PHYSICAL REVIEW B
CONDENSED MATTER

THIRD SERIES, VOLUME 31, NUMBER 4 15 FEBRUARY 1985

Localization and quantum-mechanical resonant tunneling in the presence of a time-dependent potential

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(Received 5 October 1984)

A model for tunneling in the presence of a localized, harmonic time-dependent potential is presented. The effect of dynamic oscillations on the static resonances is determined in two important cases: (1) two consecutive high barriers where resonances arise from potential energy traps, and (2) a sequence of random barriers where resonances arise from disorder-localized states. In both cases strong resonances survive even when the potential varies rapidly compared to the particle's intrinsic lifetime. Analysis of the two-barrier case reveals an interesting connection to a solvable model of the incommensurate potential problem.

I. INTRODUCTION

Resonant tunneling is a characteristic and fundamental quantum-mechanical phenomenon. A quantum-mechanical free particle of energy E incident upon a region L of average potential energy $\overline{V} \gg E$ can of course tunnel through that region; however, the probability of transmission will be exponentially small with L, *unless* the energy E is very close to a resonant energy of the potential $V(x)$. These special energies correspond to the energies of the solutions of the Schrödinger equation for the region L, which are localized in the middle and have roughly equal wave-function amplitudes at both ends. At such energies the transmission coefficient may be of order unity independent of the height of the barriers (although the resonance widths will be very small if the barriers are high). The particle transmission therefore exhibits dramatically sharp resonances at energies determined by the potential $V(x)$; this phenomenon is known as resonant tunneling.

It has recently been argued[1-4] that conduction in disordered solids, particularly quasi-one-dimensional systems, should exhibit interesting resonant tunneling effects because of the localization of states due to the random potential. The strongest, sharpest resonances would correspond to states peaked within a localization length L_0 of the center of the solid, and the nonresonant background to states localized at one end. The energies of the resonances depend on the particular impurity configuration of the sample. Moreover, there is a great deal of recent experimental work[5-9] which shows dramatic nonmonotonic variations in the conductance of quasi-one-dimensional metal-oxide-semiconductor field-effect transistor (MOSFET) devices as a function of Fermi energy. The observation that these oscillations are reproducible within a given sample upon voltage and temperature cycling but differ from sample to sample is suggestive of resonant tunneling.

However, resonant transmission is associated with tunneling through exponentially narrow quasibound states which thus have very long intrinsic lifetimes. The physical picture is of a wave packet of extremely well-defined energy (hence of great spatial extent) incident upon the scatterer and slowly leaking into its interior. Away from resonance the multiple internal reflections destructively interfere and rapidly an exponentially small "steady-state" density is established in the interior, from which a small additional flux leaks out the other side. Just at resonance, however, the internal reflections constructively interfere and a large "steady-state" density builds up inside the scatterer over a time on the order of the inverse resonance width (which typically scales exponentially with the length of the scatterer); and finally the particle, which is now mostly localized within the scatterer, leaks out the other side with high probability. This picture is entirely consistent for an electron in a static potential at zero temperature or, more generally, whenever the intrinsic reso-

nance widths are greater than the frequencies of any dynamic fluctuations in the potential barrier; however, it is problematic when applied to systems where dynamic potential fluctuations typically occur on a much shorter time scale than the inverse resonance width. The constructive interference leading to resonant transmission ought to be disrupted by inelastic scattering. The crucial question is whether resonant transmission is destroyed by dynamic fluctuations on the very-low-energy scale set by the resonance width, or only by fluctuations on a larger scale such as the resonance spacing or particle energy. Therefore we are motivated to ask a very basic physical question: What is the effect of introducing a time-dependent potential on the resonance transmission? In this paper we answer this question for the case of a localized harmonically time-dependent potential in one dimension (1D). We solve this problem exactly numerically for a particle tunneling through a random potential, and by a variety of analytic and numerical techniques for the simple case of tunneling through two high barriers. Physically this model represents the effect of a harmonic external driving force interacting with the tunneling particle. It may also be cautiously interpreted as a semiclassical model of the effect of inelastic scattering from a localized phonon mode, but it does not adequately simulate physical effects arising from the distribution of phonon energies and dissipative effects present in thermal equilibrium.

II. GENERAL MODEL

The model[10] we study is described by a time-dependent one-dimensional Schrödinger equation of the form

$$i\frac{\partial\psi(x,t)}{\partial t}=\left[-\frac{\partial^2}{\partial x^2}+u_1(x)+u_2(x)+2\delta(x)\gamma\cos(\omega t)\right]\psi(x,t)\ , \tag{1}$$

where $\hbar=2m=1$, $u_1(x)$ is an *arbitrary* static potential which is 0 for $x>0$ and goes to 0 as $x\to-\infty$, and $u_2(x)$ is an arbitrary static potential which is 0 for $x<0$ and goes to 0 as $x\to+\infty$. We wish to calculate the transmission amplitude for a particle incident from $+\infty$ to tunnel to $-\infty$, and to study its energy dependence as we vary the two parameters γ and ω, which characterize the dynamic potential. ω is of course the quantum of phonon energy, and, since we assume no particles are trapped in true bound states of the barrier, we must always assume $\omega\ll E$. The dimensionless parameter $\Gamma=\gamma/\sqrt{E}$ is essentially the probability of emission or absorption of one phonon while tunneling (when this ratio is small), and we will be mainly interested in the case when $\Gamma\omega$, the typical energy transfer in tunneling, is greater than the intrinsic resonance width but much less than the particle energy. Another important energy scale is the resonance spacing, which is basically the energy-level spacing of the scatterer, $\Delta\epsilon$, and this will enter our model in a natural way below.

