Delta|t|^{-\alpha}$. This variable decays to zero for $\alpha<0$, leaving the above asymptotic results unchanged. When $\alpha>0$, Δ is renormalized towards a fixed-point value Δ^*, and all the critical exponents are changed.

However, the asymptotic scaling forms quoted above remain unchanged.

We now turn to the scaling of $C^{(s)}$. We start in the neighborhood of the "pure" fixed point, when $\Delta \to \Delta b^{\alpha/\nu}$. The same arguments which led to Eq. (2.1) now yield

$$C^{(s)}(q,t,h,\Delta)=\zeta^2 b^{-d} C^{(s)}(bq,b^{1/\nu}t,b^{y_h}h,b^{\alpha/\nu}\Delta) . \quad (2.5)$$

The factor ζ^2 comes from having two spin (or magnetization) factors [see Eq. (1.6)], and the factor b^{-d} comes from the δ function in their momenta. Thus,

$$C^{(s)}(q,t,M,\Delta)= |t|^{-\gamma}\tilde{\mathscr{C}}(q|t|^{-\nu}, M|t|^{-\beta}, \Delta|t|^{-\alpha}) . \quad (2.6)$$

As mentioned before, we expect $C^{(s)}$ to vanish for the "pure" system, i.e., when $\Delta=0$. Assuming that $C^{(s)}$ is analytic in Δ, the leading power of Δ will be 1, and we conclude that for $\Delta|t|^{-\alpha}$ one can write

$$C^{(s)}(q,t,M,\Delta)\simeq\Delta|t|^{-\gamma-\alpha}\mathscr{C}(q|t|^{-\nu}, M|t|^{-\beta}) . \quad (2.7)$$

Such a form is borne out by all our perturbative calculations, as well as by those of GMM, and is also confirmed by the (exact) mean-field result (1.9) and large-n limit (see below). It would helpful to have a general proof of this analyticity in Δ.

At zero field, $M|t|^{-\beta}$ is again equal to the constant B, and we have

$$C^{(s)}=\Delta|t|^{-\gamma-\alpha}\mathscr{C}_0(q|t|^{-\nu}), \quad h=0 . \quad (2.8)$$

In the Ising case $n=1$ one has a correlation length of order $|t|^{-\nu}$. In the limit $q|t|^{-\nu}\ll 1$ one expects no singularity in q, and thus we expect that $\mathscr{C}_0(x)$ approaches a nonzero constant for $x\to 0$, yielding

$$C^{(s)}\propto\Delta|t|^{-\gamma-\alpha}, \quad q|t|^{-\nu}\ll 1, \quad h=0 . \quad (2.9)$$

Our detailed calculations for $n\geq 2$, which correct GMM, arrive at the same conclusion, despite the possibility of coexistence-curve singularities.

The other limit, $q|t|^{-\nu}\gg 1$, concerns the limit $t\to 0$ at fixed q. For ferromagnets (or antiferromagnets) we expect all $m(x)$'s to have the same sign as M (or the staggered magnetization). Since $M=\sum_x m(x)/N$, the vanishing of M as $|t|\to 0$ implies the vanishing of all $m(x)$, and thus for any distance $|x|\ll|t|^{-\nu}$ we expect that

$$C^{(s)}(x)=[m(0)m(x)]-M^2 \quad (2.10)$$

vanishes at T_c. If the singularities in the two terms in (2.10) do not cancel exactly, then we expect that $C^{(s)}(x)$ may have a contribution of order $M^2\propto|t|^{2\beta}$. In addition, one might expect terms analytic in temperature, i.e., of order $-t\propto M^{1/\beta}$. The correlation function $\chi_{\parallel}(x)$ also has energylike terms, of order $|t|^{1-\alpha}\propto M^{(1-\alpha)/\beta}$, coming from $[\langle S_{\parallel}(0)S_{\parallel}(x)\rangle]$ (for small x).[12] None of our explicit calculations yields such terms in $C^{(s)}$, and we would be surprised if any other singular terms appear. Fourier-transforming $C^{(s)}(x)$, a combination of these considerations with Eq. (2.8) yields

$$C^{(s)}=D_1\Delta t\, q^{-(d-2+\eta+1/\nu)}+D_2\Delta|t|^{2\beta}q^{-2/\nu}+\cdots ,$$
$$q|t|^{-\nu}\gg 1, \quad h=0 . \quad (2.11)$$

