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Periodic Laughlin-Jastrow wave functions for the fractional quantized Hall effect

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We present Laughlin-Jastrow wave functions for incompressible fluid states of two-dimensional electrons at Landau-level filling factor $1/m$ that satisfy periodic boundary conditions. This rederivation of Laughlin-type states emphasizes that it is correct short-distance behavior of the wave functions rather than angular momentum considerations that lie behind the explanation of the fractional quantized effect.

Laughlin[1] has provided the key to understanding the fractional quantized Hall effect[2] with his construction of Jastrow-type variational wave functions that describe incompressible fluid states of two-dimensional electrons in a magnetic field. As originally formulated, Laughlin's states describe circular fluid droplets containing N_e electrons that expand to provide a uniform cover of the "Hall surface" as $N_e \to \infty$. A variant formulation on a spherical surface has been described by Haldane;[3] this allows homogeneous states with *finite* N_e to be constructed. Recent finite-system studies[4] in this geometry have, we believe, conclusively confirmed that at $\frac{1}{3}$ Landau-level filling the Laughlin-Jastrow wave function describes the essential character of the ground state of systems where the interactions are sufficiently repulsive at short range, and that the Coulomb interaction belongs to this class.

Laughlin-Jastrow (LJ) states have not been previously constructed in the other popular finite-system geometry, namely, the periodic boundary conditions on the plane. In this Rapid Communication we construct such states. The philosophy of our construction is identical to that of Ref. 1; the states described here have the same thermodynamic limit as those of Ref. 1. While no new physics is being described, our construction now makes direct comparison of the LJ state with finite-size results in the periodic geometry possible. The discrete center-of-mass degeneracy of the ground state in this geometry is also made explicit.

In the Landau gauge $\mathbf{A} = -By\hat{x}$, the wave function describing a particle confined to the lowest Landau level has the analytic form

$$\psi(x,y) = \exp(-\tfrac{1}{2}y^2) f(z), \quad z = x + iy \quad , \tag{1}$$

where $f(z)$ is an *entire* (holomorphic) function, and length units $\sqrt{(\hbar/eB)} = 1$ are used. An essentially similar form occurs in the symmetric gauge,[1] and we emphasize that the following discussion can be carried out in any gauge. The particle translation operator that acts on the wave functions is given (using two-dimensional vector notation with a pseudoscalar cross product) by

$$t(\mathbf{L}) = \exp[\mathbf{L}\cdot(\nabla - ie\mathbf{A}/\hbar) - i\mathbf{L}\times\mathbf{r}] \quad . \tag{2}$$

We will impose periodic boundary conditions

$$t(\mathbf{L}_\alpha)\Psi = \exp(i\phi_\alpha)\Psi, \quad \alpha = 1,2 \quad , \tag{3}$$

where $\mathbf{L}_1 = (L_1, 0)$ and $\mathbf{L}_2 = (L_2\cos\theta, L_2\sin\theta)$ are two nonparallel displacements. For these boundary conditions to be simultaneously applicable, $t(\mathbf{L}_1)$ and $t(\mathbf{L}_2)$ must commute, i.e.,

$$|\mathbf{L}_1 \times \mathbf{L}_2| = 2\pi N_s \quad , \tag{4}$$

where N_s is an integer. This means that the total magnetic flux though the parallelogram defined by $\mathbf{L}_1$ and $\mathbf{L}_2$ is exactly N_s flux quanta and integral. This region bounded by the four points $z = \frac{1}{2}L_1(\pm 1 \pm \tau)$, $\tau = L_2 e^{i\theta}/L_1$ will be referred to as the principal region.

The boundary conditions used in the study by Yoshioka, Halperin, and Lee[5] are a special case of (3) with $\phi_\alpha = 0$. However, because of the noncommutativity of translation operators when a magnetic field is present, the choice of the ϕ_α is not invariant under continuous translations of the center of mass, and the more general form (3) is more appropriate. If the periodic boundary conditions are interpreted as imposing a toroidal topology, the phases ϕ_i can be related to "solenoid fluxes" $\Phi_i = \hbar\phi_i/e$ passing through the two periodic orbits. If the phases are allowed to vary with time, this is equivalent to a uniform electric field (E^x, E^y) where the complex drift velocity $v = (E^y - iE^x)/B$ is given by $v = (d/dt)(L_1\phi_2 - L_2 e^{i\theta}\phi_1)/2\pi N_s$.

