# Information theory and resistance fluctuations in one-dimensional disordered conductors

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A novel method is proposed to treat the problem of the random resistance of a strictly one-dimensional conductor with static disorder. For the probability distribution of the transfer matrix $R$ of the conductor we propose a distribution of maximum information entropy, constrained by the following physical requirements: (1) flux conservation, (2) time-reversal invariance, and (3) scaling with the length of the conductor of the two lowest cumulants of $\omega$, where $R=\exp(i\boldsymbol{\omega}\cdot\hat{\mathbf{J}})$. The preliminary results discussed in the text are in qualitative agreement with those obtained by sophisticated microscopic theories.

The zero-temperature dc resistance of a strictly one-dimensional conductor with static disorder is known to be nonadditive and non-self-averaging.[1-4] This is due ultimately to the coherent elastic scattering from the random static scatterers in the disordered conductor, resulting in exponential localization of the electronic eigenstates and spatial nonergodicity. Thus, the resistance fluctuates from sample to sample even when the samples are prepared macroscopically identically. It is necessary then to calculate the full probability distribution of the resistance and not just some (ensemble) averaged value.

Recently, such a distribution has indeed been obtained by several authors,[1-4] for idealized models of disorder. Generally speaking, one could summarize their procedure saying that the transfer matrix $R$ of the system is written in terms of more "microscopic" quantities (the potentials of the individual scatterers and their positions), for which a statistical law is assumed and used to evaluate the various averages of interest.

In the end, some results depend on the details of the microscopic quantities, but several others seem to be of a rather general nature.[1] It is thus tempting to try to construct the distribution of $R$ (the transfer matrix for the whole system) directly, i.e., instead of postulating a statistical law for the microscopic quantities, we would like to try and postulate a statistical law directly for $R$, a procedure which seems conceptually simpler. We want to impose on that distribution some important physical requirements and look for the "most probable distribution" (i.e., the one of maximum information-entropy[5,6]) that satisfies them, thus implying the use of a minimal set of assumptions.

From *flux conservation* and *time-reversal invariance* of the Hamiltonian, one can prove[1] that a transfer matrix $R$ must be of the form

$$R=\begin{pmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{pmatrix},\quad |\alpha|^2-|\beta|^2=1\ , \tag{1}$$

i.e., $R$ is pseudounitary and unimodular. The collection of all such matrices forms the group SU(1,1), a noncompact group homomorphic to the Lorentz group SO(2,1). The notion of equal *a priori* probability for the $R$ matrices is then naturally provided by the invariant or Haar's measure[7,8] $d\mu(R)$ of SU(1,1). This measure remains invariant when all the $R$'s are multiplied by a fixed one, $R_0$. We thus write the differential probability of our ensemble of $R$ matrices in the form

$$dP(R)=p(R)d\mu(R)\ , \tag{2}$$

and we now discuss how to choose the probability density $p(R)$.

First of all, if we specify the material that all our wires are made of, we ask that the probability density $p(R)$ be only a function of $R$ and the length $L$ of the wires, i.e., $p_L(R)$ (we may assume, specifically, that the origin of coordinates always coincides with the midpoint of the wire). We now ask for the following *combination requirement.* Consider the ensemble $p_L(R)$ associated with wires of length $L$. We assume that this same ensemble of $R$'s is generated if we pick up, from their respective ensembles, wires of lengths $L_1$ and $L_2$ (so that $L_1+L_2=L$) and put them together in pairs. We are assuming that $L_1,L_2$ are large compared with the correlation length of the electronic potential, so that $R_1,R_2$ (the transfer matrices of the individual wires centered at the origin) can be considered as statistically independent. The resulting transfer matrix $R$ takes the form

$$R=[U(-L_1/2)R_2U(L_1/2)]\times[U(L_2/2)R_1U(-L_2/2)]\ . \tag{3}$$

The $U$'s in Eq. (3) are diagonal matrices that take care of the displacement necessary to center the resulting wire at the origin [from Ref. 1, if the origin is shifted by the amount $x_0$, $U_{11}(x_0)=U_{22}^*(x_0)=\exp(ikx_0)$, $k$ being the incident wave number]. We can also write (3) as

$$R'=R_2'R_1'\ , \tag{4}$$

where

$$R'\equiv U(L/2)RU(L/2)\ , \tag{5}$$

and similarly for $R_1',R_2'$. Since $R$ and $U$ belong to SU(1,1), so does $R'$. In the notation of Eq. (1), $\alpha'=\alpha\exp(ikL)$, $\beta'=\beta$; since these relations can be trivially used to translate to $R$ all the conclusions that we shall find below for $R'$, from now on we shall drop the prime from $R'$, to simplify

 

the notation.

