PHYSICAL REVIEW B VOLUME 31, NUMBER 3 1 FEBRUARY 1985

Damping of second sound in superfluid helium by order-parameter relaxation

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(Received 24 September 1984)

The local temperature oscillations that occur in both first and second sound require a corresponding variation in the local superfluid density, which takes place by order-parameter relaxation. Therefore, the corresponding contribution to the damping rate of both first and second sound is proportional to the order-parameter relaxation time. But in second sound, part of the required variation in superfluid density is produced instantaneously by convection, thereby greatly reducing the relaxational damping of second sound. The full unmodified effect of order-parameter relaxation only sets in asymptotically very close to the λ point.

It has been emphasized[1] that there are (at least) the following four sources of damping of second sound near the λ point of superfluid ^{4}He: (1) entropy diffusion, (2) transverse order-parameter relaxation, (3) longitudinal order-parameter relaxation, and (4) normal fluid second viscosity. The last effect has recently been discussed in detail.[2] The purpose of this Rapid Communication is to present a simple and compelling treatment of the third effect. We feel that this is especially desirable at the present time because of (a) the recent precise measurements of the damping of second sound[3] and (b) the fact that the so-called renormalization-group calculation[4] omits[5] this contribution to the damping.

Our approach, which we have previously only sketched,[6,7] is basically that of Pitaevskii[8] and Khalatnikov,[9] which we, however, cast into somewhat different form. To set the stage it is necessary to review briefly the standard thermodynamic theory of the second-sound velocity u by introducing the mass fractions $X=\rho_s/\rho$ and $Y=\rho_n/\rho$ for the superfluid and normal fluid, respectively. Because of the obvious identity $X+Y=1$, only one of these variables is free. There is some conceptional advantage in working with Y and its time derivative, $\partial Y/\partial t=\dot{Y}$. In general, a fluid of local density ρ and local hydrodynamic velocity $\vec{v}$ satisfies the conservation law $\partial\rho/\partial t=-\mathrm{div}\rho\vec{v}$. For small velocities this can be written to first order as

$$\frac{\partial \ln\rho}{\partial t}=-\nabla\cdot\vec{v} \quad . \tag{1}$$

In other words, the fractional rate of increase of the density is equal to the convergence of the velocity. For the two-fluid situation it follows that the fractional rate of change of the normal-superfluid ratio is equal to the convergence of the counterflow velocity, i.e.,

$$\frac{\partial \ln(Y/X)}{\partial t}=-\nabla\cdot(\vec{v}_n-\vec{v}_s) \quad , \tag{2}$$

which can be written as

$$\dot{Y}_{\mathrm{conv}}=XY\nabla\cdot(\vec{v}_s-\vec{v}_n) \quad , \tag{3}$$

where the subscript signifies the *convective* change in Y. Because the entropy resides in the normal fluid and is carried along with it, the entropy per ^{4}He atom, s, varies in proportion to the normal fraction according to $\dot{s}/s=\dot{Y}_{\mathrm{conv}}/Y$, or

$$\dot{s}=sX\nabla\cdot(\vec{v}_s-\vec{v}_n) \quad . \tag{4}$$

Our quick review of the second-sound velocity is completed by recalling that the equation of motion for the counterflow is

$$m(\dot{\vec{v}}_s-\dot{\vec{v}}_n)=\frac{s}{Y}\nabla T \quad , \tag{5}$$

which, substituted into Eq. (4), gives

$$\ddot{s}=\frac{s^2\rho_s}{m\rho_n}\nabla^2 T \quad . \tag{6}$$

If the frequency ω is sufficiently small a variation δs produces the thermodynamic variation

$$\delta T_0=\frac{T}{C_P}\delta s \quad , \tag{7}$$

where C_P is the constant-pressure specific heat per atom. Substitution of Eq. (7) into Eq. (6) yields the wave equation

$$\ddot{s}=u^2\nabla^2 s \quad , \tag{8}$$

with the normal second-sound velocity formula

$$u^2=\frac{Ts^2\rho_s}{mC_P\rho_n} \quad . \tag{9}$$

Clearly Eq. (9) has to be modified for $\omega\neq 0$ because it is known from the theory[10] of first sound damping that the thermodynamic specific heat C_P has to be replaced by $C_P(\omega)$, with a definite frequency dependence, as confirmed by the agreement of this theory with experiment.[11] At low frequencies, $\omega<<\tau^{-1}$, where τ is the mean relaxation time for the superfluid order parameter, the hysteretic lagging response of the order parameter is the dominant frequency-dependent contribution to $C_p(\omega)$. This can be decomposed as

