Dissipative dynamics of a two-state system coupled to a heat bath

Lit-Deh Chang and Sudip Chakravarty
Physics Department, State University of New York at Stony Brook, Stony Brook, New York 11794
(Received 24 April 1984)

The dynamics of a two-state system coupled to the dissipative degrees of freedom of the environment appears in many different contexts. In this paper we have given a general formulation of the problem as well as a detailed comparison of two *a priori* very different heat baths: (a) a collection of harmonic oscillators coupled to the system, and (b) a collection of fermions coupled to the system. Consideration of such heat baths is motivated by the assertion that the weakly excited states of the environment can be modeled by a collection of elementary excitations following Landau. Our conclusion is that the case of Ohmic dissipation of harmonic-oscillator heat bath is very similar to the fermionic bath, although a crucially important dimensionless parameter α assumes a different range of values in the two different cases.

I. INTRODUCTION

It is difficult to overemphasize the fact that the dynamics of a two-state system coupled to the dissipative degrees of freedom of the environment appears in many different situations. In the past it has been discussed in its various forms in the context of a number of different problems involving defect tunneling in solids.[1] More recently, it has played an important role in the discussion of tunneling states in insulating and metallic glasses.[2] It has also been invoked to explain the unusual dielectric relaxation phenomena in solids,[3] and also as the basis for a new model for A15 superconductors.[4] Our interest in this subject was initially motivated by the recent activity in the subject of macroscopic quantum coherence in superconducting-quantum-interference devices (SQUID).[5] Such two-state models are also of importance in the theory of quantum measurement[6] and in many problems in chemical physics.[7] Furthermore, the present problem is related, in a very general sense, to the widely discussed Kondo problem[8] in solid-state physics.

The aim of the present paper is to introduce an exact and unified formalism to discuss the *real-time dynamics* of a two-state system coupled to the environment. Although our main interest would be in the case of Ohmic dissipation which raises many special interesting questions,[9] the general framework, as we show, can be readily adapted to discuss very general dissipative mechanisms. The central result of our paper is an exact formula which gives the dynamics of the two-state system coupled to a general linear heat bath (the precise meaning will be discussed below). We then apply this formula to discuss the similarities as well as the differences between *a priori* two very different heat baths: one consisting of bosons and the other consisting of fermions. This comparison, we believe, should clarify a number of interesting questions in the theory of macroscopic quantum coherence in superconducting interference devices. The general framework of the present paper is based on the formalism and the ideas of Feynman and Vernon[10] and Schwinger.[11] We believe that here we have extended their ideas in new directions.

II. THE NATURE OF THE HEAT BATH AND THE HAMILTONIAN

In a number of situations the weakly excited states of a macroscopic system can be described in terms of quasiparticles, a concept generally due to Landau. Although the interaction between the quasiparticles brings about profound changes in the properties of the macroscopic system *itself,* when the macroscopic system is merely there to act as a reservoir to extract energy, or, what is the same, to provide a dissipative mechanism to the dynamics of a subsystem (the two-state system in the present context), such a macroscopic system acts to a good approximation as a collection of independent elementary excitations coupled to the subsystem under consideration. What is important are certain general properties of the elementary excitations: spin, charge, and statistics; their influence on the subsystem is characterized by a spectral density of the excitations. To be useful such a characterization should be complete in terms of a small number of parameters. This then is our fundamental philosophy, an assertion that underlies all discussions of heat baths in all literature. In fact, we have gone one step beyond the usual discussion by giving the heat bath its proper quantum numbers as well as a (so far unspecified) spectral density. We certainly do not intend to imply that the interaction between the elementary excitations does not exist; in fact, it must, in order that the system equilibrates, but it is of no special importance to our problem except in so far as it is taken into account in the definition of the spectral density parametrized by a small number of parameters intrinsic to the heat bath. The question now arises as to what if the dynamics is such that *a priori* we have no reason to believe that the excitation is weak. This important question, we believe, cannot have a general answer, since the *particular* nature of a given heat bath may be important and should be answered on a case by case basis. However, we can argue in a fashion similar to Caldeira and Leggett[12] that a large part of the strong interaction can be incorporated as an adiabatic shift of the parameters defining the subsystem; the residual interaction between the subsystem and the heat bath could be weak.

It is important to note here that although our starting point is similar to that of Caldeira and Leggett, it is not identical. We emphasize here the concept of quasiparticles or elementary excitations in the sense of Landau, which can fall in different classes depending on their spectrum and quantum numbers, whereas Caldeira and Leggett have argued that it is sufficient to consider them as bosons for the present purpose. It is one of our intentions to find out if this difference in starting points leads to different conclusions.

It is most convenient to represent the two-state system in a pseudospin formalism; thus for the Hamiltonian we write (σ's are the usual Pauli matrices)

$$H_s = -\mu\sigma_z - \Delta\sigma_x \ . \tag{2.1}$$

For the moment the Hamiltonian of the environment, H_e, would be left arbitrary, and as mentioned earlier, represents a collection of independent elementary excitations or quasiparticles. The coupling between the pseudospin and the environment will be generally denoted by

$$H_I = \sigma_z H_c \ . \tag{2.2}$$

Here we have assumed it to be linear, which is true for a number of interesting cases (for a more general discussion, see Caldeira and Leggett[12]). The total Hamiltonian is then given by

$$H_t = H_s + H_e + H_I \ . \tag{2.3}$$

We shall chiefly consider two special cases.

(a) Bosonic bath:

$$H_t = -\mu\sigma_z - \Delta\sigma_x + \sum_\alpha \omega_\alpha b_\alpha^\dagger b_\alpha + \sigma_z \sum_\alpha f_\alpha (b_\alpha^\dagger + b_\alpha) \ , \tag{2.4}$$

where the spectral density $J(\omega)$ of the environment will be denoted by

$$J(\omega) = \sum_\alpha f_\alpha^2 \delta(\omega - \omega_\alpha) \ .$$

When $J(\omega) \sim \omega$ as $\omega \to 0$, and vanishes when ω is much larger than a microscopic cutoff ω_c, the dissipation will be referred to as Ohmic.[9]

(b) Fermionic bath:

$$H_t = -\mu\sigma_z - \Delta\sigma_x + \sum_{k,\eta} \epsilon_k c_{k\eta}^\dagger c_{k\eta} + \frac{J\sigma_z}{N} \sum_{k,k',\eta} c_{k\eta}^\dagger c_{k'\eta} \ . \tag{2.5}$$

where $b_\alpha^\dagger$ are the boson creation operators and $c_{k\eta}^\dagger$ are the fermion creation operators for wave vector k and the spin direction η. The significance of the remaining variables will become clearer later on. At this point we point out that we do not present the case of the dynamics of the phase variable in a Josephson junction since the discussion of this problem already exists in the literature.[13] However, this problem also fits very nicely into our formalism.

III. THE INCLUSIVE TRANSITION PROBABILITY

The question we ask is the following: First suppose that at time $t_i = 0$ the pseudospin is definitely known to be in a given state, say $\uparrow$, and the environment is in its thermal-equilibrium state. Then, what is the *probability* that at time t_f the spin is in the same state $\uparrow$, but with the environment in any other possible state? This probability, we emphasize, is an *inclusive* probability given by the "reaction"

$$(\text{spin})_\uparrow + (\text{env})_{\text{thermal}} \to (\text{spin})_\uparrow + (\text{env})_{\text{anything}} \ .$$

The principles of quantum mechanics tell us that we must sum over all possible final states of the environment. We thus need to calculate the probability $W_\uparrow$ given by

$$W_\uparrow = \sum_{n,m} \frac{e^{-\beta E_n}}{Z_{\text{env}}} \, | G_{m,n}^{(\uparrow)}(t_f, t_i) |^2 \ . \tag{3.1}$$

Here $1/k_B\beta$ is the temperature and E_n the energy levels of the environment, Z_{env} is the partition function. $G_{m,n}^{(\uparrow)}(t_f,t_i)$ is the probability amplitude for the transition in which at t_i the environment is in state n and the spin $\uparrow$, and at t_f the spin is $\uparrow$ but the environment is in any possible state m. A sum over the final states of the environment is denoted by m. In the next section we shall show how this inclusive probability can be expressed in terms of the dynamics of the environment plus the system along a contour in the complex-time plane.