Since the only time dependence is harmonic, we can always write a solution as

$$\psi(x,t)=\sum_{n=-\infty}^{\infty}\phi_n(x)e^{-i(E+n\omega)t}\ .$$

Substituting this into Eq. (1) gives

$$(E+n\omega)\phi_n=-\frac{\partial^2\phi_n}{\partial x^2}+(u_1+u_2)\phi_n+\gamma\delta(x)(\phi_{n-1}+\phi_{n+1})\ . \tag{2}$$

Integrating Eq. (2) over an infinitesimal region around the origin yields a linear relation between $\phi_n(0^+)$ and $\phi_n(0^-),\phi_{n-1}(0^-),\phi_{n+1}(0^-)$. At $\pm\infty$, each ϕ_n consists of a sum of two plane waves with wave vectors $\pm k_n=\pm(E_n)^{1/2}$, $E_n=E+n\omega$. For a given E_n, there always exists a linear relation[2] between the plane-wave amplitudes at $\pm\infty$ and at $0^{\pm}$, expressible in terms of 2×2 transfer matrices $\underline{M}_n^{(1)},\underline{M}_n^{(2)}$. Combining these two relations and assuming there is a monochromatic incoming wave of energy E at $+\infty$ and only outgoing waves at $-\infty$ gives us a linear relation determining $t(E+n\omega)\equiv t_n$, the transmission amplitude at outgoing energy E_n,

$$f_n t_n+\gamma h_n(g_{n+1}t_{n+1}+g_{n-1}t_{n-1})=\delta_{n0}\ , \tag{3}$$

where f_n,g_n,h_n are energy-dependent coefficients determined by $\underline{M}_n^{(1)}$ and $\underline{M}_n^{(2)}$:

$$f_n=(\underline{M}_n^{(1)}\underline{M}_n^{(2)})_{22}\ ,$$
$$g_n=(\underline{M}_n^{(1)})_{12}+(\underline{M}_n^{(1)})_{22}\ ,$$
$$h_n=-(i/2k_n)[(\underline{M}_n^{(2)})_{21}-(\underline{M}_n^{(2)})_{22}]\ .$$

Although we wish to solve for t_n, it is convenient to isolate the energy dependence in Eq. (3) by defining $c_n=t_n g_n$, giving

$$Q_n c_n+\Gamma(c_{n+1}+c_{n-1})=(1/h_0k_0)\delta_{n0}\ , \tag{4}$$

where $Q_n=f_n/(h_n g_n k_0)$, and $\Gamma=\gamma/\sqrt{E}=\gamma/k_0$.

Equation (4) is the basic equation of this model. It says that for a monochromatic incident wave of energy E, there will be transmitted waves at all energies $E\pm n\omega$ whose amplitudes are determined by the nearest-neighbor finite-difference equation in *frequency* space (4). All information about the static potential u_1+u_2 is contained in the function Q_n which depends on the elements of the transfer matrices $\underline{M}_n^{(1)},\underline{M}_n^{(2)}$ at energies E_n. The parameter Γ measures the coupling to the time-dependent potential and the static limit is obtained by letting $\Gamma\to0$. It is worth noting the formal analogy between Eq. (4) (for $n\neq0$) and the Schrödinger equation for the *real-space* wave-function amplitudes (at $E=0$) of a nearest-neighbor coupled tight-binding model. To obtain a useful representation of the diagonal element Q_n, we now introduce a completely general parametrization[2] of $\underline{M}_n^{(1)},\underline{M}_n^{(2)}$:

$$\underline{M}_n^{(j)}=\begin{pmatrix}\cosh(s_j)e^{i\alpha_j} & \sinh(s_j)e^{i\beta_j}\\ \sinh(s_j)e^{-i\beta_j} & \cosh(s_j)e^{-i\alpha_j}\end{pmatrix}\ .$$

By definition, $[\underline{M}^{(j)}(E_n)]_{11}$ is the inverse of the *static* transmission amplitude across the potential $u_j(x)$ at energy E_n; therefore e^{-s_j} measures the static tunneling probability and s_j will typically be much greater than 1, while α_j and β_j determine the phase shifts across $u_j(x)$. With this parametrization,

$$f_n = e^{i(\beta_1-\beta_2-\alpha_1-\alpha_2)}\left[\cosh(s_1+s_2)\cos\left[\frac{\alpha_1+\alpha_2+\beta_1-\beta_2}{2}\right]-i\cosh(s_1-s_2)\sin\left[\frac{\alpha_1+\alpha_2+\beta_1-\beta_2}{2}\right]\right], \tag{5a}$$

$$g_n = e^{-i[(\alpha_1-\beta_1)/2]}e^{s_1}\left[\cos\left[\frac{\alpha_1+\beta_1}{2}\right]-ie^{-2s_1}\sin\left[\frac{\alpha_1-\beta_1}{2}\right]\right], \tag{5b}$$

$$h_n = (1/2k_n)e^{-i[(\alpha_2+\beta_2)/2]}e^{s_2}\left[\sin\left[\frac{\alpha_2-\beta_2}{2}\right]+ie^{-2s_2}\cos\left[\frac{\alpha_2-\beta_2}{2}\right]\right], \tag{5c}$$

and, if $s_1, s_2 \gg 1$,

$$Q_n \approx \frac{2k_n}{k_0}\cos\left[\frac{\alpha_1+\alpha_2+\beta_1-\beta_2}{2}\right]\Big/\cos\left[\frac{\alpha_1+\beta_1}{2}\right]\sin\left[\frac{\alpha_2-\beta_2}{2}\right]. \tag{6}$$

We see that Q_n can vary between $\pm\infty$,[11] if we ignore the intrinsic resonance widths, and typically is order unity. Thus we might guess that Q_n may take arbitrary values at integer n and simulates to some extent a random 1D potential. In the static limit ($\Gamma \to 0$), from (4) we see

$$t_n = \delta_{n0}(Q_0 h_0 g_0 k_0)^{-1} \sim \delta_{n0}(e^{-s_1-s_2}/Q_0), \tag{7}$$

which tells us that static resonances, with $t_0 \sim 1$, correspond to incident energies where $Q_0 \equiv Q(E)$ is 0 to accuracy $e^{-s_1-s_2}$. Moreover, we see that the energy scale over which Q_n varies appreciably is at most the energy-level spacing of the scatterer, $\Delta\epsilon$, therefore we must always consider Eq. (4) over an energy range $n\omega \gg \Delta\epsilon$ to obtain a meaningful solution independent of boundary conditions.

