# Path-integral method for soliton-bearing systems. II. $\phi^4$ and sine-Gordon theories in the classical limit

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(Received 3 October 1984)

Making use of the collective-coordinate technique, we investigate systematically the higher-order corrections in $t$ ($=T/E_s$) to the soliton density $n_s$ for one-dimensional soliton-bearing systems in the classical limit (where $T$ is the temperature and $E_s$ the soliton energy). Under the ideal-soliton-gas approximation we obtain a general formula for $n_s$ which includes the higher-order corrections expressed by diagramatic expansions. The $n_s$ calculated for the $\phi^4$ and the sine-Gordon models give rise to corrections to the soliton free energy, which coincide exactly with the corrections found recently by the transfer-integral method. The static correlation functions are also reexamined in the same approximation, and we refine the earlier prescription in order to match them with the results of the transfer-integral method.

## I. INTRODUCTION

Since the pioneering work by Krumhansl and Schrieffer,[1] the statistical mechanics of solition-bearing systems have been studied extensively in recent years.[2] There are basically three different theoretical approaches to reveal the roles of solitons in the equilibrium statistical mechanics: the transfer-integral method,[3,4] the path-integral method,[5,6] and the Bethe-ansatz method.[7,8] The transfer-integral method (TIM) is only available in the classical limit, but it allows one to calculate the thermodynamic functions of a system exactly. Furthermore, the TIM gives criterion on the validity of other methods since any quantum theory should reproduce the results of the TIM in the classical limit. The path-integral method (PIM) is worked out within quasiclassical expansions around the solitons. The solitons are treated as classical objects and the phonons as quantum ones. The classical limit of the PIM is pursued extensively in this paper. The Bethe-ansatz method (BAM) gives the exact quantum thermodynamic functions. However, the BAM can be used only for integrable systems,[9] and so one cannot use the BAM for the $\phi^4$ model, for example, which is a nonintegrable system. Furthermore, the classical limit of the BAM is not well understood at the moment.[10]

In our previous work[11] (hereafter referred to as I), we have developed the path-integral method for the classical statistical mechanics of solitons in a one-dimensional sine-Gordon (SG) model. We have shown that the method of collective coordinates, introduced by Gervais and Sakita[12] in dealing with the quantum field theory, provides an appropriate formal scheme to deal with the higher-order corrections.

In this paper we shall generalize our method to the other one-dimensional nonlinear Klein-Gordon models.[13] The general expressions of the higher-order corrections to the soliton density are given by diagrammatic expansions. Explicit calculations are done for the $\phi^4$ model which is compared with the previous calculations of the SG model.[11] These soliton densities yield the soliton free energies consistent with the results of the TIM.[14–16] Furthermore, we improve the prescriptions given in I to calculate the static correlation functions, in order to make them consistent with the results of the TIM.[14–16] Finally we comment briefly on difficulties related to the double-quadratic (DQ) model.

## II. METHOD OF COLLECTIVE COORDINATES

The general class of one-dimensonal soliton-bearing systems which we consider is defined by the following Hamiltonian (the nonlinear Klein-Gordon model):[13]

$$H=\int_{-L/2}^{L/2}dx\left[\tfrac{1}{2}\Pi^2+\frac{1}{2}\left[\frac{\partial\phi}{\partial x}\right]^2+m^2V(\phi;g)\right] . \quad (1)$$

Here $\phi$ is a dimensionless real scalar field and $\Pi$ the field conjugate to $\phi$. $L$ is the system length and $m$ the phonon mass of the discrete system, and $g$ a dimensionless coupling constant. The local potential $V(\phi;g)$ is assumed to satisfy the following relationship:

$$V(\phi;g)=g^{-2}V(g\phi;1) . \quad (2)$$

Furthermore, we assume $V(\phi;g)$ has at least two degenerate minima, is an even function of $\phi$, and reaches its minima ($V=0$) at $\phi=\pm\phi_0$ [in the case of a periodic potential $V(\phi;g)$ has the period $2\phi_0$], and we also scale $V(\phi;1)$ so that $d^2V(\phi;1)/d\phi^2=1$ at $\phi=g\phi_0$. Examples of such potentials are listed in Table I. The soliton energy $E_s$ of this model is given by[13]

$$E_s=\alpha m/g^2,\ \ \alpha=\int_{-g\phi_0}^{g\phi_0}d\phi[2V(\phi;1)]^{1/2} , \quad (3)$$

where $\alpha$ is a model-dependent number listed in Table I. The thermodynamic properties of this system in the classical limit are determined by the partiton function $Z$ defined as

$$Z=\int\mathscr{D}[\phi]\int\mathscr{D}[\Pi]e^{-\beta H} , \quad (4)$$

 

TABLE I. The local potentials $V(\phi;1)$, the minima $\phi_0$ of $V(\phi;g)$, the one-soliton solutions $\phi_s(x)$ of $V(\phi;g)$, and the numbers $\alpha$, $A$, and $C$ which characterize the models are shown for the $\phi^4$, SG, and DQ models.