We now turn to the "random" fixed-point behavior. Here, Δ approaches a fixed-point value Δ^*, and this value replaces $b^{\alpha/\nu}\Delta$ in Eq. (2.5). [In fact, one will have $\Delta^*+(\Delta-\Delta^*)b^{\lambda_\Delta}$, with $\lambda_\Delta<0$. We ignore the corrections to scaling which arise from the expansion in $\Delta-\Delta^*$.] Repeating the same arguments as before, Eqs. (2.7), (2.8), and (2.9) are now replaced by

$$C^{(s)}(q,t,m,\Delta)\approx\Delta^*|t|^{-\gamma}\mathscr{C}_r(q|t|^{-\nu}, M|t|^{-\beta}) , \quad (2.12)$$

$$C^{(s)}=\Delta|t|^{-\gamma}\mathscr{C}_{0,r}(q|t|^{-\nu}), \quad h=0 \quad (2.13)$$

and

$$C^{(s)}=A\Delta^*|t|^{-\gamma}, \quad q|t|^{-\nu}\ll 1, \quad h=0 \quad (2.14)$$

respectively, confirming Eq. (1.10). Indeed, these results are borne out by all our explicit calculations. Similarly, Eq. (2.11) is now expected to be replaced by

$$C^{(s)}=D_1\Delta^* t q^{-(2-\eta+1/\nu)}+D_2\Delta^*|t|^{2\beta}q^{-d}+\cdots ,$$
$$q|t|^{-\nu}\gg 1, \quad h=0 . \quad (2.15)$$

Note that at fixed q, the (positive) function $C^{(s)}$ increases with decreasing $|t|$ for $q|t|^{-\nu}\ll 1$ [Eqs. (2.9) or (2.14)], and decreases to zero as $|t|\to 0$, when $q|t|^{-\nu}\gg 1$. Thus, $C^{(s)}$ should have a maximum at $|t|\propto q^{1/\nu}$, with a $C^{(s)}_{\max}$ of order $q^{-(\gamma+\alpha)/\nu}$ ("pure") or $q^{-(2-\eta)}$ ("random").

So far we have considered only scaling near the critical point, $T\simeq T_c$. A similar analysis can be carried out at low temperatures, near the zero-temperature fixed point. Since $T=0$ is a "discontinuity" fixed point,[13] one has $\beta=0$, or $\lambda_h=d$. Therefore, $\zeta^2 b^{-d}\propto b^d$, and Eq. (2.5) is replaced by

$$C^{(s)}(q,T,h,\Delta)=b^d C^{(s)}(bq,b^{\lambda_T}T,b^d h,b^{\lambda_\Delta}\Delta) . \quad (2.16)$$

The scaling of T near $T=0$ depends on n, and is believed to be with $\lambda_T=1-d$ for $n=1$, and $\lambda_T=2-d$ for $n\geq 2$.[14] The exponent λ_Δ is found as follows: An expansion of the free-energy density of (1.1) to second order in δJ_{ij} and averaging yield

$$\frac{F}{T}=\left[\frac{F}{T}\right]_0+\frac{\Delta}{2T^2}\sum_{\langle ij\rangle}\{\langle(\vec{S}_i\cdot\vec{S}_j)^2\rangle_0-\langle\vec{S}_i\cdot\vec{S}_j\rangle_0^2\}/N . \quad (2.17)$$

Each of the factors in the last term, $\langle\vec{S}_i\cdot\vec{S}_j\rangle_0$, scales as the energy density, $E\propto\partial(F/T)/\partial(1/T)$. Since $(F/T)_0$ scales as b^{-d}, and $1/T$ scales as $b^{-\lambda_T}$, we find that $E\propto b^{\lambda_T-d}$. Thus, the last term in (2.17) scales as $b^{2\lambda_T-2d}$. Comparing with $(F/T)_0\propto b^{-d}$, we identify $(\Delta/T^2)\to b^{2\lambda_T-d}(\Delta/T^2)$, and hence $\Delta\to b^{-d}\Delta$, i.e., $\lambda_\Delta=-d$, independent of n. It therefore follows that *randomness in the exchange is highly irrelevant at low temperatures.* Expanding (2.16) to leading order in Δ, setting $h=0$, and choosing $bq=1$, we thus find

$$C^{(s)}=\Delta\mathscr{C}(Tq^{-\lambda_T},\Lambda/q) . \quad (2.18)$$

The explicit dependence on Λ/q is needed for finding corrections at low T.

If the randomness is weak, so that $J_{ij}>0$ for all i,j, then as $T\rightarrow 0$ we expect $m(x)\rightarrow 1$ for all x, so that $C^{(s)}\rightarrow 0$. Explicit calculations (Sec. IV) for $n>2$ yield $C(x,y)\approx x^2(1-ay^{-\lambda_T})^2$ for some positive constant a. Hence,

$$C^{(s)}\approx T^2\Delta(a^2\Lambda^{2(d-2)}-2a\Lambda^{d-2}q^{d-2}+q^{2(d-2)}) . \quad (2.19)$$

In the percolation problem, $m(\vec{x})m(\vec{x}\,')$ is equal to 1 only if $\vec{x}$ and $\vec{x}\,'$ are on the same cluster, so that $C^{(s)}$ need not vanish at $T\rightarrow 0$. In that case, $C(x)$ may approach a nonzero constant at $T\rightarrow 0$.

III. EXACT SOLUTION FOR $n\rightarrow\infty$

In the limit of an infinite number of spin components, $n\rightarrow\infty$, $C^{(s)}(q)$ can be calculated exactly. This calculation confirms the general scaling ideas presented in the preceding section and settles some but not all of the ambiguities present in the scaling analysis. In addition, as we discuss in Sec. V, knowledge of $C^{(s)}$ in the $n\rightarrow\infty$ limit allows us to infer the correct exponentiation of the $\epsilon=4-d$ expansion of $C^{(s)}$ calculated by GMM.