Because of the generality of our formulation of Eq. (4), we are able to make some useful observations about the solutions for the transmission amplitudes independent of the form of the potential which gives rise to the static resonances. If we think of Eq. (4) as an infinite dimensional matrix equation of the form $\underline{Q}\mathbf{c} = \mathbf{V}_0$, we can immediately write down its solution in terms of a spectral representation of the operator $\underline{Q}^{-1}$

$$t_n = \frac{1}{h_0 g_n k_0}\sum_{m=0}^{\infty}\frac{\chi_m(0)\chi_m(n)}{\lambda_m}, \tag{8}$$

where $\{\lambda_m\}, \{\chi_m\}$ are the eigenvalues and eigenvectors of $\underline{Q}$, satisfying the equation

$$Q_n\chi_m(n)+\Gamma[\chi_m(n+1)+\chi_m(n-1)] = \lambda_m\chi_m(n). \tag{9}$$

Based upon the form of Eqs. (8) and (9), we can make several general comments about the nature of the solutions t_n. First, the factor $1/(h_0 g_n k_0)$ in (8) sets the scale for the nonresonant transmission; typically, $1/(h_0 g_n k_0) \sim e^{-s_1-s_2}$. If there are incident energies for which some $\lambda_m \sim e^{-s_1-s_2}$, then there will be resonances at these energies, assuming $\chi_m(0)$ is not too small. Since for each choice of incident energy one gets a continuously varying set of $\{Q_n\}$ and thus $\{\lambda_m\}$, it would seem that by varying E one could always get an eigenvalue to cross zero. This suggests that the resonances will remain for finite Γ, but at shifted incident energies. However, the strength of the resonances will depend on the amplitude the eigenvector with smallest eigenvalue has at the incident energy (site 0 in the lattice language). This depends crucially on whether the solutions to Eq. (9) are localized in energy, or extended. If they are localized, then we can predict the new resonant spectrum in the presence of inelastic scattering quite generally from a perturbative solution of Eqs. (8) and (9).

III. GENERAL PERTURBATIVE ARGUMENT

In this section we assume that the solutions of Eq. (9) are localized, and that therefore perturbation theory converges. Below we will present both numerical and analytic arguments that this assumption is justified in all relevant cases except one.

When $\Gamma = 0$ the eigenvalues of Eq. (9) are just $\{Q_n\}$ with associated eigenstates that are completely localized (i.e., unit vectors on site n). Denote the unperturbed eigenstates at incident energy E by $\{|n,E\rangle\}$ with eigenvalues $\{Q_n(E)\}$, and the perturbed states by $\{|\chi_n,E\rangle\}$ with eigenvalues $\{\lambda_n(E)\}$. Assume for the moment that $\Gamma < 1$. Suppose a static resonance occurs at E_r, so $Q_0(E_r)=0$. For $\Gamma \neq 0$, the new eigenvalue $\lambda_0(E_r) \sim \Gamma^2$; however, if the eigenvalues λ_n are a continuous function of E, then for some nearby energy E_r', with $Q_0(E_r') \sim \Gamma^2$, the shifted eigenvalue $\lambda_0(E_r')=0$. E_r' is fully determined by the usual perturbation series for $\lambda_0(E_r')$, i.e.,

$$\lambda_0(E_r') = Q_0(E_r') + \sum_{m=1}^{\infty}\lambda_0^{(m)}(E_r')\Gamma^m = 0, \tag{10}$$

where, e.g.,

$$\lambda_0^{(2)} = [Q_0(E_r')-Q_1(E_r')]^{-1}+[Q_0(E_r')-Q_{-1}(E_r')]^{-1}.$$

Note, however, $\lambda_0(E_r')$ must be zero with accuracy $e^{-s_1-s_2}$ to give a resonance of order unity, so many terms in the power series in Eq. (10) must be kept to determine E_r' precisely and in practice it is easiest just to search numerically for E_r' near E_r. To leading order in Γ the new eigenvector $|\chi_0,E_r'\rangle$ has matrix elements

$$\langle n,E_r' \mid \chi_0,E_r' \rangle = \delta_{n0} + \Gamma^n A_{n0} , \tag{11}$$

where

$$A_{n0} = \prod_{j=n}^{1} (Q_0 - Q_j^{-1}) .$$

For $n=0$ the matrix element is approximately unity and there is indeed a strong resonance (which we will call a fundamental resonance) at shifted incident energy E_r' with $t_0 \simeq 1$ (assuming $\lambda_0 \simeq e^{-s_1-s_2}$).

Furthermore, if E_r' gives $\lambda_0(E_r') \simeq 0$, it is clear that incident energy $E_r' + n\omega$ will give $\lambda_{-n}(E_r' + n\omega) \simeq 0$, since shifting incident energy by $n\omega$ generates exactly the same set $\{Q_n\}$, just relabeled. The only difference is that the overlap $\langle \chi_{-n}, E_r' + n\omega \mid 0, E_r' + n\omega \rangle \sim \Gamma^n$. Therefore the maximum transmission amplitude is now at energy $n\omega$ below the incident energy, at E_r', with a maximum amplitude $t_{-n}(E_r + n\omega) \sim \Gamma^n$. Therefore the time-dependent potential creates a set of weaker subsidiary resonances associated with each fundamental resonance.