| Models | $V(\phi;1)$ | $\phi_0$ | $\phi_s(x)$ | $\alpha$ | $A$ | $C$ |
|---|---|---|---|---|---|---|
| $\phi^4$ | $\frac{1}{8}(\phi^2-1)^2$ | $\frac{1}{g}$ | $\frac{1}{g}\tan(\frac{1}{2}mx)$ | $\frac{2}{3}$ | $2\sqrt{3/2}$ | 1 |
| SG | $1+\cos\phi$ | $\frac{\pi}{g}$ | $\frac{1}{g}[4\tanh^{-1}(e^{mx})-\pi]$ | 8 | $\sqrt{2}$ | 2 |
| DQ | $\frac{1}{2}(\lvert\phi\rvert-1)^2$ | $\frac{1}{g}$ | $\frac{1}{g}\mathrm{sgn}(x)(1-e^{-m\lvert x\rvert})$ | 1 | 1 | 1 |

where $\beta=T^{-1}$ (with units in which $k_B=1$) and $\mathscr{D}[\phi]$ and $\mathscr{D}[\Pi]$ imply functional integrals over all $\phi$ and $\Pi$ configurations with periodic boundary conditions on the interval $L$. $\phi$ and $\Pi$ depend only on $x$, the spacial coordinate.

Next we limit our analysis to the low-temperature region ($t \ll 1$), so that we can neglect the interactions between solitons and antisolitons (i.e., ideal-soliton-gas approximation).[13] Under the ideal-soliton-gas approximation one can factorize the functional space,[11] and finally we obtain the following expression for $Z$:

$$Z=Z_0\exp(Cn_sL)\,, \tag{5}$$

and

$$n_s=L^{-1}Z_1/Z_0\,. \tag{6}$$

Here $Z_0$ and $Z_1$ are the partition function of the soliton-free sector and the one-soliton sector, and $n_s$ is a soliton density. $C$ is a color of solitons listed in Table I. Our definition of $n_s$ for the $\phi^4$ and the DQ models is different by a factor of 2 from the definition given in Ref. 13, since we do not distinguish between solitons and antisolitons in these systems and we call them solitons. In other words, solitons and antisolitons with topological constraint in Ref. 13 are equivalent to the present solitons without topological constraint, which is clear from our construction of the partition function (see, for detail, Appendix A).

$Z_0$ for the soliton-free sector can be analyzed perturbatively. If we assume

$$\phi=\phi_0+\eta,\quad \Pi=\Pi_\eta\,, \tag{7}$$

the Hamiltonian in the soliton-free sector can be rewritten as

$$H=H_0^0+H_I\,, \tag{8}$$

where

$$H_0^0=\int dx\left[\tfrac{1}{2}\Pi_\eta^2+\frac{1}{2}\left[\frac{\partial\eta}{\partial x}\right]^2+\tfrac{1}{2}m^2\eta^2\right]\,, \tag{9}$$

$$H_I=\sum_{n=3}^{\infty}H_I^{(n-2)}\,, \tag{10}$$

$$H_I^{(n-2)}=\int dx\, m^2\frac{g^{n-2}}{n!}V^{(n)}(g\phi_0;1)\eta^n\,, \tag{11}$$

and $V^{(n)}(\phi;1)=d^nV(\phi;1)/d\phi^n$. $Z_0$ is calculated (with $\hbar=1$) as follows:

$$Z_0=\left[\prod_n(\beta\omega_n)\right]^{-1}\exp(\langle e^{-\beta H_I}-1\rangle^0_{\rm conn})\,, \tag{12}$$

where $\langle\cdots\rangle^0$ means the thermal average with respect to $H_0^0$ and the suffix conn means that only the connected diagrams are included; $\omega_n$ is the eigenvalue of the following eigenvalue equation:

$$\left[-\frac{d^2}{dx^2}+m^2\right]u_n(x)=\omega_n^2u_n(x)\,, \tag{13}$$

under periodic boundary conditions on the interval $L$. If we perform a diagrammatic expansion of Eq. (12),[17] it is easy to show that the free energy caused by $Z_0$ completely agrees with the corresponding terms of the TIM.

We shall now focus our attention on $Z_1$, the one-soliton contribution. As shown in I, according to the collective coordinate technique introduced by Gervais and Sakita,[12,18–20] we perform a canonical transformation[20] from the original variables $(\phi;\Pi)$ to the new variables $(X,\eta;P_T,\Pi_\eta)$ where $X$ is the soliton position and $P_T$ the total momentum of the field. The Hamiltonian in the one-soliton sector is recast as[20]

$$H=E_s+H_0^1+H_1+H_2\,, \tag{14}$$

$$H_0^1=\int dx\left[\tfrac{1}{2}\Pi_\eta^2+\frac{1}{2}\left[\frac{\partial\eta}{\partial x}\right]^2+\tfrac{1}{2}m^2V^{(2)}(g\phi_s;1)\eta^2\right]\,, \tag{15}$$

$$H_1=\sum_{n=3}^{\infty}H_1^{(n-2)}\,, \tag{16}$$

$$H_1^{(n-2)}=\int dx\, m^2\frac{g^{n-2}}{n!}V^{(n)}(g\phi_s;1)\eta^n\,, \tag{17}$$