IV. DYNAMICS ALONG A CONTOUR IN THE COMPLEX-TIME PLANE

The basic ideas in this section have already been expressed by Schwinger,[11] and Feynman and Vernon.[10] Our derivation, however, is novel in the sense that both the fermionic and the bosonic heat baths can be treated in a unified manner. The results in this section are both exact and general; in the next two sections we specialize to the two specific Hamiltonians given by Eqs. (2.4) and (2.5). Now we can write $G_{mn}^{(\uparrow)}(t_f,t_i)$ as

$$\begin{aligned} G_{mn}^{(\uparrow)}(t_f,t_i) &= \langle m\uparrow | e^{-iH(t_f-t_i)} | n\uparrow \rangle \\ &= \Big\langle m\uparrow \Big| e^{-iH_0(t_f-t_i)} T \\ &\qquad \times \exp\left[-i \int_{t_i}^{t_f} \hat{V}(t) dt \right] \Big| n\uparrow \Big\rangle \ , \end{aligned} \tag{4.1}$$

where

$$H_0 = -\mu\sigma_z + H_e + \sigma_z H_c \tag{4.2}$$

and $\hat{V}(t)$, the operator in the interaction representation, is given by

$$\hat{V}(t) = -\Delta e^{iH_0 t} \sigma_x e^{-iH_0 t} \ . \tag{4.3}$$

Note that we do not need to specify the nature of the environment yet; we therefore obtain

$$G_{mn}^{\uparrow}(t_f,t_i) = \sum_{n=0}^{\infty} (-i)^n \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_1} dt_2 \cdots \int_{t_i}^{t_{n-1}} dt_n \langle m\uparrow | e^{-iH_0(t_f-t_i)} \hat{V}(t_1)\hat{V}(t_2) \cdots V(t_n) | n\uparrow \rangle \ . \tag{4.4}$$

Now noting that $\hat{V}(t) = -(\Delta/2) e^{iH_0 t} (\sigma_+ + \sigma_-)^{-iH_0 t}$ one can trivially rewrite $W_\uparrow(t)$ as

$$W_\uparrow(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\Delta^{2i+2j}\int_0^t dt_1\cdots\int_0^{t_{2i-1}}dt_{2i}\int_0^t ds_1\cdots\int_0^{s_{2j-1}}ds_{2j}$$
$$\times\left[\sum_{m,n}\frac{e^{-\beta E_n}}{Z_{\rm env}}\langle n\,|\,(e^{iH_{0+}s_{2j}}e^{-iH_{0-}s_{2j}})\cdots(e^{iH_{0-}s_1}e^{-iH_{0+}s_1})e^{iH_{0+}t}\,|\,m\rangle\right.$$
$$\left.\times\langle m\,|\,e^{iH_{0+}t}(e^{iH_{0+}t_1}e^{-iH_{0-}t_1})\cdots(e^{iH_{0-}t_{2i}}e^{-iH_{0+}t_{2i}})\,|\,n\rangle\right], \quad (4.5)$$

where we have set $t_i=0$ and $t_f=t$. $H_{0\pm}$ are defined by

$$H_{0+}=-\mu+H_e+H_c \quad (4.6a)$$

and

$$H_{0-}=\mu+H_e-H_c\ . \quad (4.6b)$$

The sum over the intermediate states $|\,m\rangle$ in Eq. (4.5) gives unity and we obtain

$$W_\uparrow(t)=\left[\frac{1}{Z_{\rm env}}\right]\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\Delta^{2i+2j}\int_0^t dt_1\cdots\int_0^{t_{2i-1}}dt_{2i}\int_0^t ds_1\cdots\int_0^{s_{2j-1}}ds_{2j}\sum_n e^{-\beta E_n}\langle n\,|\cdots|\,n\rangle\ , \quad (4.7)$$

where the integrand is given by the expression

$$\sum_n e^{-\beta E_n}\langle n\,|\cdots|\,n\rangle={\rm Tr}[e^{-\beta H_e}(e^{iH_{0+}s_{2j}}e^{-iH_{0-}s_{2j}})\cdots(e^{iH_{0-}s_1}e^{-iH_{0+}s_1})(e^{iH_{0+}t_1}e^{-iH_{0-}t_1})\cdots(e^{-iH_{0-}t_{2i}}e^{-iH_{0+}t_{2i}})]\ . \quad (4.8)$$

Here, the exponential factors do not commute with each other. We now order the operators on the contour C_ϵ, as shown in Fig. 1(a), and define for s in segment I,

$$\zeta(s)=\zeta_+(s)=\sum_{m=1}^{2i}(-1)^{m+1}[\Theta(s-t_m)-\Theta(s-t_{m+1})]\ ; \quad (4.9)$$

for s in segment II,

$$\zeta(s)=\zeta_-(s)=\sum_{n=1}^{2j}(-1)^{n+1}[\Theta(s-s_n)-\Theta(s-s_{n+1})]\ ; \quad (4.10)$$

and $\zeta(s)=0$ for s in segment III. We can rewrite Eq. (2.8) as

$$\sum_n e^{-\beta E_n}\langle n\,|\cdots|\,n\rangle=\exp\left[i\mu\int_{C_\epsilon}ds\,\zeta(s)\right]{\rm Tr}\left[T_{C_\epsilon}\exp\left[-i\int_{C_\epsilon}ds[H_e(s)+\zeta(s)H_c(s)]\right]\right]\ . \quad (4.11)$$

For $W_\uparrow(t)$ we have

$$W_\uparrow(t)=\frac{1}{Z_{\rm env}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\Delta^{2i+2j}\int_0^t dt_1\cdots\int^{t_{2i-1}}dt_{2i}\int_0^t ds_1\cdots\int_0^{s_{2j-1}}ds_{2j}$$
$$\times\left[\exp\left[i\mu\int_{C_\epsilon}ds\,\zeta(s)\right]\right]{\rm Tr}\left[T_{C_\epsilon}\exp\left[-i\int_{C_\epsilon}ds[H_e(s)+\zeta(s)H_c(s)]\right]\right]\ . \quad (4.12)$$

The expression Tr($\cdots$) is precisely of the form discussed by Schwinger[11]—the environment described by the Hamiltonian H_e is perturbed by a time-dependent perturbation $\zeta(s)$ which is generally different along the forward and the backward directions of the time evolution; this is a manifestation of the presence of dissipation. If we define

$$Z_\lambda={\rm Tr}\left[T_{C_\epsilon}\exp\left[-i\int_{C_\epsilon}ds[H_e(s)+\lambda\zeta(s)H_c(s)]\right]\right]\ , \quad (4.13)$$

then the Tr term in Eq. (4.12) is just Z_1. It is simple to show that

$$Z_1=Z_{\rm env}\exp\left[-\int_0^1 d\lambda\int_{C_\epsilon}ds\,\zeta(s)\langle H_c(s)\rangle_\lambda\right]\ , \quad (4.14)$$

where

$$Z_\lambda\langle A(s)\rangle_\lambda=i\,{\rm Tr}\left[T_{C_\epsilon}\exp\left[-i\int_{C_\epsilon}ds'[H_e(s')+\lambda\zeta(s')H_c(s')]\right]A(s)\right]\ .$$

Equation (4.12) can be rewritten as

$$W_\uparrow(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\Delta^{2i+2j}\int_0^t dt_1\cdots\int_0^{t_{2i-1}}dt_{2i}\int_0^t ds_1\cdots\int_0^{s_{2j-1}}ds_{2j}\exp\left[i\mu\int_{C_\epsilon}dt'\zeta(t')\right]$$
$$\times\exp\left[-\int_0^1 d\lambda\int_{C_\epsilon}ds\,\zeta(s)\langle H_c(s)\rangle_\lambda\right]. \quad (4.15)$$

Up to now we have not made any assumptions about the heat bath; its effect is buried in $\langle H_c(s)\rangle_\lambda$. To calculate $W_\uparrow(t)$ we are going to use Eq. (2.4) and Eq. (2.5) for the boson and the fermion baths, respectively.