As for the widths of the new resonances, we can make the following argument: The form of the eigenvectors in Eqs. (8) and (9) does not affect the width of the resonances, only their strength. It is the dependence of the smallest eigenvalue on incident energy which determines the width. The static resonance width is obtained by expanding $Q_0(E_r)$ and is essentially $[(\partial Q_0/\partial E)(E_r)]^{-1} \times e^{-s_1-s_2}$. From Eq. (10) we see that $(\partial\lambda_0/\partial E)(E_r') \simeq (\partial Q_0/\partial E)(E_r') \simeq (\partial Q_0/\partial E)(E_r)$ to order Γ^2. Therefore, the fundamental resonances are not broadened at all in the strict sense and are still exponentially narrow; however, it is possible to regard the appearance of subsidiary resonances as a type of thermal broadening, as will be discussed in our conclusions.

When $\Gamma \geq 1$ our perturbative argument is still valid as long as the solutions of Eq. (4) are localized as $n \to \pm\infty$ and there will still be exponentially small eigenvalues determined by Eq. (10). However, their eigenvectors will be extended over many "lattice sites" and there will no longer be a clear set of fundamental resonances; instead there will be an oscillatory structure to the transmission coefficient as a function of incident energy, showing sharp, relatively weak resonances with spacing ω.

To summarize, our perturbative argument implies that for $\Gamma < 1$ the static resonance of order unity survives at a shifted energy E_r'. In addition the system develops weaker "satellite" resonances at incident energies $E_r' \pm n\omega$, in which the particle scatters inelastically, absorbing or emitting n quanta, and emerges with the resonant energy E_r'. Neither the fundamental or satellite resonances are broadened. When $\Gamma > 1$ there are still zero eigenvalues which give rise to a rapidly oscillating transmission spectrum with many weak resonances but no fundamental resonances of order unity. The validity of these arguments (except in one special case) for the two physical situations we consider is confirmed by numerical calculations, some of which will be presented below.

IV. THE TWO-BARRIER CASE

First we consider Eq. (4) for the case where we choose the static potentials u_1 and u_2 to be two consecutive strong-scattering barriers with a classically allowed region between them. By strong scattering we mean each barrier has a transmission coefficient much less than 1 in isolation and for further simplicity we will assume that the average of the barrier potential over the forbidden region is much greater than the particle energy. This latter condition ensures that the static resonance energies are shifted very little from the value they would have if the forbidden region were infinite and the resonances were true bound states. Then there are two possible situations. Either the barrier is sharp on the scale of the particle wavelength or it is smooth. In the former case the resonance energies will be well approximated by the energy levels of a particle in a box which is the length of the classically allowed region; in the latter case they will be well approximated by the energy levels determined by applying the WKB approximation to the allowed region (i.e., the Bohr quantization condition). We will treat the former case here, but extension of the analysis to the latter is straightforward.

For definiteness we consider a model consisting of two static delta-function barriers separated by a distance L, with the time-dependent delta function at an arbitrary location x_0 between them:

$$u_1(x) = V_0\delta(x + x_0), \quad u_2(x) = v_0\delta(x - L + x_0)$$

$$\text{with } V_0 \gg E ,$$

and $0 \leq x_0 \leq L$. A simple calculation of the static transfer matrices yields $\alpha_1 = \alpha_2 = \arctan(-V_0/2k) \simeq -\pi/2$, and $\beta_1 = -\pi/2 + kL$, $\beta_2 = -\pi/2 - kL$. Thus

$$Q_n = \frac{(2k_n/k_0)\sin(k_nL)}{\sin(k_nx_0)\sin[k_n(L - x_0)]} \tag{12}$$

to zeroth order in k/V_0. As noted above, this result has a simple physical interpretation. The zeros of Q_n occur at the static resonance energies, which in this case correspond to the quantum states of a particle in an infinite well and of course occur at energies where $k_nL = m\pi$. The zeros of the denominator represent resonances between the static delta functions and the oscillatory delta function, at x_0. Equation (12) holds for any choice of two high, sharp scattering barriers $u_1(x), u_2(x)$, although the first correction in k/V_0 will vary. To get a better feeling for the eigenvalues and eigenvectors of Q given this form of Q_n we expand $k_n \simeq k_0 + n\omega/2k_0$ and define $\Delta\epsilon = 4\pi k_0/L$, $\Delta\epsilon_1 = 4\pi k_0/x_0$, $\Delta\epsilon_2 = 4\pi k_0/(L - x_0)$, giving

$$Q_n \simeq \frac{\dfrac{2k_n}{k_0}\sin\left[k_0L + \dfrac{2\pi n\omega}{\Delta\epsilon}\right]}{\sin\left[k_0x_0 + \dfrac{2\pi n\omega}{\Delta\epsilon_1}\right]\sin\left[k_0(L - x_0) + \dfrac{2\pi n\omega}{\Delta\epsilon_2}\right]} . \tag{13}$$

Ignoring the relatively slow variation of k_n. Q_n is the ratio of three periodic functions of $n\omega$ with different periods, and except for special choices of x_0, the position of the oscillatory δ function, the periods are not com-

mensurate. In this case Q_n is aperiodic and we would certainly expect the eigenvectors of Eq. (9) to be localized. Even if the periods are commensurate, Q_n will not be periodic at integer n unless the driving frequency ω is commensurate with the energy-level spacing $\Delta\epsilon$. In this special case Q_n is approximately periodic and this has interesting physical implications which we will discuss in detail below. However, for the moment we wish to argue that the solutions to Eqs. (8) and (9) are localized in all other cases.