If $D$ denotes any representation of SU(1,1), (4) then gives

$$D(R)=D(R_2)D(R_1) \ . \tag{6}$$

Taking averages on both sides of this equation and using the notation

$$\langle D\rangle_L=\int D(R)p_L(R)d\mu(R) \ , \tag{7}$$

we have

$$\langle D\rangle_{L_1+L_2}=\langle D\rangle_{L_1}\langle D\rangle_{L_2} \ . \tag{8}$$

Notice the important conclusion that the matrices on the right-hand side of (8) must commute, as a consequence of our combination requirement. Equation (8) implies that $\langle D\rangle_L$ must depend on $L$ as

$$\langle D(R)\rangle_L=A^L \ , \tag{9}$$

where $A$ depends on the representation, but not on $L$.

Since (9) holds for any representation, it must hold in operator form, i.e., when we write $\hat{D}=\exp(2i\boldsymbol{\omega}\cdot\hat{\mathbf{J}})$, with $\boldsymbol{\omega}\cdot\hat{\mathbf{J}}=\omega_1\hat{J}_1+\omega_2\hat{J}_2-\omega_3\hat{J}_3$, $\hat{J}_i$ being the generators[9] of SU(1,1) and $2\omega_i$ (the factor 2 is for later convenience) three parameters of SU(1,1) analogous to those of the axis of rotation in SU(2), or $R(3)$. We then write

$$\langle e^{2i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}}\rangle_L=\hat{A}^L \ . \tag{10}$$

Let us briefly interrupt the argument to see a few applications of the parameters $\omega_i$. We first note that the $R$ matrices, themselves, can be written in terms of the $\omega_i$ as

$$R(\boldsymbol{\omega})=e^{i\boldsymbol{\omega}\cdot\boldsymbol{\tau}} \ , \tag{11}$$

where $\tau_i$ are analogous to the Pauli matrices[9]

$$\tau_1=\begin{pmatrix}0&-1\\1&0\end{pmatrix}, \quad \tau_2=\begin{pmatrix}0&-i\\-i&0\end{pmatrix}, \quad \tau_3=\begin{pmatrix}-1&0\\0&1\end{pmatrix} .$$

From (11) we find explicitly

$$R=(\cosh\omega)I+i\left(\frac{\sinh\omega}{\omega}\right)\boldsymbol{\omega}\cdot\boldsymbol{\tau}$$

$$=\begin{pmatrix}\cosh\omega+i\dfrac{\omega_3}{\omega}\sinh\omega & \dfrac{\omega_2-i\omega_1}{\omega}\sinh\omega\\ \dfrac{\omega_2+i\omega_1}{\omega}\sinh\omega & \cosh\omega-i\dfrac{\omega_3}{\omega}\sinh\omega\end{pmatrix} , \tag{12}$$

where $\omega^2=\omega_1^2+\omega_2^2-\omega_3^2$.

It is shown in Ref. 1 that $R_{11}=1/t$, $R_{12}=r/t$ (where $r,t$ represent the reflection and transmission amplitudes of the one-dimensional scattering problem) and that the "dimensionless resistance" is given by Landauer's result $\rho=|r/t|^2$. In terms of the parameters $\omega_i$, the resistance is then

$$\rho=\frac{\omega_1^2+\omega_2^2}{\omega^2}(\sinh\omega)^2 \ . \tag{13}$$

In terms of the $\omega_i$'s we can also find the invariant measure of SU(1,1) (see Ref. 7, p. 313) as

$$d\mu(R)=\left(\frac{\sinh\omega}{\omega}\right)^2 d\boldsymbol{\omega} \ , \tag{14}$$

where $d\boldsymbol{\omega}=d\omega_1\,d\omega_2\,d\omega_3$. This expression is analogous to the one obtained by Wigner[10] for the group $R(3)$.