V. DEFINITIONS AND TERMINOLOGIES

We now introduce some terminology that will be used later. As shown in Fig. 2, the arrow indicates the positive direction along the contour in the complex s plane. The right (left) side of a contour is defined as the neighboring region which is to the right (left) of the contour when one goes along the counter in the positive direction. Let s and s' be two distinct points on the contour; s is said to be to the right (left) of s' if the path going from s' to s is in the positive (negative) direction of the contour, as in Fig. 2; s (s') is to the right (left) of s' (s). Since we are only interested in open contours, this definition should not cause any confusion. The following functions are also useful:

$$\Theta_c(s,s')=\begin{cases}1 & \text{if } s \text{ is to the right of } s'\\ 0 & \text{otherwise}\,,\end{cases}$$

and

$$\text{sgn}_c(s,s')=\Theta_c(s,s')-\Theta_c(s',s)\,.$$

Let $\Phi(z)$ be an analytic function on the complex s plane except for a cut along the contour C. We will denote $\Phi^{\pm}(s)$ for s on the contour as the analytic continuation of $\Phi(z)$ from the left (+) and from the right (−). We shall refer to the contour in Fig. 1(a) as C_ϵ. The symbol $\oint ds$ will be used exclusively to mean integration along segments I and II belonging to C_ϵ, and in the direction indicated.

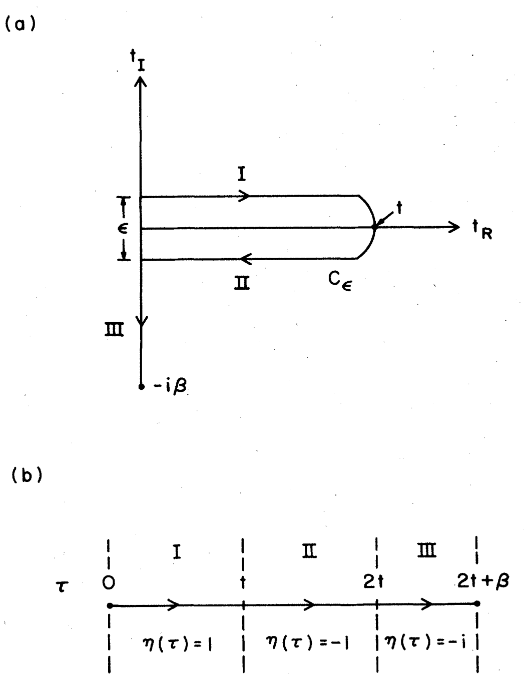

FIG. 1. (a) Complex contour C_ϵ, ϵ is infinitesimal. (b) Reparametrization of the contour in (a).

VI. BOSONIC HEAT BATH

From Eq. (2.4) we have

$$\langle H_c(s)\rangle_\lambda=\sum_\alpha f_\alpha[\langle b_\alpha^\dagger(s)\rangle_\lambda+\langle b_\alpha(s)\rangle_\lambda]\,. \quad (6.1)$$

In order to calculate this quantity conveniently, we shall introduce a function $\eta(\tau)$ where τ is a parametrization [shown in Fig. 1(b)] of the contour C_ϵ:

$$\eta(\tau)=\begin{cases}+1\,, & 0<\tau<t\\ -1\,, & t<\tau<2t\\ -i\,, & 2t<\tau<2t+\beta\,.\end{cases} \quad (6.2)$$

The relation between τ and s is as follows:

$$s=\begin{cases}\tau+i\epsilon\,, & 0<\tau<t\\ 2t-\tau-i\epsilon\,, & t<\tau<2t\\ -i(\tau-2t)\,, & 2t<\tau<2t+\beta\,.\end{cases} \quad (6.3)$$

From the definition of $\langle A\rangle_\lambda$ we then have

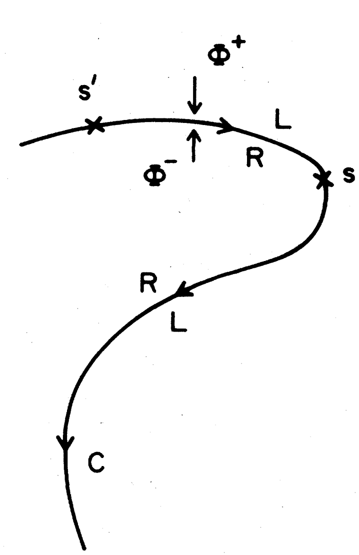

FIG. 2. Complex contour C.

$$\langle b_\alpha^\dagger(\tau)\rangle_\lambda=\frac{i}{Z_\lambda}\mathrm{Tr}\left[T_\tau\exp\left[-i\int_0^{2t+\beta}d\tau'[H_e(\tau')+\lambda\xi(\tau')H_c(\tau')]\eta(\tau')\right]b_\alpha^\dagger(\tau)\right], \tag{6.4}$$

$$\langle b_\alpha(\tau)\rangle_\lambda=\frac{i}{Z_\lambda}\mathrm{Tr}\left[T_\tau\exp\left[-i\int_0^{2t+\beta}d\tau'[H_e(\tau')+\lambda\xi(\tau')H_c(\tau')]\eta(\tau')\right]b_\alpha(\tau)\right], \tag{6.5}$$

where the definition of ξ is extended according to Eq. (6.3). From Eqs. (6.4) and (6.5) we can readily prove that

$$\langle b_\alpha^\dagger(0)\rangle_\lambda=\langle b_\alpha^\dagger(2t+\beta)\rangle_\lambda \tag{6.6}$$

and

$$\langle b_\alpha(0)\rangle_\lambda=\langle b_\alpha(2t+\beta)\rangle_\lambda . \tag{6.7}$$

Taking a derivative with respect to τ in Eqs. (6.4) and (6.5), we have

$$\frac{d\langle b_\alpha^\dagger(\tau)\rangle_\lambda}{d\tau}-i\eta(\tau)\omega_\alpha\langle b_\alpha^\dagger(\tau)\rangle_\lambda=-\lambda\eta(\tau)\xi(\tau)f_\alpha , \tag{6.8}$$

$$\frac{d\langle b_\alpha(\tau)\rangle_\lambda}{d\tau}+i\eta(\tau)\omega_\alpha\langle b_\alpha(\tau)\rangle_\lambda=\lambda\eta(\tau)\xi(\tau)f_\alpha . \tag{6.9}$$

Defining the Green's functions as

$$\frac{dG_\alpha^\pm(\tau,\tau')}{d\tau}\mp i\eta(\tau)\omega_\alpha G_\alpha^\pm(\tau,\tau')=\delta(\tau-\tau') , \tag{6.10}$$

we obtain

$$\langle b_\alpha^\dagger(\tau)\rangle_\lambda=-\lambda f_\alpha\int_0^{2t+\beta}G_\alpha^+(\tau,\tau')\eta(\tau')\xi(\tau')d\tau' , \tag{6.11}$$

$$\langle b_\alpha(\tau)\rangle_\lambda=\lambda f_\alpha\int_0^{2t+\beta}G_\alpha^-(\tau,\tau')\eta(\tau')\xi(\tau')d\tau' . \tag{6.12}$$

The periodic boundary condition has eliminated the inhomogeneous terms in these equations. $G_\alpha^\pm$ can be solved easily. We shall explicitly exhibit the solution for $G_\alpha^-(\tau,\tau')$, with $0<\tau'<\tau$, and give the solution for the general case. To solve Eq. (6.10) we integrate the corresponding homogeneous equation from $\tau=0$ with the initial value $G_\alpha^-(0,\tau')$ up to τ', add a unit jump at $G_\alpha^-(\tau'^-,\tau')$, use the result as a new initial condition, and integrate again the homogeneous equation up to $\tau=2t+\beta$. Then by applying the periodic boundary condition, $G_\alpha(0,\tau')=G_\alpha(2t+\beta,\tau')$, we can solve for $G_\alpha(0,\tau')$. Finally $G_\alpha(0,\tau')$ is set back into the formula which expresses $G_\alpha(\tau,\tau')$ in terms of $G_\alpha(0,\tau')$ to obtain the solution to Eq. (6.10). For $0<\tau'<t$ the procedure is carried out as follows:

(1) $0<\tau<t$, $\eta(\tau)=1$. For $0<\tau<\tau'$, we integrate the homogeneous equation to obtain

$$G_\alpha^-(\tau,\tau')=G_\alpha^-(0,\tau')e^{i\omega_\alpha\tau} , \tag{6.13}$$

where $G_\alpha^-(0,\tau')$ is the initial value of $G_\alpha^-(\tau,\tau')$.