$$H_2=(2E_s)^{-1}\left[\frac{P_T+\int dx\,\Pi_\eta\eta'}{1+\xi}\right]^2\,, \tag{18}$$

and

$$\xi=E_s^{-1}\int dx\,\phi_s'\eta'\,, \tag{19}$$

where $\phi_s$ is the one-soliton solution listed in Table I. The primes on $\eta$ and $\phi_s$ imply spatial derivatives. Now $Z_1$ is evaluated as follows:

$$Z_1=e^{-1/t}(2\pi)^{-1}\int dX\int dP_T\int \mathscr{D}[\eta]\delta\left[\int dx\,\phi_s'\eta\right]\int \mathscr{D}[\Pi_\eta]\delta\left[\int dx\,\phi_s'\Pi_\eta\right]\exp[-\beta(H_0^1+H_1+H_2)]\;, \tag{20}$$

where Dirac's δ functions restrict the available functional space of $\eta$ and $\Pi_\eta$ in such that $\eta$ and $\Pi_\eta$ do not contain the zero-mode component which is already extracted by the $X$ and $P_T$ integrals.[12] Since $X$ is a cyclic coordinate its integral gives simply the system length $L$. One can also integrate over $P_T$ when the configuratons of $\eta$ and $\Pi_\eta$ are fixed. After the Gaussian integral over $P_T$, we obtain

$$Z_1=L(E_s/2\pi\beta)^{1/2}e^{-1/t}\left[\prod_{l\,(\neq 0)}(\beta\omega_l)\right]^{-1}\times\exp(\langle e^{-\beta H_1}-1\rangle^1_{\rm conn})(1+\langle \xi e^{-\beta H_1}\rangle^1_{\rm conn})\;, \tag{21}$$

where $\langle\cdots\rangle^1$ means the thermal average with respect to $H_0^1$ and $\omega_l$ is the eigenvalue of the following eigenvalue equation:

$$\left[-\frac{d^2}{dx^2}+m^2V^{(2)}(g\phi_s;1)\right]u_l(x)=\omega_l^2u_l(x)\;, \tag{22}$$

under periodic boundary conditions on the interval $L$. There is always the so-called zero-mode solutions in Eq. (22), which reflects the translation invariance of the system.[13] In the following argument we have to distinguish the zero-mode solution from other eigenmode solutions. We assign $l=0$ to the zero-mode solution. The normalized zero-mode solution is given by

$$u_0(x)=E_s^{-1/2}\phi_s'(x),\quad \omega_0=0\;. \tag{23}$$

In Eq. (21) we have excluded the zero mode in the expression for the infinite product.

## III. SOLITON DENSITY

From Eqs. (6), (12), and (21), one can obtain the soliton density $n_s$ as

$$n_s=(E_s/2\pi\beta)^{-1/2}e^{-1/t}R\times\exp(\langle e^{-\beta H_1}-1\rangle^1_{\rm conn}-\langle e^{-\beta H_I}-1\rangle^0_{\rm conn})\times(1+\langle \xi e^{-\beta H_1}\rangle^1_{\rm conn})\;, \tag{24}$$

where

$$R=\left[\prod_n(\beta\omega_n)\right]\left[\prod_{l\,(\neq 0)}(\beta\omega_l)\right]^{-1}. \tag{25}$$

The value of $R$ can be calculated generally as follows:[21,22]

$$R=\beta A\sqrt{2}m\;, \tag{26}$$

where $A$ is a model-dependent number which is determined by the asymptotic form of the normalized zero-mode eigenfunction $u_0(x)$.[21,22] $A$ is given by the following limit:

$$u_0(x)\to Am^{1/2}e^{-m|x|}\quad\text{as }|x|\to\infty\;. \tag{27}$$

The explicit values of $A$ are listed in Table I. Finally we have reached the general expression for $n_s$ as

$$n_s=n_s^{(0)}\exp(\langle e^{-\beta H_1}-1\rangle^1_{\rm conn}-\langle e^{-\beta H_I}-1\rangle^0_{\rm conn})\times(1+\langle \xi e^{-\beta H_1}\rangle^1_{\rm conn})\;, \tag{28}$$

where

$$n_s^{(0)}=A\pi^{-1/2}mt^{-1/2}e^{-1/t}\;. \tag{29}$$

$n_s^{(0)}$ is the soliton density within the harmonic approximation[22] which makes the soliton free energy consistent with the main term of the results of the TIM. The higher-order corrections are expressed by the diagrammatic expansions in Eq. (28). The origin of the higher-order corrections is very clear. It is simply the contribution from the anharmonicity due to $H_I$, $H_1$, and $\xi$.