We begin with the Ginzburg-Landau-Wilson Hamiltonian (1.2) and assume, as in $n\rightarrow\infty$ calculations for the pure problem,[10,15] that the four-point coupling u is of order $1/n$. Exact calculations for the nonrandom model show that[15] $\alpha=(d-4)/(d-2)+O(1/n)$, so that randomness is irrelevant near the "pure" fixed point for $2<d<4$. It is therefore sufficient to calculate $C^{(s)}(q)$ to first order in Δ, which we do by using the decomposition (1.8) and calculating $D(q)$ to first order in Δ and $\chi_{\parallel}(q)$ to zeroth order. [In fact, the $O(\Delta)$ corrections to $\chi_{\parallel}(q)$ are $O(1/n)$.] The diagrams entering the calculation of $D(q)$ are shown in Fig. 1. Evaluating these diagrams we find

$$D(q)=\Delta M^2\{1-u_R n\Pi(0,q^2)+\tfrac{1}{4}n^2u_R^2[\Pi(0,q^2)]^2\} \quad (3.1a)$$

$$=\Delta M^2[1-\tfrac{1}{2}u_R n\Pi(0,q^2)]^2 . \quad (3.1b)$$

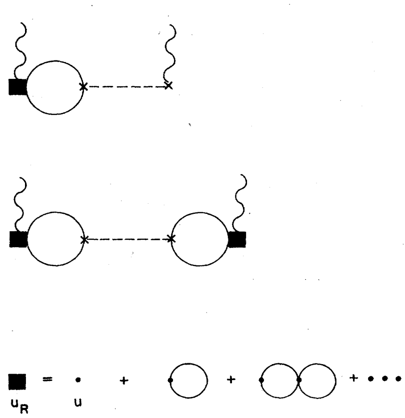

FIG. 1. Diagrams entering the $n=\infty$ calculation of $D(q)$, defined in (1.8). The crosses represent impurities, two crosses joined by a dotted line represent a factor of Δ, wiggly lines represent factors of M, solid lines are transverse, fully renormalized propagators, and the solid square represents a renormalized four-point vertex.

Here, u_R is the renormalized four-point vertex given by[10,15]

$$u_R=\frac{u}{1+\frac{1}{2}un\Pi(0,q^2)} , \quad (3.2)$$

and $\Pi(0,q^2)$ is the bubble graph shown in Fig. 1 given by

$$\Pi(0,q^2)=\int_{|p|<1}\frac{d^dp}{(2\pi)^d}\frac{1}{p^2(\vec{p}+\vec{q})^2} \quad (3.3a)$$

$$\approx q^{d-4}\Pi(0,1)-K_d/(4-d) . \quad (3.3b)$$

The zero argument in $\Pi(0,q^2)$ denotes that we are on the coexistence curve, where $h=0$ and the transverse propagators have zero mass. In evaluating (3.3a) we have assumed that the Brillouin zone is a unit sphere ($\Lambda=1$). The constant $\Pi(0,1)$ is given by

$$\Pi(0,1)=\tfrac{1}{2}B(\tfrac{1}{2}d-1,\tfrac{1}{2}d-1)K_d\pi(\tfrac{1}{2}d-1)\times\csc[\pi(\tfrac{1}{2}d-1)] , \quad (3.4)$$

where B is the beta function, and $K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(\frac{1}{2}d)$. Combining all these results we see that

$$D(q)=\Delta M^2/[1+\tfrac{1}{2}un\Pi(0,q)]^2$$
$$=\Delta M^2/\{1+\tfrac{1}{2}un[q^{d-4}\Pi(0,1)-K_d/(4-d)]\}^2 . \quad (3.5)$$

For small q this becomes

$$D(q)\approx\Delta M^2q^{2(4-d)}[\tfrac{1}{2}un\Pi(0,1)]^{-2} . \quad (3.6)$$

Since M^2 is of order n, our expansion in Δ becomes meaningful only if Δ is of order $1/n$ or smaller. When $\Delta=0$, the susceptibility $\chi_{\parallel}(q)$ is given in the $n\rightarrow\infty$ limit by[16]

$$\chi_{\parallel}(q)=\frac{1}{q^2+(2M^2/n)\Pi^{-1}(0,1)q^{4-d}} . \quad (3.7)$$

Using the $n\rightarrow\infty$ exponents, $\beta=\frac{1}{2}$ and $\gamma=2\nu=2/(d-2)$, we can write (3.7) in the scaling form (2.3), with the scaling function $X_0(x)$ given by

$$X_0(x)=\frac{1}{x^2+(2B^2/n)\Pi^{-1}(0,1)x^{4-d}} , \quad (3.8)$$

where B is the amplitude of M and $x=q\,|t|^{-\nu}$. The limiting behavior of $X_0(x)$ is

$$X_0(x)\approx\begin{cases}x^{-2}, & x\gg 1 \quad (3.9a)\\ x^{d-4}\Pi(0,1)n/2B^2, & x\ll 1 . \quad (3.9b)\end{cases}$$

Equation (3.9a) describes the critical region, with $\eta=O(1/n)$, and (3.9b) agrees with the scaling result (2.4).

