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Temperature dependence of the excitation energy spectrum of He II

P. Suebka and P. Lu
Department of Physics, Arizona State University, Tempe, Arizona 85287

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The energy excitation spectrum of He II is temperature dependent. This was shown by using the experimental data reported by Bendt, Cowan, and Yarnell [Phys. Rev. **113**, 1386 (1959)]. We can reproduce this dependence by using a modified Brueckner-Sawada method [Phys. Rev. **106**, 1128 (1957)] together with an external potential.

This note is written to report on the calculation of the excitation energy spectrum, $E(k,T)$, of quasiparticles of He II. It not only depends on $\mathbf{k}$, the wave vector, but also on T, the temperature. The experimental part of the data was given by Bendt, Cowan, and Yarnell.[1] Here, we supply a theoretical interpretation by using the Brueckner-Sawada method[2] with several modifications. This method was proven to be untrustworthy by Parry and Ter Haar;[3] however, we will see what they are doing. They used the Born approximation in evaluating the t matrices by including an outside attractive potential to the hard-core potential as in

$$t_{\mathbf{k}\mathbf{k}'}^{(l)}=4\pi\int dr\, r^2[\phi_{\mathbf{k}}^{(l)}(r)]^*v(r)\psi_{\mathbf{k}'}^{(l)}(r) \quad , \tag{1}$$

with

$$\phi_{\mathbf{k}}^{(l)}(r)=\frac{1}{\sqrt{\Omega}}j_l(kr) \quad , \tag{2}$$

and same form as the Born approximation is used,

$$\psi_{\mathbf{k}'}^{(l)}(r)=\frac{1}{\sqrt{\Omega}}j_l(k'r) \quad . \tag{3}$$

Their conclusion is that the attractive potential together with a hard-core potential gives a poorer agreement than hard core alone. However, as we have reported earlier,[4] if we had used, instead of Eq. (3),

$$\psi_{\mathbf{k}}^{(l)}(r)=\frac{1}{\sqrt{\Omega}}\left[j_l(kr)-\frac{j_l(ka)}{n_l(ka)}n_l(kr)\right]\left[1+\frac{j_l^2(ka)}{n_l^2(ka)}\right]^{-1/2} \quad , \tag{4}$$

which pays attention to the fact that the hard core with distance $a\neq0$, has a primary importance, the conclusion reached by Parry and Ter Haar would then be no longer true. The E curve, or the excitation energy spectrum, will then bend over and give us a nice roton part.

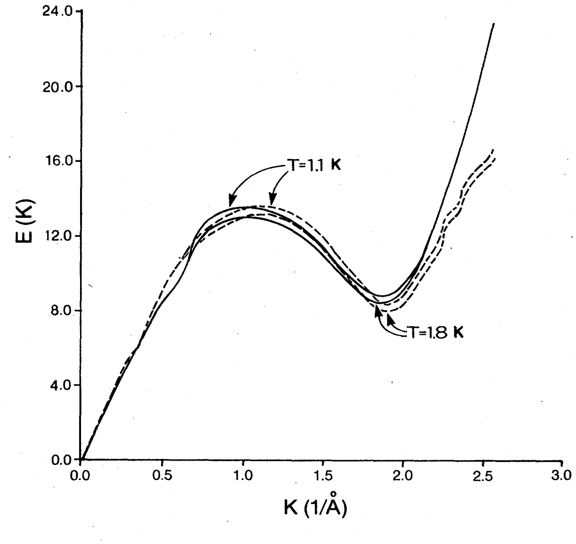

FIG. 1. $E(k,T)$ at $T=1.1$ and 1.8 K in experiment (dashed curves) and in theory (solid curves).

We start from the expression[3] with the independent variable k and T,

$$E(k,T)=\left[\left(\frac{\hbar^2k^2}{2m}+Y+N_0t_k^{(2)}\right)^2 - (Y+N_0t_{00k-k}^{(2)})(Y+N_0t_{k-k00}^{(2)})\right]^{1/2} \quad , \tag{5a}$$

with

$$Y=\frac{33\hbar^2\sin(ka)}{2kma^3} \tag{5b}$$

being for the hard-core part. Here N_0 is temperature dependent, as

$$N_0=N\left[1-\left(\frac{T}{T_\lambda}\right)^{5.6}\right] \quad . \tag{6}$$

Equation (6) is given empirically by Clow and Reppy.[5] As it was correctly pointed out by Parry and Ter Haar, that in Eq. (5a) we should use N_0, the number of the particles in the

TABLE I. $E(k,\ T=1.1\ \text{K})$ from theory and experiment.

k	$[E(k,\ T=1.1\ \text{K})]_{\text{theor}}$	$[E(k,\ T=1.1\ \text{K})]_{\text{expt}}$
0.3	5.51	5.81
0.5	8.91	9.13
0.7	12.72	11.90
0.9	13.46	13.41
1.1	13.45	13.92
1.3	12.79	12.90
1.5	11.53	11.70
1.7	9.96	10.17
1.9	8.85	8.68
2.1	9.49	10.00
2.3	12.46	13.5