The periodic boundary condition on the wave function (1) is the condition

$$\frac{f(z+L_1)}{f(z)} = e^{i\phi_1} ,$$
$$\frac{f(z+L_2e^{i\theta})}{f(z)} \exp\{i\pi N_s[(2z/L_1)+\tau]\} = e^{i\phi_2} . \quad (5)$$

Since $f(z)$ is entire, the integral of $d/dz\{\ln[f(z)]\}$ around the boundaries of the principal region counts the number of zeros of $f(z)$ inside it. The condition (5) fixes this number to be precisely N_s. The possible analytic form of $f(z)$ is thus strongly constrained, and the most general form is expressible as

$$f(z) = \exp(ikz) \prod_{\nu=1}^{N_s} \vartheta_1(\pi(z-z_\nu)/L_1|\tau) , \quad (6)$$

where the zeros z_ν are in the principal region, and k is real and in the range $0 \leq |k| \leq \pi N_s \,\mathrm{Im}(\tau)/L_1$. $\vartheta_1(u|\tau)$ are the odd elliptic theta functions.[6] Fixing the solenoid fluxes constrains k and the sum $z_0 = \sum_\nu z_\nu$ to take one of the N_s^2 sets of values satisfying

$$\exp(ikL_1) = (-1)^{N_s}\exp(i\phi_1) ,$$
$$\exp(2\pi i z_0/L_1) = (-1)^{N_s}\exp(i\phi_2 - ikL_1\tau) . \quad (7)$$

If (k, z_0) is a solution of (7), the other solutions have the form

$$(k - 2\pi n_1/L_1, z_0 + n_1 L_2 e^{i\theta} + n_2 L_1) ,$$

where n_1 and n_2 are suitable integers that keep k and z_0 in the specified ranges.

We remark that the number of linearly independent solutions of (5) is equal to the number of zeros within the principal region. The basis set of eigenstates of the translation operator $t(\mathbf{L}_1/N_s)$ (which has the action $z \rightarrow z + L_1/N_s$) is constructed by placing the N_s zeros in a string satisfying $z_{\nu+1} = z_\nu + L_1/N_s$; there are then N_s distinct orthogonal solutions of (5). An alternative way to specify states is to construct "coherent states" by placing all the zeros at the *same* point. The wave function is then maximum at the "diametrically opposed" point $z + (1+\tau)L_1/2$; there are N_s^2 nonorthogonal solutions of (5) with this form.

We now consider the many-particle wave functions for N_e particles. Translational invariance allows these to be expressed as the product of a center-of-mass term and a factor involving only relative coordinates. We follow the arguments of Ref. 1 and seek a ground-state wave function where the relative motion is described by a *Jastrow function*, i.e., a product of pair factors

$$F(\{z_i\}) = F^{\mathrm{c.m.}}(Z) \prod_{i<j} f(z_i - z_j), \quad Z = \sum_k z_k . \quad (8)$$

Application of the boundary condition for each particle gives

$$f(z+L_1)/f(z) = \eta_1 ,$$
$$f(z+L_2e^{i\theta})/f(z) = \eta_2 \exp[2\pi i(N_s/N_e)z/L_1] , \quad (9)$$

where η_1 and η_2 are constants. Integration of $d/dz \times \{\ln[f(z)]\}$ around the boundaries of the principal region shows that the number of zeros of $f(z)$ is $N_s/N_e = m$, which must be integral. We again follow Ref. 1 and seek the solution of (9) that (a) is odd under $z \rightarrow -z$, because of antisymmetry under particle exchange, and (b) has all its zeros at the point $z=0$ where the particles coincide (this eliminates "wasted" zeros). The only solutions are

$$f(z) = [\vartheta_1(\pi z/L_1|\tau)]^m, \quad m \text{ odd} . \quad (10)$$

As $z \rightarrow 0$, $f(z) \sim z^m$. The center-of-mass factor must then satisfy

$$\frac{F^{\mathrm{c.m.}}(Z+L_1)}{F^{\mathrm{c.m.}}(Z)} = (-1)^{(N_s-m)} e^{i\phi_1} ,$$
$$\frac{F^{\mathrm{c.m.}}(Z+L_2e^{i\theta})}{F^{\mathrm{c.m.}}(Z)} \exp[i\pi m(2Z/L_1)+\tau] = (-1)^{(N_s-m)} e^{i\phi_2} . \quad (11)$$

The general solution of this is characterized by a real wave vector K and m zeros $\{Z_\nu\}$

$$F^{\mathrm{c.m.}}(Z) = \exp(iKZ) \prod_{\nu=1}^{m} \vartheta_1(\pi(Z-Z_\nu)/L_1|\tau) ; \quad (12)$$

$$\exp(iKL_1) = (-1)^{N_s}\exp(i\phi_1) ,$$
$$\exp\left(2\pi i \sum_\nu Z_\nu/L_1\right) = (-1)^{N_s}\exp(i\phi_2 - iKL_1\tau) . \quad (13)$$

Thus, there is an m-fold degeneracy associated with the center-of-mass coordinate in the presence of fixed solenoid fluxes. This degeneracy of the ground state with $N_s = mN_e$ was also found in the numerical study of Yoshioka *et al.*[5]

If "coherent state" center-of-mass wave functions are constructed by placing all the zeros at the same point in the principal region, there are m^2 distinct solutions of (13) compatible with the specified solenoid fluxes. For any such solution, the amplitude of the state vanishes when the center-of-mass coordinate Z/N_e lies on one of a lattice of points $z_0 + n_1L_1/N_e + n_2L_2e^{i\theta}/N_e$. The charge density of the state will be essentially constant, but with a small superimposed periodic component that is minimized at these positions, and vanishes as $N_e \rightarrow \infty$.