A nonperturbative argument leading to this conclusion comes from making the choice $x_0=L/2$, that gives the simplest rational relation between $\Delta\epsilon_1,\Delta\epsilon_2$ and $\Delta\epsilon$:$\Delta\epsilon_1=\Delta\epsilon_2=\Delta\epsilon/2$, giving

$$Q_n=2\cot\left[\frac{k_0L}{2}+\frac{\pi n\omega}{\Delta\epsilon}\right]. \tag{14}$$

Now Q_n is a simple periodic function of $n\omega$, but unless $\omega/\Delta\epsilon$ is a rational number, Q_n will not repeat at integer n. This reveals an exact formal analogy between the eigenvalue problem of Eqs. (9) and (14), and the problem of the nature of the eigenvalues of an electron in an incommensurate spatial potential which has been of great recent interest to theoretical physicists. Moreover, for Q_n of the above form, the exact eigenvalues and eigenvectors of $\underline{Q}$ are known for $\omega/\Delta\epsilon$ *irrational*.[12] As conjectured, the exact analytic solution shows that Q_n mimics a random potential: (i) All eigenvectors are localized. The localization length is determined by Γ just as it is in a tight-binding model with a Cauchy distribution of random site energies (of unit variance) and constant nearest-neighbor hopping matrix elements Γ. (ii) In general the equation which relates the eigenvalues to the values of the initial phase of the cotangent is complicated (this phase k_0L is of course determined by the incident energy in our model, and thus is the parameter we want to vary). However, fortunately, for zero eigenvalues it simplifies greatly, and implies that there are zero eigenvalues whenever

$$k_0L=2\pi\nu_1+2\pi\nu_2\frac{\omega}{\Delta\epsilon}, \tag{15}$$

where ν_1,ν_2 are integers. Clearly, a choice of ν_1 determines the energy of the fundamental resonance, whereas a choice of ν_2 determines the energy of the subsidiary resonances spaced ω apart. (iii) The dependence of the eigenvalues λ_n on incident energy near $\lambda_n=0$ is linear and resonances are not broadened. (iv) Expansion of the solution for the eigenvectors with zero eigenvalue for $\Gamma<1$ agrees with our perturbation theory for the resonance strengths. Thus all these results are consistent with perturbation theory, and since the choice $x_0=L/2$ gives the most simple periodic form for Q_n, this strongly suggests that the general features of the new resonance spectrum are correctly described by the perturbative analysis given above. Note that this conclusion has the sensible implication that the qualitative behavior of our model does not depend at all on the position of the oscillatory delta function x_0. These conclusions are confirmed by numerical solution of Eq. (4) in the two-barrier case; see, e.g., Fig. 1 in which we have plotted the total transmission coefficient $T=\sum|t_n|^2$ versus incident energy E.

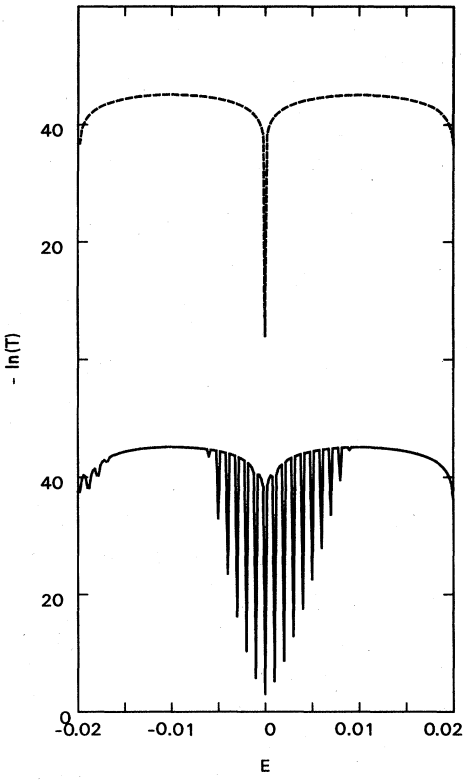

FIG. 1. Plot of $-\ln T$ versus incident energy E for two rectangular barriers. The dashed line is the static result with static resonance energy $E_0=1.0$, the solid line is for $\Gamma=0.1$. The energy range shown is two level spacings (measured from E_0) and the small energy shift of the static resonance is not shown. $\omega=0.001$, which is the spacing between resonances.

The only situation in which one would expect something new to happen on physical grounds is when ω, the driving frequency, is commensurate with $\Delta\epsilon$, the energy-level spacing of the barriers. Then the time-dependent potential can cause real energy-conserving transitions between the resonant levels with the emission or absorption of an integral number of quanta and this should have important physical consequences. This is the case we consider now.

If $\omega/\Delta\epsilon$ is rational, then Q_n is periodic and commensurate with the lattice [in the approximation of Eq. (14)]. Within this approximation we ignore all energy dependence of Q_n except for the linear expansion of k_nL, and we may write Eq. (9) in the form

$$2\cot\left[\frac{k_0L}{2}+\pi np/m\right]t_n+\Gamma(t_{n+1}+t_{n-1})=\bar{t}\,\delta_{n0}. \tag{16}$$

where $\bar{t}$ is the nonresonant transmission amplitude, and p and m are integers ($p/m=\omega/\Delta\epsilon$). Our spectral representation of the solution for t_n [Eq. (8)] is no longer meaningful, since it implicitly assumed the existence of an orthonormal set of eigenvectors. Such a set does not in general exist when Q_n is periodic. However we can still solve Eq. (4) directly by finding solutions of the equation for $n\neq0$ and then connecting two such solutions at the origin in a manner consistent with the inhomogeneous