Let us calculate the lowest-order ($t$) correction to $n_s^{(0)}$. There are only four diagrams in this order when we perform the average in the one-soliton sector. They are shown in Fig. 1. Diagrams (a)—(c) are due to the exponent in Eq. (28), and (d) is due to the last term in Eq. (28).[23]

In order to calculate these diagrams, we introduce the Green's function in the one-soliton sector in the classical limit as

$$D^1(x,y)=\langle\eta(x)\eta(y)\rangle^1=T\sum_{l\,(\neq 0)}\frac{1}{\omega_l^2}u_l(x)u_l^*(y)\;. \tag{30}$$

The explicit forms of the real-space Green's function for the $\phi^4$ and the SG models are shown in Table II, and the value of each diagram is summarized in Table III. The detailed derivations of the diagramatic calculations are given in Appendix B. Since the nonlocal terms which are proportional to $L$ in the one-soliton sector are canceled

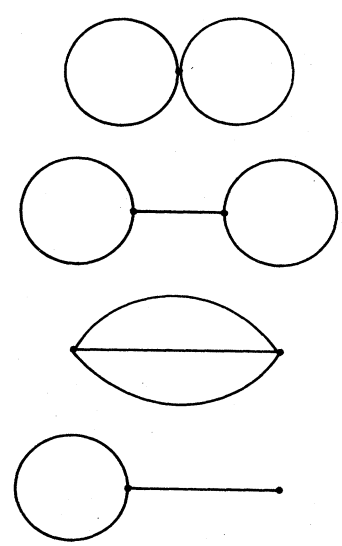


FIG. 1. The diagrams contributing to the lowest order ($t$) corrections to $n_s^{(0)}$ are shown.

TABLE II. The real-space Green's functions for the $\phi^4$ and SG models are shown.

$\Phi^4$ model

$$D^1(x,y)=\left[\frac{T}{2m}\right]\{1+T_1(\sigma)[\tanh(\tfrac{1}{2}mx)-\tanh(\tfrac{1}{2}my)] + T_2(\sigma)[\tanh(\tfrac{1}{2}mx)-\tanh(\tfrac{1}{2}my)]^2\}e^{-m|x-y|}$$

where

$$\sigma=\tfrac{1}{2}m(x-y)$$
$$T_1(\sigma)=2\mathrm{sgn}\sigma+\coth\sigma$$
$$T_2(\sigma)=-\tfrac{1}{8}[3(1+2|\sigma|)+4(4\,\mathrm{sgn}\sigma+3\sigma)\coth\sigma+(11+6|\sigma|)\coth^2\sigma]$$

SG model

$$D^1(x,y)=\left[\frac{T}{2m}\right]\{1+T(\sigma)[\tanh(mx)-\tanh(my)]\}e^{-m|x-y|}$$

where

$$\sigma=m(x-y)$$
$$T(\sigma)=-\tfrac{1}{2}[\sigma+(1+|\sigma|)\coth\sigma]$$

exactly by the contributions from the soliton-free sector diagram by diagram, we finally obtain the following expressions for $n_s$ in the $\phi^4$ and SG models.

(1) $\phi^4$ model,

$$n_s=n_s^{(0)}\exp[-\tfrac{61}{360}t+O(t^2)][1-\tfrac{49}{60}t+O(t^2)]$$
$$=n_s^{(0)}[1-\tfrac{71}{72}t+O(t^2)]\ . \qquad (31)$$

(2) SG model,

$$n_s=n_s^{(0)}\exp[-\tfrac{11}{24}t+O(t^2)][1-\tfrac{5}{12}t+O(t^2)]$$
$$=n_s^{(0)}[1-\tfrac{7}{8}t+O(t^2)]\ . \qquad (32)$$

These higher-order corrections give rise to corrections to the soliton free energy, which coincide exactly with the corrections found by the TIM.[14–16]

## IV. STATIC CORRELATION FUNCTIONS

Although the static correlation functions are calculable both in the soliton-free sector and in the one-soliton sector as shown in I, more care is required to construct the total static correlation functions. In this section we reinvestigate the static correlation functions much more carefully than before and improve the prescriptions given in I. We restrict ourselves to the SG model and choose the following correlation functions:

$$S_{s/2,s/2}(x,y)=\langle\sin[\tfrac{1}{2}g\phi(x)]\sin[\tfrac{1}{2}g\phi(y)]\rangle\ , \qquad (33)$$

TABLE III. The explicit values of each diagram in Fig. 1 for the $\phi^4$ and SG models are shown.

| Diagram | $\phi^4$ model | SG model |
|---|---|---|
| (a) | $-mL(\frac{1}{16}t)-\frac{97}{1680}t$ | $-mL(-\frac{1}{4}t)-\frac{53}{60}t$ |
| (b) | $-mL(-\frac{3}{16}t)-\frac{39}{560}t$ | $\frac{7}{20}t$ |
| (c) | $-mL(-\frac{1}{24}t)-\frac{53}{1260}t$ | $\frac{3}{40}t$ |
| (d) | $-\frac{49}{60}t$ | $-\frac{5}{12}t$ |

$$S_{c/2,c/2}(x,y)=\langle\cos[\tfrac{1}{2}g\phi(x)]\cos[\tfrac{1}{2}g\phi(y)]\rangle\ , \qquad (34)$$

where $\langle\cdots\rangle$ means the average with respect to the full Hamiltonian equation (1). $S_{s/2,s/2}$ is a soliton-sensitive correlation function, but $S_{c/2,c/2}$ is not.[24]

Let us first investigate the contributions from the soliton-free sector extensively. Using the notation of the preceding section, we obtain

$$S_{FF}^0(x,y)=\langle F(g\{\phi_0+\eta(x)\})F(g\{\phi_0+\eta(y)\})e^{-\beta H_I}\rangle_{\mathrm{conn}}^0\ , \qquad (35)$$

where $F(\phi)=\sin(\phi/2)$ or $\cos(\phi/2)$.