TABLE II. $E(k, T=1.8\text{ K})$ from theory and experiment.

k	$[E(k, T=1.8\text{ K})]_{\text{theor}}$	$[E(k, T=1.8\text{ K})]_{\text{expt}}$
0.3	5.51	5.80
0.5	8.91	9.10
0.7	11.74	11.70
0.9	12.62	13.10
1.1	12.75	13.42
1.3	12.16	12.40
1.5	10.93	11.30
1.7	9.35	9.68
1.9	8.27	8.20
2.1	9.11	9.60
2.3	12.40	13.00

TABLE III. Comparison of the differences between $E(k, T=1.1\text{ K})$ and $E(k, T=1.8\text{ K})$ in theory and experiment.

k	$[E(k, T=1.1\text{ K}) - E(k, T=1.8\text{ K})]_{\text{theor}}$	$[E(k, T=1.1\text{ K}) - E(k, T=1.8\text{ K})]_{\text{expt}}$
0.3	0.0	0.01
0.5	0.0	0.03
0.7	0.98	0.20
0.9	0.84	0.31
1.1	0.70	0.51
1.3	0.63	0.50
1.5	0.60	0.40
1.7	0.61	0.49
1.9	0.58	0.48
2.1	0.38	0.40
2.3	0.06	0.5

condensate instead of using N, the total number. Thus through Eq. (6), we will bring out the temperature dependence. In Fig. 1 we see that these excitation energy spectra are exhibited both experimentally and theoretically, and are at two different temperatures $T=1.8$ K and $T=1.1$ K, respectively. At such temperatures, we can say that we have a good check. In Tables I and II, we list the values of $E(k,T)$ for these two specific temperatures. We thus achieve in showing that this can be understood simply by noting that N_0 is temperature dependent through the relation given by Eq. (6).

Equation (6) is actually for quasiparticles. We have to use the number of helium particles here, which is an approximation. We readily had $N_0 = N[1-(T/T_\lambda)^{3/2}]$ for ideal helium gas. This can also be used instead of Eq. (6), but being an ideal gas is also an approximation. We have determined to use Eq. (6) throughout, and we think establishing this interpretation is important here. In Table III, we list the results of k vs $E(k, T=1.1\text{ K}) - E(k,T=1.8\text{ K})$. The agreements are between $\frac{4}{5}$ and $\frac{1}{5}$.

The following are two approximations that we have used in addition to what is described immediately above. We consider these minor improvements, however, they contribute to the accuracy.

We use Eq. (4) for $\phi_k^{(l)}(r)$ as $l>0$, and for $l=0$, we will consider that, besides the hard core, there is an outside potential which modifies the wave function, too. We therefore write

$$\psi_k^{(l=0)}(r) = \begin{cases} \dfrac{1}{r}[A_1\mathrm{Ai}(Z) + B_1\mathrm{Bi}(Z)], & a \leq r \leq a' \\ \dfrac{1}{r}\dfrac{C_1\sin(kr) + C_2\cos(kr)}{k\sec(ka)}, & r \geq a' \end{cases} \tag{7}$$

where we have considered that the outside potential is a linear one with

$$Z = \left[\frac{zmV_0}{\hbar^2(a-a')}\right]^{1/2}\left[r - \frac{\hbar^2k^2}{mV_0}(a'-a) - a'\right] . \tag{8}$$

If $E=0$ or $k=0$, we have to replace $v \geq a'$ with

$$\psi_{k=0}^{(l=0)} = C_1' + \frac{C_2'}{r} . \tag{9}$$

These constants A_1, B_1, C_1, C_2, and C_1' and C_2' are determined by the connection formulas at a and a' and the normalization requirement. a and a' are fixed numbers as used in Ref. 4. We can then easily evaluate the matrices $t_k^{(2)}$, $t_{00k-k}^{(2)}$, and $t_{k-k00}^{(2)}$ as given before in Ref. 4.

For k being small we use the temperature independent dispersion relation given by Maris,[6] which was later proved by Benin.[7] k is chosen to be less then, or up to, 0.8. The relation reads

$$E = (18.173\text{ K})\left[1 + 2.29\left(\frac{0.667k}{m}\right)^2 - 12.6\left(\frac{0.667k}{m}\right)^3 + \cdots\right]k . \tag{10}$$

However, as $k \geq 0.8$ the temperature dependence becomes visible. We have to use the T dependence of N_0. Then we see that we can reproduce the experimental excitation energy spectra with temperature dependence for the entire k range.

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[1] P. J. Bendt, R. D. Cowan, and J. L. Yarnell, Phys. Rev. **113**, 1386 (1959).

[2] K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1128 (1957).

[3] W. E. Parry and D. Ter Haar, Ann. Phys. (N.Y.) **19**, 496 (1962).

[4] P. Lu and C. K. Chan, Phys. Rev. B **20**, 3709 (1979).

[5] J. R. Clow and J. D. Reppy, Phys. Rev. Lett. **16**, 887 (1966).

[6] N. J. Maris, Phys. Rev. A **9**, 1412 (1974).

[7] D. Benin, Phys. Rev. B **11**, 145 (1975).