term. The solutions away from the origin must reflect the translational symmetry of the "lattice," i.e., the solution in one unit cell must be a constant times the solution in an adjacent cell (it may be worth reminding the reader that we are always referring to positions in energy when we use the terms "site" and "lattice"). If the constant has modulus unity the solution lies in a band of the lattice and must be of the Bloch form; if it has a modulus not equal to unity the solution lies in a gap and must be exponentially growing to the right or left. In a perfect lattice such gap solutions are not allowed; however, because the equation we are solving is inhomogeneous it is possible to connect a solution which decays exponentially to the left of the origin with one that decays exponentially to the right of the origin and get a full solution which does not blow up at $\pm\infty$. An examination of Eq. (16) for various values of m, p, and Γ reveals that both types of solutions can occur. For gap solutions we impose the boundary conditions that $t_n \to 0$ as $n \to \pm\infty$. Then the three joining equations for t_n at $n=0,\pm 1$ fully specify the solution for t_n. The details of such a solution are worked out in the Appendix; however, the results are not too interesting, because it results that there is in general no broadening or damping of the static resonances. The case where the solution lies in the band is physically much more interesting because it corresponds to the situation where the particle makes real transitions to other energy levels of the barrier while it is being transmitted, which is what we usually mean by inelastic scattering. Unfortunately, with the approximation of Eq. (16), we cannot obtain a meaningful solution for t_n. This is because when the solutions are in the band we cannot impose the boundary condition that $t_n \to 0$ as $n \to \pm\infty$, and therefore in order to fully specify a solution we need to impose a further arbitrary boundary condition at some finite N. But then our solution will depend on what boundary condition we choose and will not be physically meaningful.

To get around this difficulty we must abandon the approximation of Eq. (16), and insert the next energy-dependent terms in Q_n, which destroys its perfect periodicity. Now the solutions will be localized in energy as in the incommensurate case, but on a much broader energy scale. We can estimate very crudely the new energy localization length by looking at the most rapidly varying energy-dependent correction to Q_n, which is the next term in the expansion of the argument $k_n L$ in the cotangent, and arguing that when this is of order π the approximate periodicity of Q_n is lost. This predicts an energy localization length $N_0 \sim (E\Delta\epsilon/\omega^2)^{1/2}$. If we now return to our spectral representation of the solution, it is clear that we can still get zero eigenvalues by searching in incident energy. The spacing between eigenvalues whose eigenvectors have significant overlap with site 0 will now be on the order of $1/N_0$, and since this is typically much greater than the resonance width there will still be only one zero eigenvalue for each strong resonance. However, now for a monochromatic incident wave at the resonant energy there will be outgoing waves of equal strength at all energies E_n within $N_0\omega$ of the incident energy. If we again look at the spectral representation for t_n in Eq. (8), assuming one smallest eigenvalue $\lambda_s \simeq e^{-s_1 - s_2}$ dominates and estimating the total transmission, we find that $|\sum_{n=0}^{\infty} t_n|^2$ is now reduced by a factor $1/N_0$, assuming the normalized eigenvector $\chi_s(n) \sim N_0^{1/2}$. A similar crude estimate suggests that $\partial\lambda_s/\partial E$ is increased by a factor $N_0^{1/2}$, which causes a broadening of the resonance. Although this result makes some sense qualitatively, it should not be taken too seriously, since it only applies to the case where there are many levels with spacing exactly equal to ω, whereas one might suppose that the possibility of inelastic scattering to even one other level would disrupt the phase coherence needed for resonant tunneling. However, we note that if the time-dependent potential is due to an external field, our model makes the physically sensible prediction that in this case the transmitted electron emerges with much greater mean squared energy than the incident one, and this means the external field heats the electron distribution much more efficiently. This kind of resonant effect is often observed in atomic or molecular physics where the external field is provided by a laser; it would be interesting to observe such an effect in a semiconductor heterojunction at low temperature.

V. THE RANDOM SYSTEM

Now we turn to the case where the static potentials u_1 and u_2 are sequences of N_1 and N_2 random barriers which represent a disordered solid. Informed by our work

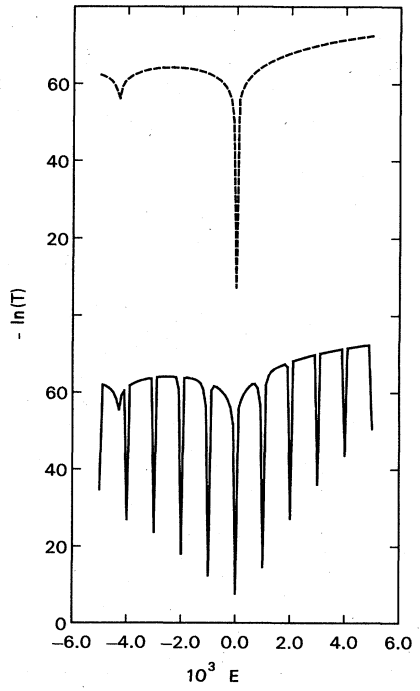

FIG. 2. Plot of $-\ln T$ vs E for two sequences of 1000 random barriers with $w=1.0$ and $E_0 \simeq 2.19$. The dashed line is the static result and the solid line the result for $\Gamma=0.1$. The average level spacing is about 0.002, $\omega=0.001$, which is the spacing between resonances, and the actual width of all resonances shown is of order 10^{-14}.

on the previous model we can immediately see the only essential difference between localized states arising from the strong scattering barriers and those arising from a random potential. In the latter case, *the energy levels are not equally spaced;* rather they are randomly distributed over some energy interval. Therefore, there should be no analog of the case where ω is commensurate with $\Delta\epsilon$, and a pure harmonic driving force can never broaden the resonances of a disordered scatterer. However, we do expect satellite resonances to appear here in essentially the same way as in the previous model when ω is incommensurate with $\Delta\epsilon$.