The diagrammatic technique allows one to calculate the above expression perturbatively. We have calculated up to order $t^3$ and obtain the following results:

$$S_{s/2,s/2}^0(x,y)=A_0(t)+\sum_{n=1}^{\infty}A_{2n}(t)\exp[-2nB_{2n}(t)m|x-y|]\ , \qquad (36)$$

$$S_{c/2,c/2}^0(x,y)=\sum_{n=1}^{\infty}C_{2n-1}(t)\exp[-(2n-1)D_{2n-1}(t)m|x-y|]\ , \qquad (37)$$

where the first few terms of $\{A_{2n}\}$, $\{B_{2n}\}$, $\{C_{2n-1}\}$, and $\{D_{2n-1}\}$ are shown in Table IV.

If we compare these with the results of TIM (see Appendix C), we conclude that $S_{c/2,c/2}^0$ agrees but $S_{s/2,s/2}^0$ does not, especially as far as the $A_0(t)$ term is concerned. This indicates that for the soliton-insensitive correlation functions the contribution from the soliton-free sector is sufficient, but on the other hand, the multisoliton contribution is essential for the soliton-sensitive ones.[24]

From the above argument we proceed to investigate the one-soliton contributions to $S_{s/2,s/2}$, which are written as

TABLE IV. $\{A_{2n}\}$, $\{B_{2n}\}$, $\{C_{2n-1}\}$, and $\{D_{2n-1}\}$ appearing in Eqs. (36) and (37) are shown.

| | |
|---|---|
| $A_0(t)=1-t-\frac{1}{2}t^2-t^3+O(t^4)$ | |
| $A_2(t)=\frac{1}{2}t^2[1+\frac{3}{2}t+O(t^2)]$ | $B_2(t)=1-\frac{3}{2}t+O(t^2)$ |
| $C_1(t)=t[1-\frac{1}{8}t^2+O(t^3)]$ | $D_1(t)=1-t-t^2+O(t^3)$ |
| $C_3(t)=\frac{3}{8}t^3[1+O(t)]$ | $D_3(t)=1+O(t)$ |

$$S^1_{s/2,s/2}(x,y) = \frac{1}{QL}\int dX \langle \sin[\tfrac{1}{2}g\{\phi_s(x-X)+\eta(x-X)\}] \times \sin[\tfrac{1}{2}g\{\phi_s(y-X)+\eta(y-X)\}] \times (1+\xi)e^{-\beta H_1}\rangle^1 , \quad (38)$$

and

$$Q=\langle (1+\xi)e^{-\beta H_1}\rangle^1 . \quad (39)$$

Agian the diagrammatic technique yields[11]

$$S^1_{s/2,s/2}(x,y)=[1-(2/L)\,|x-y|\,]A_0(t) + \frac{1}{mL}\sum_{n=1}^{\infty} P_{2n}(t;m\,|x-y|) \times \exp(-2nm\,|x-y|) , \quad (40)$$

where $P_{2n}(t;m\,|x-y|)$ is a polynomial in $t$ and $m\,|x-y|$.[11] Next we have to consider the question of how to construct $S_{s/2,s/2}$ in terms of $S^0_{s/2,s/2}$ and $S^1_{s/2,s/2}$ under the same approximation as in the preceding section. Now we shall define $S^{00}_{s/2,s/2}$ and $S^{10}_{s/2,s/2}$ as follows:

$$S^{00}_{s/2,s/2}(x,y)=A_0(t)$$

and (41)

$$S^{10}_{s/2,s/2}(x,y)=[1-(2/L)\,|x-y|\,]A_0(t) .$$

These are the parts of the correlation functions which survive in the limit $|x-y|$ tends to infinity, while other terms vanish exponentially with $|x-y|$. The more precise prescription to construct the correlation function is given by

$$S_{s/2,s/2}(x,y)=S^{00}_{s/2,s/2}(x,y)\exp\left[2n_sL\left[\frac{S^{10}_{s/2,s/2}(x,y)}{S^{00}_{s/2,s/2}(x,y)}-1\right]\right]+\sum_{n=1}^{\infty} A_{2n}(t)\exp[-2nB_{2n}(t)m\,|x-y|\,] . \quad (42)$$

Noticing that $A_0$ in $S^{00}_{s/2,s/2}$ and $S^{10}_{s/2,s/2}$ cancels out, we obtain

$$S_{s/2,s/2}(x,y)=A_0(t)\exp(-4n_s\,|x-y|\,)+\sum_{n=1}^{\infty} A_{2n}(t)\exp[-2nB_{2n}(t)m\,|x-y|\,] , \quad (43)$$

which completely agrees with the results of TIM. Although we eliminate completely the $P_{2n}$ terms in Eq. (40), and their roles are unclear for the moment, we can reproduce the results of TIM and obtain the correlation length of the soliton as

$$\xi_s^{-1}=4n_s . \quad (44)$$

In the case of the $\phi^4$ model the situation is quite similar, so one can easily generalize the argument in this section to other nonlinear Klein-Gordon models.