Inserting (3.6) and (3.7) in (1.8), we find that to $O(\Delta)$, $C^{(s)}(q)$ is given by

$$C^{(s)}=\Delta\,|t|^{-1}\mathscr{C}_0(q\,|t|^{-1/(d-2)}) , \quad (3.10)$$

where

$$\mathscr{C}_0(x)=(2/un)^2\Pi^{-2}(0,1)B^2 \times\frac{x^{2(4-d)}}{[x^2+(2B^2/n)\Pi^{-1}(0,1)x^{4-d}]^2}\ . \tag{3.11}$$

This confirms the scaling form (2.8), since $\alpha+\gamma=1$ and $\nu=1/(d-2)$. The scaling function $\mathscr{C}_0(x)$ has the following limiting behavior:

$$C_0(x)\approx\begin{cases}1/u^2B^2, & x\ll 1 \quad (3.12a)\\ (2/un)^2\Pi^{-2}(0,1)B^2x^{2(2-d)}, & x\gg 1\ . \quad (3.12b)\end{cases}$$

Equation (3.12a) is in agreement with the asymptotic behavior displayed in (2.9) and indicates that there are no coexistence-curve singularities in $C^{(s)}(q)$, contrary to the results of GMM. Comparing (3.10b) to (2.11), we see that we cannot distinguish between the two terms in (2.11), since $2\beta=1$, and thus the result (3.12b) is consistent with both terms (but not with a term of order $|t|^{1-\alpha}$).

IV. LOW-TEMPERATURE AND $d=2+\tilde{\epsilon}$ EXPANSIONS

For $n\geq 2$ we can calculate the behavior of $C^{(s)}(q)$ in any dimension in the low-temperature regime $q|t|^{-\nu}\ll 1$ by constructing a low-temperature renormalization group (RG). In $d=2+\tilde{\epsilon}$ dimensions and for $n>2$, this calculation should also, in principle, give the behavior for $q|t|^{-\nu}\gg 1$, since $T_c=O(\tilde{\epsilon})$.[17] However, as we will explain below, there are difficulties in extracting the behavior of $C^{(s)}$ in this limit.

The low-temperature renormalization group begins with the fixed-length spin Hamiltonian (1.1). Following the conventional notation,[17] we write $\vec{S}=(\sigma,\vec{\pi})$ and assume that $\vec{M}$ is along the σ direction. The vector $\vec{\pi}$ has $n-1$ components and is a measure of the fluctuations about uniform ordering. Using the constraint $\vec{S}^2=1=\sigma^2+\vec{\pi}^2$, we can write the continuum version of (1.1) (measured relative to the ground state) as

$$H=\tfrac{1}{2}\int d^dx(\{[J]+\delta J(\vec{x})\}[\vec{\nabla}\vec{\pi}(\vec{x})]^2 +\{\vec{\nabla}[1-\vec{\pi}^2(\vec{x})]^{1/2}\}^2)\ . \tag{4.1}$$

In writing the reduced Hamiltonian $\overline{H}=H/T$, we will absorb the constant $[J]$ into the temperature prefactor. To construct low-temperature recursion relations, we expand the nonlinear interaction in (4.1) in powers of $\vec{\pi}$. To obtain recursion relations to $O(T^2,T\Delta)$, we can drop the factor $\delta J(\vec{x})$ multiplying the nonlinear term and simply expand the nonlinear term to $O(\pi^4)$. In Fourier space, we then have

$$\overline{H}=-\frac{1}{2T}\int\frac{d^dq}{(2\pi)^d}q^2\vec{\pi}(\vec{q})\cdot\vec{\pi}(-\vec{q})+\frac{1}{2T}\int\frac{d^dq}{(2\pi)^d}\int\frac{d^dq'}{(2\pi)^d}\vec{q}\cdot\vec{q}\,'\delta J(-\vec{q}-\vec{q}\,')\vec{\pi}(\vec{q})\cdot\vec{\pi}(\vec{q}\,')$$
$$+\frac{1}{2T}\int\frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\vec{q}_1\cdot\vec{q}_2\vec{\pi}(\vec{q}_1)\cdot\vec{\pi}(\vec{q}_3)\vec{\pi}(\vec{q}_2)\cdot\vec{\pi}(-\vec{q}_1-\vec{q}_2-\vec{q}_3)+O(\vec{\pi}^6)\ , \tag{4.2}$$

and the probability distribution of $\delta J(q)$ obeys

$$[\delta J(q)]=0\ , \tag{4.3a}$$

$$[\delta J(\vec{q})\delta J(\vec{q}\,')]=\Delta\delta(\vec{q}+\vec{q}\,')\ . \tag{4.3b}$$