It may be appropriate to replace the "solenoid flux" boundary condition (3) with the less restrictive condition

$$t_i(\mathbf{L}_\alpha) = t_j(\mathbf{L}_\alpha), \quad \text{all } i,j , \quad (14)$$

where t_i is the translation operator of the ith particle. This is a selection rule that requires all particles to satisfy the *same* boundary condition, but leaves the ϕ_α unspecified. In this case, the restrictions (13) are lifted, and z_0 can be chosen arbitrarily. Since the eigenvalue spectrum of a translationally invariant Hamiltonian is independent of the ϕ_α, there is a continuous degeneracy associated with the center-of-mass coordinate if (14) is used.

Following Laughlin's treatment on the open plane, we exhibit wave functions describing fractionally charged "hole" defects. The hole state is given by

$$F(\{z_i\};\bar{z}) = F^{\mathrm{c.m.}}(Z) \prod_k \vartheta_1(\pi(z_k-\bar{z})L_1|\tau) \prod_{i<j} [\vartheta_1(\pi(z_i-z_j)/L_1|\tau)]^m, \quad Z = \sum_k z_k + m^{-1}\bar{z} . \quad (15)$$

N_s is given by mN_e+1. $F^{\text{c.m.}}(Z)$ are again solutions of (11). The defect is centered at the point $z=\bar{z}$. Since the amplitude of the wave function vanishes if any electron coordinate is at $\bar{z}$, this state has vanishing charge density at that point, which can be chosen without restriction.

As a function of the *hole* coordinate $\bar{z}$, the wave function (15) satisfies the boundary condition

$$\frac{F(\{z_i\};\bar{z}+mL_1)}{F(\{z_i\};\bar{z})}=(-1)^{m-1}e^{i\phi_1} ,$$
$$\frac{F(\{z_i\};\bar{z}+mL_2e^{i\theta})}{F(\{z_i\};\bar{z})}\exp\{i\pi N_s[(2\bar{z}/L_1)+m\tau]\}=(-1)^{m-1}e^{i\phi_2} . \tag{16}$$

If $m>1$, the function is not periodic in $\bar{z}$ with the fundamental periods, but only with longer periods so the repeat distances are m times those of the electronic wave functions. The boundary conditions show that the number of zeros in the enlarged "principal region" bounded by $\bar{z}/L_1=\frac{1}{2}m\times(\pm1\pm\tau)$ which contains m^2N_s flux quanta is mN_s. Since the flux quantum for a charge q particle is $1/q$ times the flux quantum for an electron, the wave function for such a particle would have $m^2(qN_s)$ zeros in this region. The boundary conditions (16) thus indicate that the "hole" carries fractional charge[1] $|q|=1/m$. This seems to be essentially the same argument as the "adiabatic transport" argument of Arovas, Schrieffer, and Wilczek.[7]

The model "particle" defect state seems less easy to construct in the periodic geometry. The defect creation operator would have to remove one zero from the wave function as a function of each particle coordinate in each repetition of the fundamental region. An ansatz involving $\vartheta_1(\pi(d/dz_i)/L_1|\tau)$ seems the likely solution, but the choice of ordering is nontrivial, and we leave the construction of the periodic analog of Laughlin's "particle" defect as an open problem.

Numerical studies by Su[8] with square boundary conditions ($\tau=i$) identified defect states in the form of a line defect, which were eigenstates of the many-particle translation operator $\prod t_n(L_1/N_s)$. These are the analogs of the momentum-basis states usually used in Landau-gauge calculations. For direct comparison with the model states (15) which are equivalent to Laughlin's states in the thermodynamic limit, the coherent-state linear combination of the line-defect states would have to be formed. The relation between these types of states is analogous to that described above for the electron wave functions.

Finally, we note that while the above discussion is mainly formal, it does allow one physical point to be made. The original formulation[1] made use of arguments based on angular momentum conservation. The above formalism shows that Laughlin's construction can just as easily be carried out in a geometry that *does not conserve angular momentum*, and it is instead *correct behavior of the wave functions as particles approach* that is the key principle of its success. There is an analogy with the original BCS formulation of the superconducting ground state, which made use of momentum conservation, while the basic principle of pairing of time-reversed states can of course be implemented under the more general conditions of "dirty superconductivity."

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