To confirm this we choose $u_1(x)=\sum_{a=1}^{N_1} V_a\delta(x+a)$, $u_2(x)=\sum_{b=1}^{N_2} V_b\delta(x-b)$, where $\{V_{a,b}\}$ are random variables uniformly distributed with $-W\leq V_{a,b}\leq W$. We numerically generated the transfer matrices $\underline{M}_n^{(1)}$ and $\underline{M}_n^{(2)}$ for a particular impurity configuration and generated the $\{Q_n\}$ needed for Eq. (4). As long as the solutions of (4) are localized, they may be found with exponential accuracy by truncating the finite difference equation at $E\pm N\omega$ (for some large N) and inverting the resulting matrix equation. In order to consider fairly large N we used the iterative inversion algorithm for tridiagonal matrices first employed by Thouless and Kirkpatrick.[13] Not surprisingly, we find the solutions of (4) *are* localized and typical results for the new transmission spectrum are shown in Fig. 2. The numerical results also show that there is no resonance broadening, only a shifted resonance with satellites. This is fully consistent with our earlier perturbative arguments.

VI. CONCLUSIONS

The physical interpretation of our results is thus straightforward. For $\Gamma<1$, the harmonic driving field does not significantly alter the intrinsic level structure of the system, even though the potential varies significantly during a particle lifetime, *unless* ω is commensurate with the intrinsic level spacing $\Delta\epsilon$. This cannot occur for a random system, since $\Delta\epsilon$ is not constant. The only alteration in the resonant spectrum arises from a small shift in the resonance energies due to the presence of the time-dependent potential, and the fact that now a particle incident at a gap energy exactly $n\omega$ from a level can make a transition there and be resonantly transmitted. This phenomena is analogous to thermal broadening in that it allows an electron with the "wrong" initial energy to still fit into the resonance. It seems clear that a time-dependent potential with a continuous distribution of frequencies will lead to a continuous broadening of this type, and not just discrete sidebands. However, the more interesting effect physically is not the smearing of the incident electron energy, but the effect of inelastic scattering on the quasi-bound state itself. This effect is only seen when ω is commensurate with $\Delta\epsilon$ so that the particle can make real transitions from one level to another; this phenomenon may lead to a true increase in the intrinsic resonance width. Our model does not allow a detailed, physically meaningful calculation of this broadening.

Finally, we emphasize that a thermal equilibrium *distribution* of oscillating potentials may have a very different effect on the resonances of a random system in part because such a potential can cause real transitions to any level within k_BT of the incident energy. However, it is by no means clear that even an arbitrary time-dependent potential (with a cutoff frequency which plays the role of the temperature) is a fully adequate model for inelastic scattering in a solid. This is because such a model still may not simulate the effect of the exchange of energy with a thermal reservoir; which introduces irreversibility and therefore true dissipation into the system. The question of how inelastic processes affect resonant tunneling appears to be closely related to the general question of how interaction with a macroscopic system can destroy quantum-mechanical interference phenomena. A first-principles theoretical answer to this question is a goal we have not yet attained.[14]

ACKNOWLEDGMENTS

We would like to acknowledge the patient assistance of K. Rabe in computational aspects of this work, as well as a useful discussion with D. Grempel. Part of this work was supported by National Science foundation MRL Grant No. DMR81-19295, and by Office of Naval Research Grant No. N00014-77-C-0132. One of us (M. Ya. A.) is very grateful to M. H. Cohen for his hospitality at Exxon Research Corporation.

APPENDIX: SOLUTION OF EQUATION (16) IN GAP

In this Appendix we outline the solution of Eq. (16) when the solutions of the homogeneous ($\bar{t}=0$) equation lie in a gap. For $n\neq 0$ any solution must be a sum of the two linearly independent solutions:

$$t_n=A_L\Phi_L(n)+A_R\Phi_R(n)\ , \tag{A1}$$

where Φ_L and Φ_R are the solutions of the homogeneous equation which decay exponentially to the left and right, respectively, and A_L, A_R are constants. Write $n=qm+p$, where q and p are integers (m is the number of sites in the unit cell). Then by translational symmetry

$$\begin{aligned}\phi_L(n)&=(\lambda_L)^qU^L(p)\ ,\\ \phi_R(n)&=(\lambda_R)^qU^R(p)\ ,\end{aligned} \tag{A2}$$

where $\lambda_{L,R}$ are eigenvalues of the translation operator, and $U^{L,R}(p)$ are periodic functions satisfying $U(p+m)=U(p)$. By definition $|\lambda_L|>1$, $|\lambda_R|<1$. We shall see below that $\lambda_L\lambda_R=1$.

We can determine $\lambda_{L,R}$ as follows: Inserting the form (A2) into the homogeneous equation yields m equations for $U(p)$ (we will now suppress the indices L,R). However, only the two equations at the boundaries of the unit cell involve λ. If we define

$$\epsilon_p=2\cot\left[\frac{k_0L}{2}+\frac{p\pi}{qm}\right],\quad p=0,1,\ldots,m-1\ ,$$

then these equations are

$$\lambda\epsilon_0 U(0)+\Gamma[\lambda U(1)+U(m-1)]=0\ ,$$
$$\epsilon_{m-1}U(m-1)+\Gamma[\lambda U(0)+U(m-2)]=0\ . \tag{A3}$$

To eliminate two unknowns from this equation we note that the other $m-2$ equations can be rewritten in the form

$$\begin{bmatrix} U(m-1) \\ U(m-2) \end{bmatrix} = \prod_{p=1}^{m-2} \underline{T}^{(p)} \begin{bmatrix} U(1) \\ U(0) \end{bmatrix} \equiv \underline{T}_m \begin{bmatrix} U(1) \\ U(0) \end{bmatrix} \tag{A4}$$

where the $\underline{T}^{(p)}$ are 2×2 matrices given by

$$\underline{T}^{(p)}=\begin{bmatrix} -\epsilon_p/\Gamma & -1 \\ 1 & 0 \end{bmatrix}.$$