The vertices for this theory are shown in Fig. 2. Recursion relations for T and Δ are constructed without replicas in two steps.[3(a),18] First, one integrates out the short-wavelength components of $\vec{\pi}(q)$ for an arbitrary function of $\delta J(\vec{q})$ to obtain recursion relations for $\delta J(\vec{q})$ and $T(\vec{q})$ (at this intermediate stage the temperature varies spatially). Subsequently, one forms a recursion relation for $\delta J(\vec{q})\delta J(\vec{q}\,')$, which—upon configurational averaging—produces a recursion relation for Δ. Configurational averaging of the recursion relation for $T(\vec{q})$ yields a recursion relation for the average temperature T. This procedure is illustrated graphically in Fig. 3, with the results

$$\frac{dT}{dl}=-(d-2)T+\frac{n-2}{2\pi}T^2+O(T^3,T^2\Delta)\ , \tag{4.4a}$$

$$\frac{d\Delta}{dl}=-d\Delta+\frac{n-2}{\pi}T\Delta+O(\Delta^2,T^2\Delta)\ , \tag{4.4b}$$

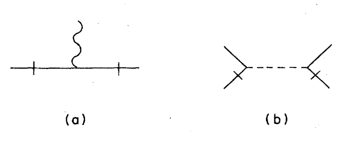

FIG. 2. Vertices entering the low-temperature renormalization-group theory. (a) The interaction corresponding to the second term on the right-hand side of (4.2). The slashes denote $\vec{q}$, and the wiggly line is $\delta J(\vec{q})$. Solid lines are $\vec{\pi}$ fields. (b) The third term on the right-hand side of (4.2). The dashed line is an expanded vertex which separates $\vec{\pi}$ fields with different vector components.

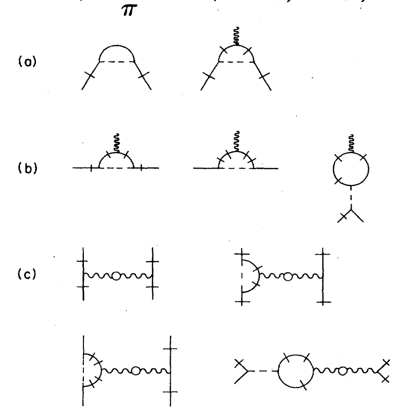

FIG. 3. Graphs contributing to the recursion relations (4.4). (a) Graphs renormalizing $T(\vec{q})$. The second graph averages to zero and does not contribute to T. (b) Graphs renormalizing $\delta J(\vec{q})$. (c) Graphs renormalizing Δ, found by squaring those in (b) (including the tree graph for δJ not shown) and averaging, which is denoted by an open circle.

where the rescaling factor b is now replaced by e^l. To study the low-temperature, noncritical properties, it suffices to keep only the lowest-order terms on the right-hand side of (4.4), which also represents the naive dimension of the operators. These latter terms describe the behavior around the fixed point $T^*=\Delta^*=0$, which is the sink for the low-temperature phase even when randomness is present, and which controls any coexistence-curve singularities which may be present. Note the agreement with our general scaling arguments, following Eq. (2.16). In $d=2+\tilde{\epsilon}$ dimensions, the recursion relations (4.4) exhibit a "pure" fixed point $T^*=2\pi\epsilon/(n-2), \Delta^*=0$, which is stable under the random perturbation[19] with an eigenvalue $\lambda_\Delta=-2+\tilde{\epsilon}+O(\tilde{\epsilon}^2)$, which, to this order in $\tilde{\epsilon}$, equals α/ν, in agreement with the expected scaling of Δ.

We first examine the behavior of $\chi_{\parallel}(q)$ in the pure system, to demonstrate how the scaling behavior described in Sec. II is confirmed by the low-temperature RG analysis. As noted in Sec. I, $\chi_{\parallel}(q)$ is a connected correlation function of the σ field, and, hence, as shown in Ref. 17, it obeys the homogeneous scaling relation (2.1). Using $b=e^l$ in (2.1), we then have

$$\chi_{\parallel}(q,T)=\zeta^2(l)e^{-dl}\chi_{\parallel}(qe^l,T(l)) \,, \tag{4.5}$$

where the spin rescaling factor $\zeta(l)$ is given by[20]

$$\zeta(l)=\exp\left[dl+\frac{1}{2}\frac{n-1}{n-2}\ln\left[1+\frac{n-2}{2\pi\tilde{\epsilon}}T(e^{-\tilde{\epsilon}l}-1)\right]\right] , \tag{4.6}$$

and[20]

$$T(l)=\frac{Te^{-\tilde{\epsilon}l}}{1+[(n-2)/2\pi\tilde{\epsilon}]T(e^{-\tilde{\epsilon}l}-1)} \,, \tag{4.7}$$

correct to leading order in $\tilde{\epsilon}$. We choose $qe^l=1$, and evaluate $\chi_{\parallel}(1,T(l))$ using an ordinary low-temperature perturbation expansion, since there are no infrared problems. To extract the q dependence of $\chi_{\parallel}(q)$ it is sufficient to keep the lowest term in this expansion, which corresponds to Fig. 4, i.e.,

$$\begin{aligned}\chi_{\parallel}(1,T(l))&\simeq\tfrac{1}{4}\int d^dx\, e^{i\hat{n}\cdot\vec{x}}[\langle\pi^2(x)\pi^2(0)\rangle-\langle\pi^2\rangle^2]\\ &=\tfrac{1}{2}(n-1)T^2(l)\frac{1}{(2\pi)^d}\int\frac{d^dp}{p^2(\vec{p}+\hat{n})^2}\\ &=\frac{1}{4\pi\tilde{\epsilon}}(n-1)T^2(l)+O(\tilde{\epsilon}^2,T^2(l)) \,. \end{aligned}\tag{4.8}$$