(2) For $\tau'<\tau<t$ we have

$$G_\alpha^-(\tau,\tau')=[G_\alpha^-(0,\tau')e^{-i\omega_\alpha\tau'}+1]e^{-i\omega_\alpha(\tau-\tau')} . \tag{6.14}$$

The term in square brackets is obtained by adding 1 to $G_\alpha^-(\tau'^-,\tau')$.

(3) $t<\tau<2t$, $\eta(\tau)=-1$. From above we obtain

$$G_\alpha^-(t,\tau')=[G_\alpha(0,\tau')e^{-i\omega_\alpha\tau'}+1]e^{-i\omega_\alpha(t-\tau')} . \tag{6.15}$$

Using this as the new initial condition, we integrate over the present segment to obtain

$$G_\alpha^-(\tau,\tau')=G_\alpha^-(0,\tau')e^{i\omega_\alpha(\tau-2t)}+e^{i\omega_\alpha(\tau-2t)}e^{i\omega_\alpha\tau'} . \tag{6.16}$$

(4) $2t<\tau<2t+\beta$, $\eta(\tau)=-i$. Using the result from (3), in this segment we have

$$G_\alpha^-(2t,\tau')=G_\alpha^-(0,\tau')+e^{i\omega_\alpha\tau'} . \tag{6.17}$$

Using this as the initial condition we obtain

$$G_\alpha^-(\tau,\tau')=G_\alpha^-(0,\tau')e^{-\omega_\alpha(\tau-2t)}+e^{-\omega_\alpha(\tau-2t)}e^{i\omega_\alpha\tau'} . \tag{6.18}$$

Now by setting $\tau=2t+\beta$, we obtain the relation between $G_\alpha^-(2t+\beta,\tau')$ and $G_\alpha^-(0,\tau')$, given by

$$G_\alpha^-(2t+\beta,\tau')=[G_\alpha^-(0,\tau')+e^{i\omega_\alpha\tau'}]e^{-\omega_\alpha\beta} . \tag{6.19}$$

Together with the periodic boundary conditions we can solve for $G_\alpha^-(0,\tau')$ to obtain

$$G_\alpha^-(0,\tau')=\frac{e^{i\omega_\alpha\tau'}}{e^{\beta\omega_\alpha}-1} . \tag{6.20}$$

Now that $G_\alpha^-(0,\tau')$ is known, we can write the complete solution for $0<\tau'<t$ as follows:

$$G_\alpha^-(\tau,\tau^-)=\begin{cases} n_\alpha e^{-i\omega_\alpha(\tau-\tau')} , & 0<\tau<\tau' \\ (n_\alpha+1)e^{-i\omega_\alpha(\tau-\tau')} , & \tau'<\tau<t \\ (n_\alpha+1)e^{+i\omega_\alpha(\tau-2t)+i\omega_\alpha\tau'} , & t<\tau<2t \\ (n_\alpha+1)e^{-\omega_\alpha(\tau-2t)+i\omega_\alpha\tau'} , & 2t<\tau<2t+\beta \end{cases} \tag{6.21}$$

where $n_\alpha=1/(e^{\beta\omega_\alpha}-1)$. Using Eq. (6.3) we can now return to the original variables s,s':

$$G_\alpha^\pm(s,s')=\left[\frac{1}{e^{\mp\beta\omega_\alpha}-1}+\Theta_c(s,s')\right]e^{\pm i\omega_\alpha(s-s')} . \tag{6.22}$$

Finally, from Eqs. (6.11), (6.12), (6.22), and (4.14), we obtain

$$Z_1=Z_{\mathrm{env}}\exp\left[-\sum_\alpha f_\alpha^2\left[\int_0^t ds\int_0^s ds'\xi_+(s)\xi_+(s')e^{-i\omega_\alpha(s-s')}+\int_0^t ds\int_0^s ds'\xi_-(s)\xi_-(s')e^{i\omega_\alpha(s-s')}\right.\right.$$
$$-\int_0^t ds\int_0^t ds'\xi_-(s)\xi_+(s')e^{-i\omega_\alpha(s-s')}$$
$$\left.\left.+\frac{1}{e^{\beta\omega_\alpha}-1}\int_0^t ds\int_0^t ds'[\xi_+(s)-\xi_-(s)][\xi_+(s')-\xi_-(s')]e^{-i\omega_\alpha(s-s')}\right]\right] . \tag{6.23}$$

The exponential term in Eq. (6.23) is nothing but the influence functional obtained previously.[10,11] Thus for the bosonic environment we have

$$W_{\uparrow}(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\Delta^{2i+2j}\int_0^t dt_1\cdots\int_0^{t_{2i-1}}dt_{2i}\int_0^t ds_1\cdots\int_0^{s_{2j-1}}ds_{2j}\exp\left[i\mu\int_0^t ds[\zeta_+(s)-\zeta_-(s)]\right]\left[\frac{Z_1}{Z_{\text{env}}}\right], \quad (6.24)$$

where Z_1/Z_{env} is given by Eq. (6.23). Equation (6.24) can now be rewritten in the form of Eq. (1) in Chakravarty and Leggett.[5] This is done by relabeling the path by the time sequence $t_1,t_2,\ldots,t_{2n}$ such that there is a "blip" in the internal (t_{2j-1},t_{2j}) and a sojourn in (t_{2j},t_{2j+1}).

VII. FERMIONIC BATH

The calculation of $\langle H_c\rangle_\lambda$ for the fermionic bath is lengthier than the corresponding part for the bosonic bath. Sections VII A—VII D will be devoted only to the problem of the fermionic bath. We first introduce the Green's function, then set up the Dyson's equation along the contour in Fig. 1(a) (Sec. VII A), solve it (Secs. VII B and VII C), and finally evaluate Z_1 (Sec. VII D).

The quantity of interest is $\langle H_c\rangle_\lambda$; from Eq. (2.5) we have

$$\langle H_c(\tau)\rangle_\lambda=\frac{J}{N}\sum_{\substack{k,k'\\ \sigma}}\langle c^\dagger_{k'\sigma}(\tau)c_{k\sigma}(\tau)\rangle_\lambda\,. \quad (7.1)$$

Notice that the spin-up electrons do not mix with the spin-down electrons; we can separate $\langle\ \rangle_\lambda$ into a spin-up part and a spin-down part. Hence,

$$H_e=H_{e,\uparrow}+H_{e,\downarrow}\,, \quad (7.2)$$

$$H_c=H_{c,\uparrow}+H_{c,\downarrow}\,, \quad (7.3)$$

$$Z_\lambda=Z_{\lambda,\uparrow}Z_{\lambda,\downarrow}\,, \quad (7.4)$$

and

$$\langle c^\dagger_{k'\sigma}(\tau)c_{k\sigma}(\tau)\rangle_\lambda=\frac{i}{Z_{\lambda,\sigma}}\mathrm{Tr}_\sigma\left[T\exp\left[-i\int d\tau'[H_{e,\sigma}(\tau')+\lambda\zeta(\tau')H_{c,\sigma}(\tau')]\eta(\tau')\right]c^\dagger_{k'\sigma}(\tau)c_{k\sigma}(\tau)\right], \quad (7.5)$$

where Tr_σ means all states corresponding to σ are summed over. The definitions of $H_{e,\sigma}$, $H_{c,\sigma}$, and $Z_{\lambda,\sigma}$ are obvious. For convenience we shall omit the spin index; the *parameter* λ *will also be omitted.* We now define the Green's function $G_{kk'}(\tau,\tau')$ as follows:

$$G_{kk'}(\tau,\tau')=-i\Theta(\tau-\tau')\mathrm{Tr}\left[T\exp\left[-i\int d\tau''[H_e(\tau'')+\zeta(\tau'')H_c(\tau'')]\eta(\tau'')\right]c_k(\tau)c^\dagger_{k'}(\tau')\right]$$
$$+i\Theta(\tau'-\tau)\mathrm{Tr}\left[T\exp\left[-i\int d\tau''[H_e(\tau'')+\zeta(\tau'')H_c(\tau'')]\eta(\tau'')c^\dagger_{k'}(\tau')c_k(\tau)\right]\right], \quad (7.6)$$

where we have used the same parametrization τ as in Eq. (6.3) and the function η defined in Eq. (6.2). We also define the local Green's function $G(\tau,\tau')$ as

$$G(\tau,\tau')=\sum_{k,k'}G_{kk'}(\tau,\tau')\,. \quad (7.7)$$

Let $G^0_{kk'}$ be the free Green's function, i.e., $H_c=0$, and also let G^0 be the free local Green's function. The label τ in $\zeta(\tau)$, $H(\tau)$, etc. is simply a parameter to define the order in which the operators act. Equation (7.1) can be written in terms of G as

$$\langle H_c(\tau)\rangle=\frac{J}{N}G(\tau,\tau^+)\,. \quad (7.8)$$

Therefore we shall set up the Dyson's equation for $G(\tau,\tau')$ and solve it to obtain the answer we are pursuing. Let us now derive $G^0(\tau,\tau')$ which will be needed to solve the Dyson's equation. $G^0_{kk'}(\tau,\tau')$ can be obtained by directly evaluating Eq. (7.6) with $\zeta=0$. We find that

$$G^0_{kk'}(\tau,\tau')=-i\delta_{kk'}[\Theta(\tau-\tau')(1-f_k)e^{-i\epsilon_k\phi(\tau,\tau')}$$
$$-\Theta(\tau'-\tau)f_ke^{-i\epsilon_k\phi(\tau,\tau')}]\,, \quad (7.9)$$

where $\phi(\tau,\tau')=\int_\tau^{\tau'}d\tau''\eta(\tau'')$, and

$$f_k=\frac{1}{e^{\beta\epsilon_k}+1}\,. \quad (7.10)$$

If we express $G^0_{kk'}(\tau,\tau')$ in terms of the variables s,s' [refer to Eq. (6.3)] along the contour C_ϵ, we find

$$G^0_{kk'}(s,s')=-i\delta_{kk'}[\Theta_c(s,s')(1-f_k)e^{-i\epsilon_k(s-s')}$$
$$-\Theta_c(s',s)f_ke^{-i\epsilon_k(s-s')}]\,. \quad (7.11)$$

One should not forget that s,s' are now defined on the contour C_ϵ, which includes sections from both the real and the imaginary axes. From Eqs. (7.7) and (7.11) we obtain $G^0(s,s')$:

$$G^0(s,s')=-i\int_{-\infty}^{\infty}d\epsilon\rho(\epsilon)e^{-i\epsilon(s-s')}\times\left[\frac{-1}{e^{\beta\epsilon}+1}+\Theta_c(s,s')\right], \quad (7.12)$$

$\rho(\epsilon)$ is the density of states. We are going to assume $\rho(\epsilon)$ to be a constant throughout the conduction band of bandwidth $2/\delta$, e.g., $\rho(\epsilon)=\rho_0e^{-\delta|\epsilon|}$, $\delta>0$. We obtain, in the limit $\delta/\beta\ll1$ and $\beta-\mathrm{Im}(s-s')\gg\delta$, G^0 as

$$G^0(s,s')=-\rho_0\left[\frac{\pi}{\beta}\right]\mathrm{csch}\left[\frac{\pi}{\beta}[s-s'-i\delta\,\mathrm{sgn}_c(s,s')]\right]. \quad (7.13)$$

A. Dyson's equation

The Hamiltonian corresponding to the Green's function $G_{kk'}(\tau,\tau')$ is

$$h(\tau)=H_e+\zeta(\tau)H_c \ . \tag{7.14}$$

In the operator form, Dyson's equation reads

$$\hat{G}=\hat{G}^0+\hat{G}^0(\zeta\hat{H}_c)\hat{G} \ .$$

In the functional representation we have

$$G_{kk'}(s,s')=G^0_{kk'}(s,s') + \sum_{k_1,k_2} \int_{C_\epsilon} du\, du' G^0_{kk_1}(t,u)(\zeta H_c)_{k_1k_2}(u,u') \times G_{k_2k'}(u',s') \ . \tag{7.15}$$

The integration path is the contour C_ϵ. If we sum over k,k' on both sides of Eq. (7.14) we obtain

$$G(s,s')=G^0(s,s')+\frac{J}{N}\int_{C_\epsilon} du\, G^0(s,u)\zeta(u)G(u,s') \ . \tag{7.16}$$

The boundary condition is fixed by G^0.

B. Long-time approximation

In principle we can use Eq. (7.13) for G^0 to solve the Dyson's equation. However, we do not know how to do that exactly. Nonetheless, if we approximate G by its "long-time" behavior, i.e., take G^0 to be given by

$$G^0(s-s')=-\rho_0\left[\frac{\pi}{\beta}\right]\frac{P}{\sinh[(\pi/\beta)(s-s')]} \ , \tag{7.17}$$

we can solve the Dyson's equation to obtain the correct answer for the transient part of $G(s,s')$. This is the method proposed by Nozières and De Dominicis.[14] The instantaneous response (the adiabatic part of G) to ζ is not well captured by the approximation; we shall deal with this problem later. We now examine in more detail the way the cutoff should be handled, and what we mean by the long-time approximation. We first examine the long-time approximation. Let us go back to Eq. (7.13) and consider how $G^0(s-s')$ behaves as s' sweeps from $s'=s-\epsilon$ (which we will designate as s^-) to $s'=s+\epsilon$ (which we will denote as s^+). Let $G^0_R=\mathrm{Re}(G^0)$, $G^0_I=\mathrm{Im}(G^0)$.

Consider both s and s' on the same branch, I, II, or III of C_ϵ; take segment I for example. When s' approaches s from the left, G^0_R increases up to a large number and then decreases back to zero till $s'=s$; after that s' goes to the right of s, G^0_R then becomes negative, and its absolute value becomes large, it finally approaches zero. Figure 3(a) shows schematically the behavior of G^0_R and Fig. 3(b) shows the behavior of G^0_I.

When s is in segment I and s' in segment II, G_R retains the same form as shown in Fig. 3(a). Since $\mathrm{sgn}_c(s,s')$ is always negative in this case, i.e., s' is to the right of s although its value ranges from s^+ to s^-, G_{I} changes to a more symmetric form shown in Fig. 3(b). In Fig. 1 the distance ϵ between segments I and II is an infinitesimal

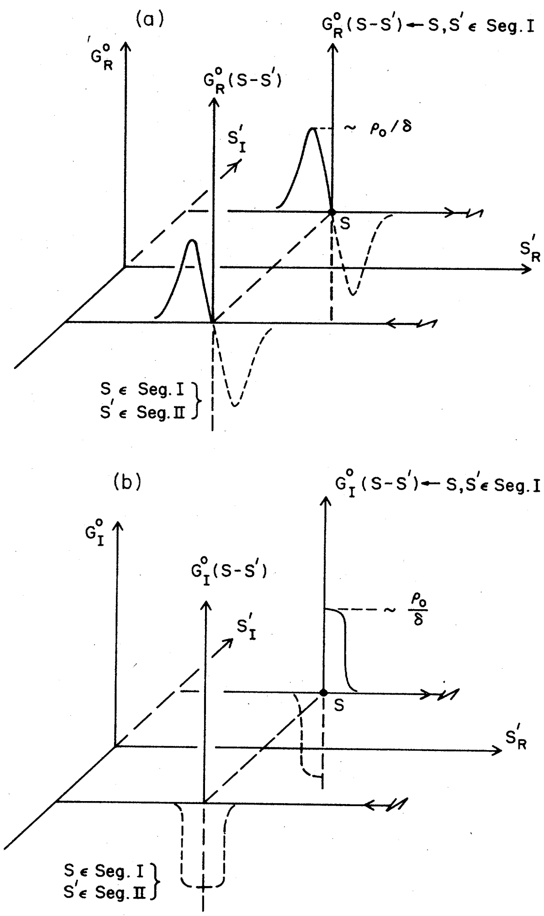

FIG. 3. (a) Schematic plot of $G^0_R(s-s')$. (b) Schematic plot of $G^0_I(s-s')$.