## V. CONCLUDING REMARKS

Limiting ourselves to the classical limit of the nonlinear Klein-Grodon models, we have examined systematically the higher-order corrections in $t$ to the soliton density $n_s$ in the ideal-soliton-gas approximation. We have generalized the previous calculations for the SG model[11] to the nonlinear Klein-Gordon models and obtained a general formula for $n_s$ which contains the higher-order corrections expressed by diagrammatic expansions. Explicit calculations have been done for the $\phi^4$ and the SG models. Both higher-order corrections to $n_s$ yield corrections to the soliton free energy, which are consistent with the corrections found by the TIM.[14–16] These agreements have proved the validity of the ideal-soliton-gas approximation, even when we include the higher-order corrections in both models. Furthermore, it is confirmed that the method of the collective coordinate provides a general framework for studying the higher-order corrections. We have also improved the previous analysis[11] for the static correlation functions and proposed prescriptions to construct them, and compared them with the results of the TIM.

Our calculations in the classical limit can be easily extended to semiclassical calculation,[5] where we treat only phonons as quantum objects. According to Matsubara's technique at finite temperature, the Green's function in Eq. (30) is simply replaced by a thermal Green's function. After some renormalization[5] one can obtain the higher-order corrections. The results will be reported in future publications.

Finally we comment on the obstacles which prevent us from performing a similar analysis for the DQ model. The local potential $V(\phi;g)$ in the DQ model has a cusp at $\phi=0$.[25] If we formally expand $V(\phi;g)$ around the one-soliton solution $\phi_s$, the interaction Hamiltonian $H_1$ in Eq. (14) contains higher derivatives of Dirac's $\delta$ function. Since these singular vertices cause divergences when we evaluate the diagrams shown in Fig. 1, we have not yet succeeded to get the higher-order corrections to $n_s^{(0)}$ in this special model. Even if we were able to get the finite corrections to the soliton density, there is another problem for the soliton correlation lengths $\xi_s$. According to the exact results of the TIM, $\xi_s^{-1} \propto n_s$ is no longer valid for the DQ model if the higher-order corrections are included.[15] On the other hand, according to the argument in the preceding section, we always obtain $\xi_s^{-1} \propto n_s$ under the ideal-soliton-gas approximation. This discrepancy may indicate a possible breaking down of the ideal-soliton-gas approximation for this special model.[15] Although the DQ model is a mathematical toy, further investigations are clearly desirable.


## ACKNOWLEDGMENTS

One of us (T.M.) wishes to thank the Department of Physics of the University of Southern California for its kind hospitality. We have benefited from discussions with Professor S. Takada on related subjects. This work is supported by the National Science Foundation under Grant No. DMR-82-14525 and by a Grant-in-Aid for Fundamental Scientific Research of the Ministry of Education, Science and Culture (Japan).


## APPENDIX A: PARTITION FUNCTIONS FOR SOLITON-BEARING SYSTEMS

For partition functions for the SG, $\phi^4$ and DQ models are given as

$$Z_{\rm SG} = \sum_{n,m=0}^{\infty} Z_{n,m} , \tag{A1}$$

$$Z_{\phi^4,{\rm DQ}} = \sum_{n=0}^{\infty} Z_n , \tag{A2}$$

respectively, where for the SG model suffix $n,m$ indicates the sector with $n$ solitons and $m$ antisolitons, while for the $\phi^4$ and the DQ models suffix $n$ means the $n$-soliton sector. In contrast to Ref. 13, we do not distinguish between solitons and antisolitons in the latter systems. Since the soliton is always followed by the antisoliton, we consider them as the same particle without any topological constraint.

Now within the ideal-soliton-gas approximation, we obtain

$$Z_{n,m} = Z_0 \frac{(n_s L)^{n+m}}{n!m!} , \tag{A3}$$

$$Z_n = Z_0 \frac{(n_s L)^n}{n!} , \tag{A4}$$

where

$$n_s L = Z_1 / Z_0 , \tag{A5}$$

and

$$Z_1 = Z_{1,0}, \quad Z_0 = Z_{0,0} . \tag{A6}$$

From these follow Eq. (5) in text.

## APPENDIX B: CALCULATION OF THE DIAGRAMS IN FIG. 1

Since the evaluations of the integrals which are represented by Figs. 1(a) and 1(d) is simple, we concentrate on the integral corresponding to Figs. 1(b) and 1(c). These terms are due to the second-order contribution in $H_1^{(1)}$ [which is given by Eq. (17)], and we obtain

$$J = \frac{1}{2!} \beta^2 \langle H_1^{(1)} H_1^{(1)} \rangle_{\rm conn}^1 . \tag{B1}$$

This is expressed in terms of the Green's function $D^1(x,y)$ given in Eq. (30) as

$$J = \frac{1}{2!} \beta^2 (m^2 g/3!)^2 \int dx \int dy \, V^{(3)}(g\phi_s(x);1) V^{(3)}(g\phi_s(y);1) \{ 9D^1(x,x) D^1(x,y) D^1(y,y) + 6[D^1(x,y)]^3 \} , \tag{B2}$$

where the first term corresponds to Fig. 1(b) and the second to Fig. 1(c). In Appendix A in I, we have performed the $k$-space calculations for the SG model. The $k$-space calculations for the $\phi^4$ model are so difficult that we rely on the real-space calculations to evaluate $J$, where $D^1(x,y)$ is explicitely given in Table II.