$$C_P(\omega)=C_{P,Y}+\frac{C_s}{1-i\omega\tau}\cong C_{P,\Delta}+i\omega\tau C_s = C_{P,\Delta}(1+i\omega\tau') \quad , \tag{10}$$

where the renormalized mean time is

$$\tau' = \frac{C_s}{C_{P,\Delta}}\tau \tag{11}$$

and

$$C_s = C_{P,\Delta} - C_{P,Y} \tag{12}$$

is the contribution of the order parameter to $C_{P,\Delta} = C_P$, the $\omega = 0$ thermodynamic specific heat. Δ is the difference in chemical potential between the superfluid and normal fluid and vanishes in the $\omega \to 0$ limit. By neglecting the weaker frequency dependence coming from the fluctuations, we can identify $C_{P,Y}$ as essentially the $\omega \to \infty$ high-frequency limit of the specific heat, for clamped order parameter.

It would be natural to treat second sound analogously to first sound and, wherever C_P appears in the velocity formula, to substitute in its place the frequency-dependent generalization $C_P(\omega)$ from Eq. (10). Equation (9) would then yield

$$u^2(\omega \neq 0) \cong (1 - i\omega\tau')u^2(\omega = 0) \ . \tag{13}$$

The small imaginary part of the complex second-sound velocity would then correspond to a frequency breadth of $\omega^2\tau'/2$ and to a contribution to the damping coefficient of

$$\Delta D_2^{\text{asymp}} = u^2\tau' \ . \tag{14}$$

This, in fact, is the formula advanced by Khalatnikov[8] as a result of an approximation that is asymptotically valid in the immediate neighborhood of the λ point. Equation (14) is, however, not valid in the temperature range in which the experimental measurements have been carried out.[3] As we now demonstrate, Eq. (14) undergoes a substantial reduction by a factor f^2, where f is a kind of "coupling constant" that is introduced below.

The error in simply substituting $C_P(\omega)$ in place of C_P in Eq. (9) lies in the fact that $C_P(\omega)$ describes the oscillatory response of the superfluid in the absence of any convective "bunching" produced by the counterflow. In this case thermal equilibrium requires

$$\dot{Y}_0 = \left(\frac{\partial Y}{\partial s}\right)_{P,\Delta} \dot{s} \ , \tag{15}$$

with the entire change in Y_0 supplied by supernormal conversion by the relaxation process. In the case of second sound, however, convection, according to Eqs. (3) and (4), already supplies the change

$$\dot{Y}_{\text{conv}} = \frac{Y}{s}\dot{s} \ . \tag{16}$$

Thus, the net amount of supernormal relaxation that is required corresponds to the difference

$$\dot{Y}_0 - \dot{Y}_{\text{conv}} = \left(1 - \frac{\dot{Y}_{\text{conv}}}{\dot{Y}_0}\right)\dot{Y}_0 = f\dot{Y}_0 \ , \tag{17}$$

reduced by the factor

$$f = 1 - \frac{Y}{s}\left(\frac{\partial s}{\partial Y}\right)_{P,\Delta} \ . \tag{18}$$

For $\omega \neq 0$ the equilibration is not complete, resulting in the small out-of-equilibrium values $\Delta, \delta Y_1 = \delta Y - \delta Y_0$, and $\delta T_1 = \delta T - \delta T_0$. The relaxation of δY toward its instantaneous equilibrium value δY_0 is governed by the rate equation

$$\delta\dot{Y} = -\gamma''(\delta Y - \delta Y_0) + \delta\dot{Y}_{\text{conv}} \ , \tag{19}$$

where

$$\gamma'' = \frac{C_{P,\Delta}}{C_{P,Y}}\gamma \ . \tag{20}$$

Substitution of Eq. (17) puts Eq. (19) into the form

$$\left(\frac{\partial}{\partial t} + \gamma''\right)\delta Y_1 = -f\delta\dot{Y}_0 = i\omega f\delta Y_0 \ . \tag{21}$$