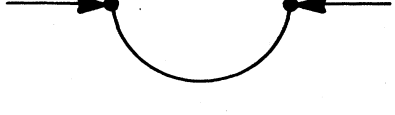

FIG. 4. Lowest-order (in T) graph contributing to $\chi_{\parallel}(q=1,T(l))$, evaluated in (4.8). The arrows represent external momenta $\vec{q}$. Solid lines are $\vec{\pi}$ propagators.

The integration of p is over the entire Brillouin zone, and $\hat{n}$ is a unit vector along $\vec{q}$. Higher-order diagrams correspond to higher powers of $T(l)/\tilde{\epsilon}$. These diagrams then will affect the amplitude of $\chi_{\parallel}(q,T)$ near T_c but not the q dependence. Using (4.5)—(4.8) we find

$$\chi_{\parallel}(q)=\frac{n-1}{2\pi}\frac{T^2}{\tilde{\epsilon}}q^{-2+\tilde{\epsilon}}\left[1-\frac{T}{T_c}+\frac{T}{T_c}q^{\tilde{\epsilon}}\right]^{(3-n)/(n-2)} , \tag{4.9}$$

with the limiting behavior

$$\chi_{\parallel}(q)\approx\begin{cases}\dfrac{1}{2\pi}(n-1)\dfrac{T^2}{\tilde{\epsilon}}q^{d-4}\left[1-\dfrac{T}{T_c}\right]^{(3-n)/(n-2)} , & q\,|\,t\,|^{1/\tilde{\epsilon}}\ll 1 \quad (4.10a)\\ \dfrac{1}{2\pi}(n-1)\dfrac{T_c^2}{\tilde{\epsilon}}q^{-2+\tilde{\epsilon}/(n-2)}, & q\,|\,t\,|^{1/\tilde{\epsilon}}\gg 1 \,. \quad (4.10b)\end{cases}$$

Recalling that $\eta=\tilde{\epsilon}/(n-2)+O(\tilde{\epsilon}^2)$,[17] we see that (4.10) gives the expected limiting behavior. In particular, (4.11a) confirms Eq. (2.4), since

$$(4-d)\nu-\gamma\simeq(3-n)/(n-2)+O(\tilde{\epsilon}) \,.$$

Equation (4.11a) also agrees with the low-temperature behavior of the $n\to\infty$ result (3.7) since $\pi(0,1)\approx 1/\pi\tilde{\epsilon}$.

A similar analysis can be carried out for $C^{(s)}(q)$, although due to the difficulties mentioned below (4.8), regarding $\chi_{\parallel}(q)$ when $T\approx T_c$, we will not be able to see the expected vanishing of $C^{(s)}(q)$ as $T\to T_c$. We restrict our attention then to low temperatures, but arbitrary dimensionality. Since, upon configurational averaging, $C^{(s)}(q)$ is a connected two-point function of the σ field [see (1.5)], it obeys a homogeneous scaling relation:

$$C^{(s)}(q,T,\Delta,\Lambda)=\zeta^2(l)e^{-dl}C^{(s)}(qe^l,T(l),\Delta(l),\Lambda e^l) \,. \tag{4.11}$$

At low temperatures we have [see (4.4b), (4.6), and (4.7)] $\zeta(l)=e^{dl}$, $T(l)=Te^{-(d-2)l}$, and $\Delta(l)=\Delta e^{-dl}$. Choosing $qe^l=1$, we have

$$C^{(s)}(q,T,\Delta,\Lambda)=q^{-d}C^{(s)}(1,Tq^{d-2},\Delta q^d,\Lambda/q) \,. \tag{4.12}$$

We evaluate $C^{(s)}$ on the right-hand side of (4.12) using ordinary perturbation theory. The lowest-order diagram is shown in Fig. 5 and is given by

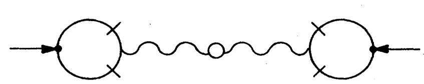

FIG. 5. Lowest-order graph contributing to $C^{(s)}(q=1)$; see (4.13). See the captions of Figs. 2—4 for the graphical conventions used here.

$$C^{(s)}(1,T(l),\Delta(l),\Lambda e^{l})\approx\tfrac{1}{4}\int d^dx\, e^{i\hat{n}\cdot\vec{x}}\{[\langle\pi^2(x)\rangle\langle\pi^2(0)\rangle]-[\langle\pi^2(x)\rangle]^2\}$$
$$=\tfrac{1}{4}(n-1)^2\frac{\Delta(l)T^2(l)}{(d-2)^2}[(\Lambda e^{l})^{d-2}-A]^2\,, \tag{4.13}$$

where A is a positive constant dependent on dimensionality. Combining (4.12) and (4.13) we find

$$C^{(s)}(q,T,\Delta,\Lambda)=\frac{1}{4}\left[\frac{n-1}{d-2}\right]^2\Delta T^2(\Lambda^{2(d-2)}-2A\Lambda^{d-2}q^{d-2}+A^2q^{2(d-2)})\,. \tag{4.14}$$

Thus, there are no coexistence-curve singularities in $C^{(s)}$. In fact, the renormalization-group analysis is not even needed here since the straightforward perturbation-theory result (4.13) is finite even if $\vec{q}=\vec{0}$. We also note that (4.14) is in agreement with $n\rightarrow\infty$ result (3.11).