quantity. If Eq. (7.17) is evaluated on the contour C_ϵ with arbitrarily small ϵ, i.e., s,s' are in C_ϵ, it is clear that G^0_R is well simulated by it. But for G^0_I, it is also clear that Eq. (7.17) cannot describe the characteristic shape of $G^0(s-s')$ when s is in segment I, s' is in segment II, and vice versa. Fortunately there is an easy way out of this difficulty. All we have to do is to evaluate Eq. (7.17) on the contour c_δ, δ designating the cutoff. For example, if s is in segment I of C_ϵ, s' is in segment II of $C_\epsilon(\epsilon \ll \delta)$, the points corresponding to s and s' on C_δ are $s+i\delta/2$ and $s'-i\delta/2$. Equation (7.17) then gives

$$-\left[\frac{\pi\rho_0}{\beta}\right]\frac{P}{\sinh\{(\pi/\beta)[(s+i\delta/2)-(s'-i\delta/2)]\}} = -\left[\frac{\pi\rho_0}{\beta}\right]\frac{P}{\sinh[(\pi/\beta)(s-s'+i\delta)]} \ .$$

In fact the formula above is exactly the same as Eq. (7.13). Other cases are also easy to check. Although Eq. (7.17) does not always give the exact result as the case above, it is always a good approximation for the long-time behavior of Eq. (7.13). Now the meaning of the phrase "long time" becomes clear: It simply means that $\Delta\tau$ between two points is $\gg\delta$. One can consider Eq. (7.17) as the result of (a) a change of variable, $C_\epsilon \to C_\delta$, and (b) the long-time approximation.

From this discussion it is natural to find $G(s,s^+)$, s,s' are in C_ϵ as follows.

(i) Start from the Dyson's Eq. (7.16) and the Eq. (7.13); the variables in these equations are on C_ϵ.

(ii) Change the variables from C_ϵ to C_δ, and make the long-time approximation on G^0; i.e., replace Eq. (7.13) by Eq. (7.17).

(iii) Solve the singular integral equation on C_δ and find $G(s,s^+)$.

(iv) Change the variables from C_δ back to C_ϵ. Replace the short-time cutoff which was ignored in the long-time approximation.

This completes the discussion of how to handle the cutoff and the meaning of long-time approximation. We have seen that due to the particular contour that we have in our problem, the meaning of the long-time approximation introduced by Nozières and De Dominicis[14] has to be reinterpreted.

C. The equal-time limit of $G(s,s')$

From Eqs. (7.16) and (7.17) we obtain

$$G(s,s')=G^0(s,s')-\frac{\rho_0 J}{N}P \times \int_{C_\delta} du\left[\frac{\pi}{\beta}\right]\text{csch}\left[\frac{\pi}{\beta}(s-u)\right] \times \zeta(u)G(u,t') . \tag{7.18}$$

Introducing the following functions,

$$B(s)=i\xi(s) , \tag{7.19}$$

$$\xi(s)=\frac{\pi\rho_0 J}{N}\zeta(s) , \tag{7.20}$$

$$\psi(s)=G(s,s') , \tag{7.21}$$

$$g(s)=G^0(s-s') , \tag{7.22}$$

Eq. (7.18) can be written as

$$\psi(s)-\frac{1}{i\pi}P\int_{C_\delta}\left[\frac{\pi}{\beta}\right]\text{csch}\left[\frac{\pi}{\beta}(s-u)\right] \times B(u)\psi(u)du=g(s) . \tag{7.23}$$

This equation is in Muskhelishvili's standard form. The method of solving this equation is the same as the Appendix of Hamann.[15] However, one may observe the following differences: (1) The kernel here is csch instead of csc; (2) the contour here is C_δ instead of a straight-line segment; (3) the function $B(t)$ is discontinuous here rather than continuous. Observations (1) and (2) do not make any difference in solving the equation. Observation (3) may cause some problems, since the Cauchy integral is logarithmically divergent at those points where the discontinuities of $B(s)$ occur. We are going to assume that $B(s)$ is continuous in solving the equation, and then let $B(s)$ tend to a step function at the end of the calculation; a direct solution for discontinuous $B(s)$ is also available.[16] It is now straightforward to obtain $G(t,t^+)$. The arithmetic is the same as Hamann's Appendix,[15] and we shall not repeat it here. The answer is (G_{al} and G_{tr} signify the adiabatic and the transient parts G as explained in Sec. VII B)

$$G(s,s^+)=G_{\text{ad}}(s,s^+)+G_{\text{tr}}(s,s^+) , \tag{7.24}$$

$$G_{\text{ad}}(s,s')=\frac{-\rho_0}{1+\xi^2(s)}P\left[\frac{\pi}{\beta}\text{csch}\left[\frac{\pi}{\beta}(s-s')\right]\right] - \frac{\xi(s)}{1+\xi(s)^2}[\pi\rho_0\delta(s-s')] , \tag{7.25}$$

$$G_{\text{tr}}(s,s^+)=\frac{\pi\rho_0}{\xi(s)}\left[-\frac{1}{4\pi^2}\right]\left[\frac{1}{X^+(s)}-\frac{1}{X^-(s)}\right] \times\int_{C_\delta}du\left[\frac{\pi}{\beta}\right]^2\frac{P}{\sinh^2\left[\frac{\pi}{\beta}(u-s)\right]} \times[X^+(u)-X^-(u)] , \tag{7.26}$$

where

$$X(z)=\exp\left\{\frac{1}{2\pi i}\int_{C_\delta}\left[\frac{\pi}{\beta}\coth\left[\frac{\pi}{\beta}(u-z)\right]\right] \times \ln\left[\frac{1-B}{1+B}\right]du\right\} . \tag{7.27}$$

$X^\pm(t)$ are defined in Sec. V. They are

$$X^\pm(s)=e^{\pm i\Theta(s)}e^{+\Gamma(s)} , \tag{7.28}$$

$$\Theta(s)=-\tan^{-1}\xi(s) , \tag{7.29}$$

$$\Gamma(s)=\frac{P}{\pi}\int_{C_\delta}\left[\frac{\pi}{\beta}\right]\coth\left[\frac{\pi}{\beta}(u-s)\right]\Theta(u)du . \tag{7.30}$$

Clearly $G_{\text{ad}}(s,s')$ does not have a finite limit as $s'\to s^+$. As discussed in Sec. VIII B the adiabatic part is expected to go wrong due to the long-time approximation. Figure 3 shows that the original G^0 has a cutoff of order $1/\delta$. If we replace

$$-\rho_0\left[\frac{\pi}{\beta}\text{csch}\left[\frac{\pi}{\beta}(s-s')\right]\right]$$

by G^0 of Eq. (7.7), and $\delta(s-s')$ by i/δ, we have

$$G_{\text{ad}}(s,s^+)=\frac{iN}{2}\left[\frac{1}{1+\xi^2(s)}-\frac{\pi\xi(s)}{1+\xi^2(s)}\right]+O\left[\frac{\delta^2}{\beta^2}\right]\cdots . \tag{7.31}$$

$N=2\rho_0/\delta$ equals number of electron states in the band. This indicates that $G_{\text{ad}}(s,s^+)$ is, up to the order $(\delta/\beta)^2$, temperature independent. We can do better than Eq. (7.31) by solving $G_{\text{ad}}(s,s')$ at zero temperature exactly and then take $s'\to s^+$ limit. To find the adiabatic part of G we should first find the Green's function for the constant potential problem. For a constant potential Eq. (7.6) reduces to the usual definition of the Green's function. Dyson's equation for real time s,s' is

$$G(s-s')=G^0(s-s')+V\int_{-\infty}^{\infty}du\,G^0(s-u)G(u-s')\ , \tag{7.32}$$

where $V=J/N$ or $-J/N$. Taking the Fourier transform of both sides, we have

$$G(\omega)=\frac{1}{[G^0(\omega)]^{-1}-V}\ . \tag{7.33}$$

Hence,

$$G(s)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega s}\frac{1}{G^0(\omega)^{-1}-V}\ . \tag{7.34}$$