We now go back to the requirement (10) and write it as (if the $\hat{J}_\alpha$ were commuting operators, this would be the familiar cumulant expansion)

$$\ln\langle e^{2i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}}\rangle_L=2ig_\alpha\bar{\omega}_\alpha\hat{J}_\alpha+\frac{(2i)^2}{2!}g_\alpha g_\beta(\overline{\omega_\alpha\omega_\beta}-\bar{\omega}_\alpha\bar{\omega}_\beta)\hat{J}_\alpha\hat{J}_\beta$$

$$+\frac{(2i)^3}{3!}g_\alpha g_\beta g_\gamma(\overline{\omega_\alpha\omega_\beta\omega_\gamma}-\tfrac{3}{2}\bar{\omega}_\alpha\overline{\omega_\beta\omega_\gamma}-\tfrac{3}{2}\overline{\omega_\alpha\omega_\beta}\bar{\omega}_\gamma+2\bar{\omega}_\alpha\bar{\omega}_\beta\bar{\omega}_\gamma)\hat{J}_\alpha\hat{J}_\beta\hat{J}_\gamma+\cdots=L\cdot\text{Operator} \ , \tag{15}$$

where the bars on the rhs denote the same ensemble average as on the lhs and a sum over repeated indices is understood. In Eq. (15), $g_1=g_2=1$, $g_3=-1$. We can satisfy (15) by requesting that the various independent terms in the expansion be proportional to $L$. In practice we do not have information on all the moments of the distribution of $\boldsymbol{\omega}$. We thus propose to fix the first two cumulants

$$\bar{\omega}_\alpha=a_\alpha L \ , \tag{16a}$$

$$\overline{\omega_\alpha\omega_\beta}-\bar{\omega}_\alpha\bar{\omega}_\beta=b_{\alpha\beta}L \ , \tag{16b}$$

and look for that distribution that has maximum information entropy $\mathcal{S}$, among those distributions that satisfy (16a) and (16b). $\mathcal{S}$ is defined as[5,6]

$$\mathcal{S}[p]\equiv-\int p(R)\ln p(R)d\mu(R) \ . \tag{17}$$

One finds the result[5,6]

$$dP(\boldsymbol{\omega})=C\exp(\lambda_\alpha\omega_\alpha-\lambda_{\alpha\beta}\omega_\alpha\omega_\beta)\left(\frac{\sinh\omega}{\omega}\right)^2 d\boldsymbol{\omega}$$

$$\equiv p(\boldsymbol{\omega})d\boldsymbol{\omega} \ . \tag{18}$$

Loosely speaking, among all the distributions whose two lowest cumulants scale with $L$, (18) represents the "most probable one," or the one that treats $R$ "most randomly."

Equation (18) gives a prediction for the joint distribution of $\omega_1,\omega_2,\omega_3$ [and hence for the real and imaginary parts of $R_{ij}$ [see Eq. (12)]] that can be compared with detailed Monte Carlo calculations and with other theoretical models. Some preliminary results are examined in what follows.

We first consider the large-$L$ limit where, from (16), the distribution of $\boldsymbol{\omega}$, Eq. (18), should be concentrated in a region far away from the origin. Let us assume that this region does not cross the "light cone," so that it is well within the domain $\omega^2<0$, or $\omega^2>0$. Suppose $\omega^2<0$ first. Then $\omega=i\phi$. From (13),

$$\rho=(\omega_1^2+\omega_2^2)\sin^2\phi/\phi^2\leq(\omega_1^2+\omega_2^2)/\phi^2$$

and

$$\bar{\rho}\sim[(\bar{\omega}_1)^2+(\bar{\omega}_2)^2]/|(\bar{\omega}_1)^2+(\bar{\omega}_2)^2-(\bar{\omega}_3)^2| \ ,$$

independent of $L$ from Eq. (16). The physical situation[1] will then arise when $\omega^2(\bar{\boldsymbol{\omega}})>0$, which we now consider.

One can calculate the $n$th moment of the dimensionless resistance $\rho$ of Eq. (13) with the distribution (18), by using the saddle-point approximation. The result turns out to be of the form

$$\langle \rho^n \rangle = A_n e^{B_n L} , \tag{19}$$

which has a structure similar to that found in Ref. 11. In particular, $(\mathrm{var}\rho)/\langle\rho\rangle^2 \sim c' \exp(cL)$, which, in general, diverges with $L$; occasionally it could be constant, when $c = 0$; this agrees with the remarks made in Ref. 1, Sec. 9.3.