V. EXPANSION IN $d=4-\epsilon$ DIMENSIONS

The diagrams needed for the ϵ expansion of $D(q)$, defined via Eq. (1.8), were calculated by GMM. Their results are summarized in their Eq. (4.33), which reduces, in the limit $h\rightarrow 0$, to

$$D(q)=\Delta M^2[1-\tfrac{1}{2}K_4(4\Delta-3u)\ln(uM^2)+3(\Delta-u)Q(uM^2/q^2)+K_4u(n-1)\ln q]\,, \tag{5.1}$$

with

$$Q(x^2)=\tfrac{1}{2}K_4\left[(1+4x^2)^{1/2}\ln\left[\frac{(1+4x^2)^{1/2}-1}{2x}\right]+1\right] \tag{5.2}$$

and $K_4=1/8\pi^2$. (The factor $\frac{1}{2}$ was missing in GMM.)

It is now easy to see that, at this order in u and Δ,

$$D(q)\approx\Delta M^2[1-K_4(4\Delta-3u)\ln M+K_4u(n-1)\ln q] \tag{5.3}$$

for $q \ll M$ (i.e., $q\ll|t|^{\nu}$, to leading order), and

$$D(q)\approx\Delta M^2\{1-K_4\Delta\ln M+K_4[(n+2)u-3\Delta]\ln q\} \tag{5.4}$$

for $q\gg M$ ($\propto|t|^{\nu}$). We are now faced with the problem of exponentiating these forms, after substituting the appropriate fixed-point values, u^* and Δ^*.

In the Ising case, $n=1$, Eq. (5.3) reduces to

$$D(q)\approx\Delta M^2[1-K_4(4\Delta^*-3u^*)\ln M]\,. \tag{5.5}$$

Assuming a single power law, this becomes (to order ϵ)

$$D(q)\approx\Delta M^{(\gamma-\alpha)/\beta}\propto|t|^{\gamma-\alpha} \tag{5.6}$$

for the "pure" fixed point ($\Delta^*=0$, $K_4^*u^*=2\epsilon/9$, $\gamma=1+\epsilon/6$, $\alpha=\epsilon/6$, and $1/\beta=2+2\epsilon/3$) and (to order $\epsilon^{1/2}$)

$$D(q)\approx\Delta M^2\propto|t|^{\gamma} \tag{5.7}$$

for the "random" (Khmelnitzkii) point [$K_4u^*=4(2\epsilon/159)^{1/2}$, $K_4\Delta^*=3(2\epsilon/159)^{1/2}$, and $\gamma\approx 2\nu\approx 2\beta=1+\frac{1}{2}(6\epsilon/53)^{1/2}$; note an error in Eq. (4.16c) of GMM]. Both (5.6) and (5.7) are consistent, to this order, with our Eqs. (2.9) and (2.14) (using $\chi\propto|t|^{-\gamma}$).

GMM also calculate the equation of state, which (to leading order in $\epsilon^{1/2}$) yields $\chi^{-1}(0)=r=uM^2$. Thus, Eq. (5.7) is equivalent to $D(q)=\Delta r/u$, i.e., $C^{(s)}(0)=\Delta\chi(0)/u$, identifying A of Eq. (1.10) as $1/u$. We also note that $D(q)$ has no q dependence for small q. Thus, to leading order,

$$\mathcal{S}(q)=\chi(q)+(\Delta/u)r\chi(q)^2$$
$$\approx[\chi(q)^{-1}-(\Delta/u)r]^{-1}$$
$$\approx[\chi(0)^{-1}(1-\Delta/u)+Cq^2]^{-1}\,, \tag{5.8}$$

and the measured amplitudes of both $\chi(0)$ and the correlation length will be modified.

In the other limit, $q\gg|t|^{\nu}$, the assumption of a single power law yields

$$D(q)\propto\Delta M^2q^{4-2/\nu} \tag{5.9}$$

near the "pure" point, consistent with the second term in Eq. (2.11) [using $\chi\propto q^{-(2-\eta)}\approx q^{-2}$, since $\eta=O(\epsilon^2)$]. Similarly, such an exponentiation near the "random" point yields

$$D(q)\propto\Delta^*M^{1/\beta}q^{2+\eta-1/\nu}\,, \tag{5.10}$$

consistent with the first term (analytic in t) in Eq. (2.15). It must be emphasized that both terms in Eqs. (2.11) and (2.15) have the form $M^{2+O(u^*,\Delta^*)}q^{O(u^*,\Delta^*)}$, so that the above identifications may be finalized only by checking terms of order $u^2(\ln M)^2$.