Let us calculate the second term of $J$ which we call $J_c$ for the $\phi^4$ model, which is the most cumbersome,

$$J_c = 3(\beta m^2 g/2)^2 \int dx \int dy \tanh(\tfrac{1}{2}mx) \tanh(\tfrac{1}{2}my) \times [D^1(x,y)]^3 . \tag{B3}$$

After changing the integral variables $x$ and $y$ to $\sigma = m(x-y)/2$ and $\rho = my/2$, we first perform the $\rho$ integral. We encounter the following type of integral:

$$P_n(\sigma) = \int d\rho [\tanh(\sigma+\rho)] \tanh\rho] \times [\tanh(\sigma+\rho) - \tanh\rho]^n, \quad n = 0,1,\ldots \tag{B4}$$

where in the case of $n=0$ we obtain a nonlocal term which is proportional to $L$.

In order to evaluate $P_n(\sigma)$, we introduce the fundamental integral $R_n(\sigma)$ as follows:

$$R_n(\sigma) = \int d\rho [\operatorname{sech}(\sigma+\rho) \operatorname{sech}\rho]^n, \quad n = 1,2,\ldots \tag{B5}$$

which is easily evaluated as

$$R_n(\sigma) = 2^n \frac{1}{(n-1)!} \frac{1}{(\sinh\sigma)^n} S_n(\sigma) , \tag{B6}$$

and

$$S_n(\sigma) = (\sinh\sigma)^n \left[ -\frac{1}{\sinh\sigma} \frac{\partial}{\partial\sigma} \right]^{n-1} \frac{\sigma}{\sinh\sigma} . \tag{B7}$$

$S_n(\sigma)$ is a polynomial of $\sigma$ and $\coth\sigma$ and satisfies the following recurrence formulas which are essential to our calculations:

$$S_{n+1}(\sigma) = (2n-1)(\coth\sigma) S_n(\sigma) - (n-1)^2 S_{n-1}(\sigma) , \tag{B8}$$

$$S_n'(\sigma) = -(n-1)(\coth\sigma)S_n(\sigma) + (n-1)^2 S_{n-1}(\sigma) \ . \quad \text{(B9)}$$

The remaining $\sigma$ integral is given by the Laplace transform of $S_n(\sigma)$ multiplied by polynomials in $\sigma$ and $\coth\sigma$, which one can calculate by the use of the recurrence formulas, Eqs. (B8) and (B9). The tedious but straightforward calculations yield the results listed in Table III. The first term of $J$ for the $\phi^4$ model is also calculated in a similar manner.

## APPENDIX C: CALCULATION OF THE CORRELATION FUNCTIONS BY THE USE OF THE TIM

According to the well-established TIM,[13] the static correlation function for $F(g\phi)$ is exactly given by as follows:

$$S_{FF}(x,y) = \langle F(g\phi(x))F(g\phi(y))\rangle = \sum_{n=0}^{\infty} \frac{|\langle n|F(\phi)|0\rangle|^2}{\langle n|n\rangle\langle 0|0\rangle} \times \exp\left[\frac{-1}{\alpha t}(\epsilon_n - \epsilon_0)m|x-y|\right] , \quad \text{(C1)}$$

where $\epsilon_n$ is the eigenvalue of the following pseudo-Schrödinger eigenvalue equation:[13]

$$\left[-\frac{(\alpha t)^2}{2}\frac{d^2}{d\phi^2} + V(\phi;1)\right]\Phi_n(\phi) = \epsilon_n \Phi_n(\phi) , \quad \text{(C2)}$$

and the bra-ket notation in Eq. (C1) means

$$\langle n|F(\phi)|m\rangle = \int d\phi\, \Phi_n^*(\phi)F(\phi)\Phi_m(\phi) , \quad \text{(C3)}$$

and $\alpha$ is the model-dependent number listed in Table I. In the low-temperature region ($t \ll 1$) we assume the eigenvalue structure $\epsilon_n$ has a pair structure[13]

$$\epsilon_{2l} = \alpha(E_l - \tau_{2l}) , \quad \text{(C4)}$$

$$\epsilon_{2l+1} = \alpha(E_l + \tau_{2l+1}), \quad l = 0,1,\ldots \quad \text{(C5)}$$

where $E_l$ is the $l$th eigenvalue of the isolated well of $V(\phi;1)$ (order 1) and $\tau_{2l}$ and $\tau_{2l+1}$ are the tunneling contributions (order $e^{-1/t}$). Let us introduce a subsidiary differential equation in order to solve Eq. (C2):[15]

$$\left[-\frac{(\alpha t)^2}{2}\frac{d^2}{d\phi^2} + V(\phi;1)\right]\Psi_l(\phi) = \alpha E_l \Psi_l(\phi) \quad \text{(C6)}$$