Using the saddle-point approximation one can also calculate the average of the transmission amplitude $t = 1/R_{11}$ with the result

$$\langle t \rangle = A e^{-BL} . \tag{20}$$

One can prove, and this will be reported in a future publication, that the same form (20) arises from a microscopic calculation, in which the potential is assumed to be a Gaussian random process with the properties $\langle V(x) \rangle = 0$, $\langle V(x) V(x') \rangle = K \delta(x - x')$; in that microscopic calculation the resulting exponent is $B = K(2m\hbar^{-2}k^{-1})^2$, with $m$ the mass and $k$ the wave number of the electron.

To be more specific about the above results, as well as to have a preliminary suggestion about the physical interpretation of the parameters appearing in Eq. (16), we consider a particular case which can be related to other publications. For $a_1 = a$, $a_2 = a_3 = 0$, $b_{11} = b$, $b_{22} = b_{33} = b_{\alpha\neq\beta} = 0$ [since only $\omega_1$ survives, the restrictions imposed by the noncommutativity of the $\hat{J}_\alpha$'s in Eq. (15) do not appear], the distribution of $\ln(1+\rho)$ reduces, for large $L$, to that of Ref. 13, i.e., a Gaussian with centroid $2aL$ and variance $2bL$. This also agrees with the results of Refs. 2 and 12. The corresponding distribution of $\rho$ then becomes

$$p(\rho) \approx \frac{e^{-a^2 L/2b}}{\sqrt{2\pi bL}} \rho^{a/2b - 1} e^{-(\ln\rho)^2/8bL} , \tag{21}$$

whose general structure is similar to that found in Ref. 4 assuming a Gaussian random process for the potential. For the coefficients $A_n, B_n$ associated with the $n$th moment of the resistance, Eq. (19), one finds explicitly

$$A_n = 4^{-n}, \quad B_n = 2n(a + nb) , \tag{22}$$

so that the $c$ and $c'$ mentioned right after Eq. (19) in connection with $\mathrm{var}\rho/\langle\rho\rangle^2$ take the values $c' = 1$, $c = 4b$. In this same model, the coefficient $B$ of Eq. (20) is $a - b/2$; this then connects the parameters $a, b$ with a statistical property of the potential, through the result indicated right after Eq. (20). Comparing these results with those of Refs. 1–4 and 11–13, we see that we have to choose our input parameters $a$ and $b$ as proportional to the inverse of the localization length $L_0$. We can thus be more explicit about the above large-$L$ approximation and interpret it as meaning $L \gg L_0$.

In the opposite extreme, in which $L$ is small compared with the localization length, other authors[2–4] recover the linearity of $\langle\rho\rangle$ with $L$. In our model, we expect $p(\omega)$ to be concentrated near the origin, around which we can make a series expansion. Considering for the parameters of Eq. (16) the particular case of the previous paragraph, one can show

$$\langle \rho \rangle = bL + (a^2 + b^2)L^2 + \cdots , \tag{23}$$

where the leading term is indeed linear in $L$.

To summarize, we have suggested the joint distribution of Eq. (18) for the matrix elements of the transfer matrix (a more complete distribution than what normally appears in the literature), based only upon the following physical information: (1) flux conservation, (2) time-reversal invariance, and (3) scaling with $L$ of the two lowest cumulants of the $\omega_\alpha$'s. The specific physical system in question enters through the specific value of the scaling constants. The final expectation values are expressed as three-dimensional integrals, instead of involving the much more numerous microscopic variables of the standard treatments.

The preliminary results mentioned in the text agree with the general trends of other models. A more detailed account of them will be given in a future publication. More extensive calculations are needed, however, and these we plan to perform in the future. Of course, more information could be added to the problem by scaling, in Eq. (15), terms that contain higher cumulants of the $\omega_\alpha$'s.

We may also mention that an analysis of this problem very much connected with the group SU(1,1) was given in Refs. 13 and 14 and a study of its relation with the present approach would also be interesting. A clearer understanding of that relation might also be helpful in understanding the physical significance of the parameters of Eq. (16).

As a final observation, the reader may find applications of information theory to other physical problems in Refs. 5 and 6.

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