For simplicity of notation we define $(\underline{T}_m)_{11}=T_1$, $(\underline{T}_m)_{12}=T_2$, $(\underline{T}_m)_{21}=T_3$, $(\underline{T}_m)_{22}=T_4$. Then using (A4) to eliminate $U(1), U(m-2)$ in (A3), we write (A3) as a matrix equation and set the determinant equal to 0. This gives a quadratic equation for λ, with solutions

$$\lambda=-b\pm(b^2-1)^{1/2}\ ,$$
$$b=\frac{1}{2}\left[T_1+\frac{1}{T_1}\right]-\frac{1}{2}\left[\frac{\epsilon_0 T_1}{\Gamma}-T_2\right]\left[\frac{\epsilon_{m-1}}{\Gamma}+\frac{T_3}{T_1}\right]. \tag{A5}$$

Note, the product of the roots is unity, as claimed. Knowing λ, one can now plug back into the m equations for $\{U(p)\}$ and solve for them up to a normalization factor. Now we return to solve the inhomogeneous equation. We assume

$$t_n=\begin{cases} A_L\phi_L(n), & n\le -1 \\ A_R\phi_R(n), & n\ge 1\ . \end{cases} \tag{A6}$$

Thus $t_n\to 0$ as $n\to\pm\infty$, as it must. We choose the normalization $\phi_L(-1)=1$, $\phi_R(1)=1$. With this choice the equations for t_0, t_1, t_{-1} are

$$t_0=\frac{-\Gamma(A_L+A_R)}{\epsilon_0}+\frac{\bar{t}}{\epsilon_0}\ ,$$
$$t_1=A_R=\frac{-\Gamma}{\epsilon_1}(t_0+r_1 t_1)\ , \tag{A7}$$
$$t_{-1}=A_L=\frac{-\Gamma}{\epsilon_{m-1}}(t_0+r_2 t_{-1})\ ,$$

with $r_1=U^R(2)/U^R(1)$, $r_2=U^L(m-2)/U^L(m-1)$. Solving these gives

$$t_0=\frac{\bar{t}/\epsilon_0}{1+(\Gamma/\epsilon_0)\beta} \tag{A8}$$

with

$$\beta=\frac{-\Gamma/\epsilon_{m-1}}{1+r_2\Gamma/\epsilon_{m-1}}+\frac{-\Gamma/\epsilon_1}{1+r_1\Gamma/\epsilon_1}\ . \tag{A9}$$

Static resonances of order unity occur when $k_0 L$ is near enough to an odd multiple of π that $\epsilon_0\simeq\bar{t}$. We assume $\epsilon_0\ll\Gamma\ll 1$. If β were of order unity we would then predict a weakened, broadened resonance. In fact, since the cotangent is antisymmetric around its zero, $|\epsilon_{m-1}-\epsilon_1|\simeq 2|\epsilon_0|$, and $\beta\sim\Gamma/\epsilon_0$. Thus $t_0\sim\bar{t}/\epsilon_0$, and we find no damping or broadening of the static resonance for this case. Numerical solutions for t_n in Eq. (16) by direct diagonalization (not using this approach), for various values of m, show that these gap solutions do exist and do not give damped or broadened resonances, in agreement with the above argument.

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[1]I. M. Lifshitz and V. Ya. Kirpichenkov, Sov. Phys.—JETP **50**, 499 (1979).

[2]M. Ya. Azbel, Phys. Rev. B **28**, 4106 (1983); and Solid State Commun. **45**, 527 (1983).

[3]M. Ya. Azbel and P. Soven, Phys. Rev. B **27**, 831 (1983).

[4]D. P. DiVincenzo and M. Ya. Azbel, Phys. Rev. Lett. **50**, 2102 (1983).

[5]A. B. Fowler, A. Hartstein, and R. A. Webb, Phys. Rev. Lett. **48**, 196 (1982).

[6]R. G. Wheeler, K. K. Choi, A. Goel, R. Wisnieff, and D. E. Prober, Phys. Rev. Lett. **49**, 1674 (1982).

[7]W. J. Skocpol, L. D. Jackel, E. L. Hu, R. E. Howard, and L. A. Fetter, Phys. Rev. Lett. **49**, 951 (1982).

[8]M. Pepper and M. J. Uren, J. Phys. **C15**, L617 (1982).

[9]R. F. Kwasnick, M. A. Kastner, J. Melngailis, and P. A. Lee, Phys. Rev. Lett. **52**, 224 (1984).

[10]Nonresonant tunneling through a time-dependent barrier has been studied by M. Buttiker and R. Landauer [Phys. Rev. Lett. **49**, 1739 (1982)], by considering a similar model.

[11]In Eq. (6) and subsequently in the text we have dropped the exponentially small imaginary part of Q_n. This imaginary part prevents the transmission coefficient from becoming greater than unity when the real part of Q_n is zero; and this approximation is equivalent to ignoring the intrinsic resonance widths. We do not make this approximation in the numerical solution of Eq. (4), and it has no effect on our conclusions.

[12]D. R. Grempel, S. Fishman, and R. E. Prange, Phys. Rev. Lett. **49**, 833 (1982).

[13]D. J. Thouless and S. Kirkpatrick, J. Phys. C **14**, 235 (1981).

[14]Since completion of this work there have been two attempts to determine the effect of inelastic processes on resonant tunneling in 1D disorder-localized systems based on phenomenological theories, which have reached somewhat different conclusions. See M. Ya. Azbel, A. Hartstein, and D. P. DiVincenzo, Phys. Rev. Lett. **52**, 1641 (1984); A. D. Stone and P. A. Lee (unpublished).