At zero temperature $G^0(\omega)$ is easily found to be

$$G^0(\omega)=\int_{-\infty}^{\infty}d\epsilon\frac{\rho(\epsilon)}{\omega-\epsilon+i\eta\,\mathrm{sgn}\epsilon}=\frac{N}{\omega+iD\,\mathrm{sgn}\omega}\ , \tag{7.35}$$

where for convenience we have used

$$\rho(\epsilon)=\frac{N}{\pi}\frac{D}{\epsilon^2+D^2}$$

to obtain the result. We shall see that $G(0^-)$ depends only on ρ_0, V_0, and N. When D is large the Lorentzian cutoff should be as good as the exponential cutoff. We substitute Eq. (7.35) into Eq. (7.34) and carry out the integration to obtain for $s<0$ [si(z) and ci(z) below are the usual sine and cosine integrals[17]],

$$G(s)=\frac{N}{2\pi}[f(\gamma)-f(-\gamma^*)]+\frac{Ni}{2\pi}[g(\gamma)-g(-\gamma^*)]\ , \tag{7.36}$$

where

$$\gamma=-Ds(\pi\rho_0V+i)\ , \tag{7.37}$$

$$\rho_0=\rho(\epsilon=0)=\frac{N}{\pi D}\ , \tag{7.38}$$

$$f(z)=\sin z\,\mathrm{si}(z)+\cos z\,\mathrm{ci}(z)\ , \tag{7.39}$$

$$g(z)=\sin z\,\mathrm{ci}(z)-\cos z\,\mathrm{si}(z)\ . \tag{7.40}$$

Taking the $s=0^-$ limit we obtain

$$G(0^-)=\frac{Ni}{2}\left[1-\frac{2}{\pi}\tan^{-1}(\pi\rho_0V)\right]\ . \tag{7.41}$$

Thus $G_{\mathrm{ad}}(s,s^+)$ is given by

$$G_{\mathrm{ad}}(s,s^+)=\frac{Ni}{2}\left[1-\frac{2}{\pi}\tan^{-1}\xi(s)\right]\ . \tag{7.42}$$

We now replace Eq. (7.31) by Eq. (7.42). From Eq. (7.36) we can also get the long-time behavior of G_{ad}. Since $f(z)\sim 1/z$ and $g(z)\sim 1/z^2$ for large $|z|$, we have for large $s\gg 1/D$,

$$G(s)\sim\frac{-\rho_0}{1+\xi^2}\frac{1}{s}\ ;$$

this is exactly the same as G_{ad} in Eq. (7.25) in the zero-temperature limit.

One may also notice that G_{tr} in Eq. (7.26) is not well defined in the form it is written, since $P/\sinh^2[(\pi/\beta)(u-s)]$ is really not a well-defined object. Replacing the cutoff as described in Sec. VII B, the divergence can be eliminated. As we show below, this is not yet necessary if we only need to evaluate Z_1.

D. Evaluation of Z_1/Z_{env}

From Eqs. (4.14) and (7.8) we have to evaluate the following integral:

$$I\equiv\int_0^1 d\lambda\int_{C_\epsilon}\frac{J}{N}\zeta(s)G(s,s^+)=I_{\mathrm{ad}}+I_{\mathrm{tr}}\ , \tag{7.43}$$

where I_{ad} is the contribution from G_{ad} and I_{tr} is the contribution from G_{tr}. Z_1/Z_{env} is simply e^{-2I}. The factor of 2 comes from combining the contribution from spin-up and spin-down states. I_{ad} is easy to find:

$$I_{\mathrm{ad}}=\frac{iJ}{2}\oint\zeta(s)ds\ . \tag{7.44}$$

The $\tan^{-1}\xi(s)$ term in Eq. (7.42) does not contribute. Since $\zeta(s)=0$ along the imaginary axis the integral $\int_{C_\epsilon}ds$ reduces to $\oint ds$; also since $\zeta(s)\tan^{-1}[\xi(s)\lambda]$ is independent of s, the integral $\oint ds\,\zeta(s)\tan^{-1}[\xi(s)\lambda]$ is equal to zero. Equation (7.44) gives an adiabatic energy shift of the two-level system. To calculate I_{tr} we have to do the integral

$$I'_{\mathrm{tr}}=\int_{C_\delta}ds\int_{C_\delta}ds'\left[\frac{1}{X^+(s)}-\frac{1}{X^-(s)}\right]\left[\frac{\pi}{\beta}\right]^2\times\frac{P}{\sinh^2[(\pi/\beta)(s'-s)]}[X^+(s')-X^-(s')]\ . \tag{7.45}$$

The relation between I_{tr} and I'_{tr} is given by

$$I_{\mathrm{tr}}=\int_0^1\left[\frac{-1}{4\pi^2\lambda}\right]I'_{\mathrm{tr}}\,d\lambda\ . \tag{7.46}$$

We have changed $\int_{C_\epsilon}ds$ in Eq. (7.43) to $\int_{C_\delta}ds$. This is perfectly alright since $G(s,s')$ is defined along C_δ. We still think of $\Theta(s)$ as a continuous function on C_δ; hence $X^\pm(s)$ are also continuous. We perform an integration by part with respect to the variable s' and obtain

$$I'_{\mathrm{tr}}=\int_{C_\delta}ds'\frac{d}{ds'}[X^+(s')-X^-(s')]\times\int_{C_\delta}ds\left[\frac{1}{X^+(s)}-\frac{1}{X^-(s)}\right]P\frac{\pi}{\beta}\times\coth\left[\frac{\pi(s'-s)}{\beta}\right]\ . \tag{7.47}$$

The integral $\int_{C_\delta}ds$ can be found using the Sokhotski formula.[16] For

$$\Phi(z)=\frac{1}{2\pi i}\int_L\frac{\phi(s)}{s-z}ds$$

and $\phi(s)$ Hölder continuous along the smooth contour L, Sokhotski's formulas states that

$$\Phi^+(s)-\Phi^-(s)=\phi(s) \tag{7.48}$$

and

$$\Phi^+(s)+\Phi^-(s)=\frac{P}{\pi i}\int_L \frac{\phi(s)}{s-t}ds \, . \tag{7.49}$$

Therefore, in our case we have

$$\frac{1}{X^+(s)}+\frac{1}{X^-(s)}=\frac{P}{\pi i}\int_{C_\delta} ds \frac{\pi}{\beta}\coth\left[\frac{\pi}{\beta}(s-s')\right] \times\left[\frac{1}{X^+(s)}-\frac{1}{X^-(s)}\right] . \tag{7.50}$$

Thus,

$$I'_{\rm tr}=-\pi i\int_{C_\delta} ds\left[\frac{1}{X^+(s)}+\frac{1}{X^-(s)}\right] \times\frac{d}{ds}[X^+(s)-X^-(s)]$$
$$=4\pi\int_{C_\delta} ds\left[\cos\theta\frac{d\sin\theta}{ds}+\cos\theta\sin\theta\frac{d\Gamma(s)}{ds}\right] . \tag{7.51}$$

The first term does not contribute due to the periodic boundary condition on θ. Remember that we have embedded the parameter λ in ξ throughout the calculation. Extracting λ explicitly, we have, from Eq. (7.30),

$$I'_{\rm tr}=4\int_{C_\delta} ds\int_{C_\delta} ds' P\left[\frac{\pi}{\beta}\right]^2 \operatorname{csch}^2\left[\frac{\pi}{\beta}(s-s')\right]\frac{\lambda\xi(s)\tan^{-1}[\lambda\xi(s')]}{1+\lambda^2\xi^2(s)} . \tag{7.52}$$