We now turn to the case $n\geq 2$. We start with the q dependence. GMM studied the small q dependence only for the "random" fixed point, $K_4u^*\simeq\epsilon/2(n-1)$, $K_4\Delta^*\simeq\epsilon(4-n)/8(n-1)$, which is *unstable* for $n\geq 2$ and $d=3$. They derived the order-ϵ^2 terms, and found [their Eq. (4.35)] that

$$D(q)\propto\left[1+\frac{\epsilon}{2}\ln q-\frac{\epsilon^2}{16}\ln^2 q\right], \tag{5.11}$$

which they exponentiated as

$$D(q)\propto\tfrac{5}{2}/(\tfrac{3}{2}+q^{-5\epsilon/4})\,.$$

Based on our experience in Secs. III and IV [Eqs. (3.5) and (4.14)], we argue that Eq. (5.10) should really be exponentiated as

$$D(q)\propto 16/(3+q^{-\epsilon})^2\,, \tag{5.12}$$

yielding a q-independent constant for $C^{(s)}(0)$.

Since the stable behavior is governed by the "pure" point, $\Delta^*=0$, $K_4u^*=2\epsilon/(n+8)$, we calculated the analog of Eq. (5.11) for that fixed point and found

$$D(q)\propto\left[1+\frac{2(n-1)}{n+8}\epsilon\ln q+\frac{(n-1)(2n-11)}{(n+8)^2}\epsilon^2\ln^2 q\right], \tag{5.13}$$

which can be exponentiated in the form

$$D(q) \propto \left[\frac{n+8}{n-1}\right]^2 \Big/ \left[\frac{9}{n-1}+q^{-\epsilon}\right]^2 . \qquad (5.14)$$

This is the only exponentiation which is consistent with the large-n limit, Eq. (3.5). (Note that for $n \to \infty$, to leading order in ϵ, both equations have no constant in the denominator.)

For small q we thus conclude that both fixed points yield $D(q) \propto q^{2\epsilon}$. Combining this with Eq. (1.8), we conclude that $C^{(s)}(q)$ approaches a limit independent of q as $q \to 0$.

We now turn to the t dependence of $C^{(s)}(0)$. Equation (5.3) can be rewritten in the form

$$D(q) \approx \Delta M^{2+K_4(n+2)u^* - 4K_4\Delta^*}[1+K_4u^*(n-1)\ln(q/M)] . \qquad (5.15)$$

Treating the second factor as before, it becomes of the form $A/[C+(q/M)^{-\epsilon}]^2$, which turns into $A(q/M)^{2\epsilon}$ for $q \ll M$. Combining this result with Eqs. (1.8) and (2.4), and substituting the appropriate fixed-point values and exponents, we confirm that $C^{(s)}(0)$ obeys Eqs. (2.9) and (2.14). Note that our exponentiation procedure can be checked by deriving the terms of order $\epsilon^2(\ln M)^2$. However, the agreement with the scaling predictions and with Secs. III and IV gives strong support for our results.

Finally, we consider the limit $q \gg |t|^\nu$. Near the "pure" fixed point, it is easy to check that a single power-law exponentiation of Eq. (5.4) again yields Eq. (5.9), for all n. However, for the "random" fixed point we find that Eq. (5.4) is consistent with Eq. (2.15) only if we include *both* terms,

$$C^{(s)} \propto \Delta^* \left[M^{1/\beta} q^{-(2-\eta+1/\nu)} - \frac{4(n-1)}{4-n} M^2 q^{-d} \right] . \qquad (5.16)$$

Again, a complete proof of Eq. (2.15) or (5.16) will follow from checking the terms of order ϵ^2. However, the consistency of Eqs. (5.9) and (5.16) with their n-dependent amplitudes for all n is reassuring.

VI. CONCLUSIONS

Theoretically, our paper again shows the danger in exponentiating low-order ϵ expansions without some additional information, either from an exact limit (e.g., $n \to \infty$) or from physical reasoning [e.g., the behavior of $C^{(s)}(q)$ as M^2 or $M^{1/\beta}$ for large q].

For large $q|t|^{-\nu}$, i.e., close to T_c, we predict that $C^{(s)}$ approaches zero. Since $\chi_{\parallel}$ approaches a nonzero value, i.e., $q^{-2+\eta}$, $C^{(s)}$ will become negligible as $T \to T_c$. However, it will add up to the corrections to $\chi_{\parallel}$, e.g., $|t|^{1-\alpha}$,[12] and will introduce a modified coefficient to the analytic correction, of order t, and a new term, of order $M^2 \propto |t|^{2\beta}$.

For $n \geq 2$, $C^{(s)}(q)$ is also small when $q|t|^{-\nu} \ll 1$ since it approaches a temperature-dependent constant, while $\chi_{\parallel}$ diverges as q^{d-4} However, this coexistence-curve divergence is very difficult to observe anyway, and the addition of $C^{(s)}$ will only make it more complicated.

Our main new practical message concerns the Ising case, $n=1$. We emphasize again that $C^{(s)}$ must be included in any analysis of critical amplitudes.

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