For $\omega/\gamma'' = \omega\tau'' \ll 1$ the time derivative on the left-hand side of Eq. (21) can be dropped, giving

$$\delta Y_1 = i\omega\tau'' f\delta Y_0 \tag{22}$$

and

$$\Delta = \left(\frac{\partial \Delta}{\partial Y}\right)_{P,s}\delta Y_1 = i\omega\tau'' f\left(\frac{\partial \Delta}{\partial Y}\right)_{P,s}\left(\frac{\partial Y}{\partial s}\right)_{P,\Delta}\delta s \ . \tag{23}$$

A thermodynamic identity then yields

$$\Delta = -i\omega\tau'' f\left(\frac{\partial \Delta}{\partial s}\right)_{P,Y}\delta s \ . \tag{24}$$

The small supernormal chemical potential difference expressed by Eq. (24) corresponds to the first-order temperature deviation

$$\delta T_1 = \left(\frac{\partial T}{\partial \Delta}\right)_{P,s} = -i\omega\tau'' f\left(\frac{\partial T}{\partial \Delta}\right)_{P,s}\left(\frac{\partial \Delta}{\partial s}\right)_{P,Y} \ . \tag{25}$$

This can be simplified by the substitution of the thermodynamic identity

$$\left(\frac{\partial T}{\partial \Delta}\right)_{P,s}\left(\frac{\partial \Delta}{\partial s}\right)_{P,Y} = \frac{T}{C_{P,Y}} - \frac{T}{C_{P,\Delta}} = \frac{TC_s}{C_{P,Y}C_{P,\Delta}} \tag{26}$$

to yield

$$\delta T_1 = -i\omega\tau'' f\frac{C_s}{C_{P,Y}}\frac{T\delta s}{C_{P,\Delta}} = -i\omega\tau' f\delta T_0 \ , \tag{27}$$

by virtue of Eqs. (20), (12), and (7). Equation (27) would signify that the effective frequency-dependent specific heat that regulates the second-sound oscillations is proportional to $1 - i\omega\tau' f$, which, according to Eq. (13), would add one factor of f to the right-hand side of Eq. (14). But this is not the whole story. Because of the imbalance in the chemical potential the equation of motion acquires the additional force, $-\text{grad}\Delta$, so that Eq. (5) becomes

$$m(\dot{\vec{v}}_s - \dot{\vec{v}}_n) = \frac{s}{Y}\nabla T - \nabla\Delta = \frac{s}{Y}\nabla\left(\delta T - \frac{Y}{s}\Delta\right)$$

$$= \frac{s}{Y}\nabla\left\{\delta T_0 + \left[1 - \frac{Y}{s}\left(\frac{\partial \Delta}{\partial T}\right)_{P,s}\right]\delta T_1\right\} \ . \tag{28}$$

But by means of the Maxwell relation

$$\left(\frac{\partial T}{\partial \Delta}\right)_{P,s} = \left(\frac{\partial Y}{\partial s}\right)_{P,\Delta} \ , \tag{29}$$

we see that the quantity in square brackets in Eq. (28) is nothing other than the "coupling constant" defined in Eq.

(18). Thus, the effective hysteretic temperature deviation from δT_0 is not δT_1 but $f\delta T_1$. It follows by substitution of Eq. (27) that the correct effective frequency-dependent specific heat, for the second-sound dynamics, is not given by Eq. (10), but rather (for $\omega\tau' \ll 1$) by

$$C_P^{\rm eff}(\omega) = (1 + i\omega\tau' f^2) C_P \ . \tag{30}$$

Therefore, the frequency-dependent factor in Eq. (13) should be $1 - i\omega\tau' f^2$, so that the final version of Eq. (14), valid also in the nonasymptotic region, is

$$\Delta D_2 = u^2 \tau' f^2 \ . \tag{31}$$

Although $f^2 \rightarrow 1$ asymptotically as $t \rightarrow 0$, the approach is very slow. Even at $t = 10^{-5}$, the closest approach that is experimentally feasible,[3] $f^2 \cong 0.7$, representing a 30% reduction of ΔD_2 below its asymptotic expression in Eq. (14). The reduction increases to approximately 50% at $t = 10^{-4}$ and tends toward 100% further below the λ point, as $\partial Y/\partial s$ tends toward Y/s, and as illustrated in Fig. 2 of Ref. 6.

This work has been supported by the U. S. National Science Foundation under Grants No. DMR 7901172 and No. 8205356.

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