From (7.46) we have

$$I_{\rm tr}=-\frac{1}{\pi^2}\int_{C_\delta} ds\int_{C_\delta} ds' P\left[\frac{\pi}{\beta}\right]^2 \operatorname{csch}^2\left[\frac{\pi}{\beta}(s-s')\right]\xi(s)\xi(s')\int_0^1 d\lambda\frac{\xi_0\tan^{-1}(\lambda\xi_0)}{1+\lambda^2\xi_0^2}$$
$$=-\frac{1}{8}\left[\frac{2}{\pi}\tan^{-1}\xi_0\right]^2\int_{C_\delta} ds\int_{C_\delta} ds' P\left[\frac{\pi}{\beta}\right]^2 \operatorname{csch}^2\left[\frac{\pi}{\beta}(s-s')\right]\xi(s)\xi(s') , \tag{7.53}$$

where $\xi_0=\pi\rho_0 J/N$. From Eqs. (7.43), (7.44), and (7.53) we finally have

$$2I=iJ\oint\xi(s)ds-\frac{1}{4}\left[\frac{2}{\pi}\tan^{-1}\xi_0\right]^2\oint ds\oint ds'\left[\frac{\pi}{\beta}\right]^2\frac{\xi(s)\xi(s')}{\sinh^2\{(\pi/\beta)[s-s'-i\delta\,{\rm sgn}_c(s-s')]\}} , \tag{7.54}$$

where we have reintroduced the cutoff $i\delta$; $\oint ds\oint ds'$ is now the double integral along the path arbitrarily close to the real axis. In principle we could stop at Eq. (7.54), but in order to compare the present result with the result for the bosonic bath we shall try to rewrite Eq. (7.54) in a form similar to the exponent of Eq. (6.23). We introduce $J(\omega)=\omega e^{-\delta\omega}$ and obtain the result [to order $(\delta/\beta)^2$]:

$$2I=iJ\oint\xi(s)ds+\frac{1}{2}\left[\frac{2}{\pi}\tan^{-1}\xi_0\right]^2\int_0^\infty d\omega J(\omega)\oint ds\oint ds'\xi(s)\xi(s')$$
$$\times\left[\tfrac{1}{2}e^{-i\omega(s-s')}[1+\Theta_c(s,s')-\Theta_c(s',s)]+\frac{e^{-i\omega(s-s')}}{e^{\beta\omega}-1}\right] . \tag{7.55}$$

Aside from the adiabatic shift, Eq. (7.55) is the same as Eq. (6.23) when we identify

$$\frac{1}{2}\left[\frac{2}{\pi}\tan^{-1}\xi_0\right]^2\int_0^\infty d\omega J(\omega)$$

with $\sum_\alpha f_\alpha^2$. Note that since we need $J(\omega)=\omega e^{-\delta\omega}$ in order to match the fermionic result to the bosonic result, the corresponding bosonic bath is necessarily Ohmic [cf. the statements after Eq. (2.4)]. Therefore the electrons behave collectively as if they were bosons. However, for the dimensionless dissipation coefficient α, as defined in Eq. (3) of Chakravarty and Leggett,[5] we have

$$\alpha=\left[\frac{2}{\pi}\tan^{-1}\xi_0\right]^2 . \tag{7.56}$$

VIII. CONCLUSION

In this paper we addressed the problem of the dynamics of a two-state system coupled to the dissipative degrees of freedom of the environment (heat bath) and have shown that the inclusive transition probability of a two-state system can be expressed in a simple formula Eq. (4.15) independent (to a large extent) of the nature of the heat bath. We then explicitly compared two different heat baths: fermionic and bosonic. Our conclusion is that the case of Ohmic dissipation of the bosonic bath is remarkably similar to the fermionic case provided one remembers that a crucial dimensionless parameter α (see Refs. 5) appearing in the bosonic case is to be identified with $[(2/\pi)\tan^{-1}\xi_0]^2$ [Eq. (7.56)] of the fermionic case. Since $(2/\pi)\tan^{-1}\xi_0$ is never greater than unity, no matter how large $\xi_0(=\pi\rho_0 J/N)$ is, the dimensionless coupling constant never exceeds unity in the fermionic case. This conclusion is similar to that arrived at by Yu and Anderson[4] for a *similar* Hamiltonian; an important difference between their work and our's is that we have explicitly considered the real-time finite-temperature dynamics, whereas they have considered the thermodynamics of such a two-state system. In the language of macroscopic quantum coherence (Refs. 5), it is clear that for the fermionic bath considered here the system could never be in the broken-symmetry regime at zero temperature.

The formulation of the paper, as mentioned earlier, extends the ideas of Schwinger[11] and Feynman and Vernon[10] by presenting a unifying method to treat both bosonic and fermionic environments. The fermionic case also required a careful interpretation of the long-time approximation of Nozières and De Dominicis[14] as well as the extension of the well-known singular-integral-equation techniques applied to similar problems[14,15,18] to a contour on the complex plane. Although the mathematics of certain singular integral equations on a complex contour is well known,[16] we believe that here we have made effective use of such mathematics.

Although we started with the idea that even if any one degree of freedom is weakly excited it does not necessarily follow that the environment can be considered as a collection of harmonic oscillators, the curious feature, however, is that for the problem studied here, the bosonic heat bath with Ohmic dissipation behaves in a way very similar to the fermionic bath. However, an important parameter, as discussed above, has a different range.

In the future we hope to discuss the role of tunneling states in the metallic glasses using the present formalism.

ACKNOWLEDGMENTS

We sincerely thank Dr. Clare Yu for sending us results prior to publication. One of us (S.C.) would like to thank the Alfred P. Sloan Foundation for support. This work was also supported by a grant from the National Science Foundation (Grant No. DMR-83-01519).

[1]For a recent discussion, see J. P. Sethna, Phys. Rev. B **24**, 698 (1981); *ibid.* **24**. 5050 (1981).

[2]For a review, see J. Black, in *Metallic Glasses,* edited by H. J. Güntherodt (Springer, New York, 1981), p. 167.

[3]K. L. Ngai and C. T. White, Phys. Rev. B **20**, 2475 (1979).

[4]C. C. Yu and P. W. Anderson, Phys. Rev. B **29**, 6165 (1984).

[5]S. Chakravarty, Phys. Rev. Lett. **49**, 681 (1982); S. Chakravarty and S. Kivelson, *ibid.* **50**, 1811 (1983); **51**, 1109(E) (1983); S. Chakravarty and A. J. Leggett, *ibid.* **52**, 5 (1984); A. J. Bray and M. A. Moore, *ibid.* **49**, 1546 (1982); W. Zwerger, Z. Phys. B **54**, 87 (1983); *ibid.* **53**, 53 (1983).

[6]W. H. Zurek, in Proceedings of the International Symposium on the Foundations of Quantum Mechanics (Tokyo, 1983) (unpublished).

[7]R. Harris and R. Silbey, J. Chem. Phys. Rev. **78**, 7330 (1982), and references cited therein; also unpublished.

[8]The discussion of the Kondo problem relevant to our work is given in P. W. Anderson, G. Yuval, and D. R. Hamann, Phys. Rev. B **1**, 4464 (1970), and also in V. J. Emery and A. J. Luther, Phys. Rev. B **9**, 215 (1974).

[9]See Refs. 5, particularly, Chakravarty and Leggett.

[10]R. P. Feynman and F. L. Vernon, Jr., Ann. Phys. (N.Y.) **24**, 118 (1963).

[11]J. Schwinger, J. Math. Phys. **2**, 407 (1961).

[12]A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983).

[13]V. Ambegaokar, in *The Proceedings of the NATO-ASI Summer School Lectures on Percolation, Localization and Superconductivity* (Plenum, New York, 1984).

[14]P. Nozières and C. De Dominicis, Phys. Rev. **178**, 1097 (1969).

[15]D. R. Hamann, Phys. Rev. B **2**, 1373 (1970).

[16]N. I. Muskhelishvili, *Singular Integral Equations,* translated by J. R. M. Radok (Nordhoff, Leyden, The Netherlands, 1977).

[17]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).

[18]G. Yuval and P. W. Anderson, Phys. Rev. B **1**, 1522 (1970).