under the proper boundary conditions discussed in detail in Ref. 15. We solve the differential equation (C6) by the use of the so-called modified WKB method.[15] We assume

$$\Psi_l(\phi) = \lambda_0^{(l)}(\phi)\exp[(1/t)\lambda_{-1}^{(l)}(\phi)]\left[1 + \sum_{n=1}^{\infty} \lambda_n^{(l)}(\phi)t^n\right] , \quad \text{(C7)}$$

and solve Eq. (C6) under the given $E_l$,

$$E_l = \sum_{n=1}^{\infty} a_n^{(l)} t^n \ . \quad \text{(C8)}$$

If we neglect $e^{-1/t}$ compared with 1, the eigenvalues and eigenfunctions are well approximated by

$$\epsilon_{2l} = \epsilon_{2l+1} = \alpha E_l, \quad l \neq 0 \quad \text{(C9)}$$

$$\Phi_{2l}(\phi) = \Phi_{2l+1}^{(\phi)} = \Psi_l(\phi) \quad \text{for } 0 \le \phi \le 2\pi , \quad \text{(C10)}$$

which we need to evaluate the correlation functions. Let us restrict ourselves to the SG model and choose $F(\phi)$ as $\sin(\phi/2)$ or $\cos(\phi/2)$.

The general expressions for $E_l$ and the first four $\Psi_l$ are summarized by[15]

$$E_l = \frac{2l+1}{2}t - \frac{2l^2+2l+1}{2^2}t^2 - \frac{2l^3+3l^2+3l+1}{2^3}t^3 + O(t^4) , \quad \text{(C11)}$$

$$\Psi_0(\phi) = \frac{1}{w}\exp[-(1-w^2)/t]\left[1 + \frac{t}{2^2}\frac{1}{w^2} + \frac{t^2}{2^4}\left[\frac{1}{w^2} + \frac{3}{w^4}\right] + \frac{t^3}{2^6}\left[\frac{6}{w^2} + \frac{6}{w^4} + \frac{15}{w^6}\right] + O(t^4)\right] , \quad \text{(C12)}$$

$$\Psi_1(\phi) = \frac{(1-w^2)^{1/2}}{w^2}\exp[-(1-w^2)/t]\left[1 + \frac{t}{2^2}\frac{3}{w^2} + \frac{t^2}{2^4}\left[\frac{9}{w^2} + \frac{15}{w^4}\right] + \frac{t^3}{2^6}\left[\frac{90}{w^2} + \frac{90}{w^4} + \frac{105}{w^6}\right] + O(t^4)\right] , \quad \text{(C13)}$$

$$\Psi_2(\phi) = \frac{1-w^2}{w^3}\exp[-(1-w^2)/t]\left[1 + \frac{t}{2^2}\left[-\frac{1}{1-w^2} + \frac{6}{w^2}\right] + \frac{t^2}{2^4}\left[-\frac{11}{1-w^2} + \frac{24}{w^2} + \frac{45}{w^4}\right] + \frac{t^3}{2^6}\left[-\frac{159}{1-w^2} + \frac{339}{w^2} + \frac{405}{w^4} + \frac{420}{w^6}\right] + O(t^4)\right] , \quad \text{(C14)}$$

$$\Psi_3(\phi) = \frac{(1-w^2)^{3/2}}{w^4}\exp[-(1-w^2)/t]\left[1 + \frac{t}{2^2}\left[-\frac{3}{1-w^2} + \frac{10}{w^2}\right] + \frac{t^2}{2^4}\left[-\frac{51}{1-w^2} + \frac{40}{w^2} + \frac{105}{w^4}\right] + \frac{t^3}{2^6}\left[-\frac{1065}{1-w^2} + \frac{645}{w^2} + \frac{1155}{w^4} + \frac{1260}{w^6}\right] + O(t^4)\right] , \quad \text{(C15)}$$

where

$$w=\cos\left[\frac{\pi-\phi}{4}\right]. \tag{C16}$$

This $\{\Psi_l\}$ is the correct low-temperature expansion of the Mathieu functions.[26]

The above expressions allow us to calculate the matrix elements in Eq. (C1) as expansions in $t$. After symmetry arguments which determine the selection ruled for the matrix elements, we finally obtain

$$S_{s/2,s/2}(x,y)=A_0(t)\exp\left[-\frac{1}{t}(\tau_0+\tau_1)m\,|x-y|\right]+\sum_{n=1}^{\infty}A_{2n}(t)\exp[-2nB_{2n}(t)m\,|x-y|\,], \tag{C17}$$

$$S_{c/2,c/2}(x,y)=\sum_{n=1}^{\infty}C_{2n-1}(t)\exp[-(2n-1)D_{2n-1}(t)m\,|x-y|\,], \tag{C18}$$

where $\{A_{2n}\}$, $\{B_{2n}\}$, $\{c_{2n-1}\}$, and $\{D_{2n-1}\}$ are shown in Table IV, and

$$\frac{1}{t}(\tau_0+\tau_1)=4n_s. \tag{C19}$$

Here $n_s$ is the soliton density under the ideal-soliton-gas approximation given in Eqs. (32). Equations (C17)—(C19) are compared with the results of the PIM in Sec. IV.

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