Off-axis correlation functions in the isotropic $d=2$ Ising model

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We present exact explicit expressions for the spin-spin correlation functions $\langle \sigma_{0,0}\sigma_{m,n} \rangle$ for $(m,n)=(2,1)$, (3,1), (3,2), (4,1), (4,2), and (4,3) in the isotropic $d=2$ Ising model. We also infer a general structural formula for arbitrary $\langle \sigma_{0,0}\sigma_{m,n} \rangle$ in terms of complete elliptic integrals K and E.

The two-dimensional Ising model remains of great importance as one of the very few interacting many-body systems which is exactly soluble.[1-6] The spin-spin correlation functions $S_{m,n}=\langle \sigma_{0,0}\sigma_{m,n} \rangle$ are of particular interest, because they express in a precise way the effect of the interactions on the dynamical variables and because, in principle, at least, they can be calculated exactly. A general method for computing these correlation functions in terms of Pfaffians was developed by Montroll, Potts, and Ward[4] for the square lattice, following previous work by Kaufmann and Onsager.[2] This method was extended to a lattice with an additional diagonal coupling by Stephenson.[5] However, rather surprisingly, the only explicit calculations of specific $S_{m,n}$ which were performed using this or any other method were for the nearest-neighbor row (or, equivalently, column) and diagonal correlation functions $S_{1,0}$ and $S_{1,1}$. Accordingly, we recently calculated $S_{n,n}$ in the general, anisotropic Ising model and $S_{n,0}$ in the isotropic model, for n up to 6.[7,8] We also discovered general structural formulas for diagonal and row and column correlation functions.[7,8] These results show a number of interesting features, such as a striking hierarchical structure involving levels composed of homogeneous polynomials in the complete elliptic integrals K and E.

The present paper presents exact explicit expressions, calculated via the Pfaffian method, for the off-axis correlation functions $S_{m,n}$ with $(m,n)=(2,1)$, (3,1), (3,2), (4,1), (4,2), and (4,3) in the isotropic $d=2$ Ising model. In combination with our previous results for diagonal and row correlation functions, these yield a complete, explicit determination of $S_{m,n}$ for $\max(|m|,|n|)\leq 4$ and, in addition, of $S_{n,n}$ for $|n|\leq 6$ and of $S_{n,0}$ for $|n|\leq 6$. We also infer a general structural formula for arbitrary $S_{m,n}$ in this model.

The Hamiltonian for the isotropic $d=2$ Ising model is

$$H=-\sum_{(i,j)\in \mathbf{Z}^2}(J_1\sigma_{i,j}\sigma_{i+1,j}+J_2\sigma_{i,j}\sigma_{i,j+1}) , \quad (1)$$

with $\sigma_{i,j}=\pm 1\in \mathbf{Z}_2$ and $J_1=J_2\equiv J$. Define the elliptic moduli

$$k_>(\beta)=\sinh^2(2\beta J) , \quad (2a)$$

applicable for $T>T_c$, and

$$k_<=k_>^{-1} , \quad (2b)$$

applicable for $T<T_c$, with $\beta=(k_BT)^{-1}$ and T_c defined by

$$[k_>(\beta_c)]^2=[k_<(\beta_c)]^2=1 . \quad (3)$$

As indicated in the notation, because of the homogeneity of the lattice, $\langle \sigma_{i,j}\sigma_{i+m,j+n} \rangle=\langle \sigma_{0,0}\sigma_{m,n} \rangle$; furthermore, $\langle \sigma_{0,0}\sigma_{-m,-n} \rangle=\langle \sigma_{0,0}\sigma_{m,n} \rangle$, so that, with no loss of generality, m and n may be taken to be positive. In addition, the isotropy property $J_1=J_2$ implies that $S_{m,n}=S_{n,m}$, so that one may take $m\geq n$ with no loss of generality. If one uses the square-lattice Pfaffian method of Ref. 4, the calculation of $S_{m,n}$ requires the evaluation of the Pfaffian of a $2(m+n)\times 2(m+n)$ (antisymmetric) matrix. A useful simplification is obtained if one first calculates the correlation function for a more complicated Hamiltonian with the diagonal interaction term $J_d\sigma_{i,j}\sigma_{i+1,j+1}$ added to the row and column spin-spin interaction terms in (1), and then lets $J_d=0$.[9] The reason for this is that with the more complicated Hamiltonian one can take advantage of diagonal steps to achieve a shorter route linking the two spins at the points (0,0) and (m,n). With the conventions $m,n\geq 0$, $m\geq n$ chosen above, a minimal route is to travel from (0,0) by $\min(m,n)$ steps up along the diagonal and then to go out $m-n$ steps in a horizontal direction to (m,n). This path contains $\max(m,n)$ steps in all, with the result that the corresponding Pfaffian involves a $2\max(m,n)\times 2\max(m,n)$ antisymmetric matrix. Thus with this method the path length and resultant size of the Pfaffian are less than or equal to the values which they would have if one used the square-lattice method, and are strictly less if $\min(m,n)\equiv n\neq 0$. This reduction in the size of the Pfaffian is important for both theoretical analysis and actual calculations, since the number of terms in a $2r\times 2r$ Pfaffian, $(2r-1)!!$, rapidly becomes unmanageably large as r increases.

We find that $S_{m,n}$ has a structure which depends on whether $m-n$ is even or odd. Thus the generic form of the correlation functions is similar along diagonals parallel to the line $m=n$. From our explicit results and some inductive analysis of the Pfaffians involved, we infer the following general structural formula for the $S_{m,n}$ [where

$S_{m,n,\pm} \equiv S_{m,n}$ (for $T > T_c$ and $T < T_c$, respectively)]. For $m-n$ even,

$$S_{m,n,\pm} = D_{m,n} k^{-p_{m,n,\pm}} \sum_{l=l_L}^{l_U} \left[\frac{1+(-1)^{l-l_L}}{2} \right] \pi^{-l} \sum_{r=0}^{l} \mathscr{P}_{l-r,r}^{(m,n,\pm)}(k)(k-1)^r [E(k)]^{l-r} [K(k)]^r , \tag{4}$$

and, for $m-n$ odd,

$$\left. \begin{matrix} S_{m,n,+} \\ S_{m,n,-} \end{matrix} \right\} = [\text{sgn}(J)]^{m+n} D_{m,n} k^{-p_{m,n,\pm}} \times \left\{ \begin{matrix} (1+k_>^{-1})^{1/2} \\ (l+k_<)^{1/2} \end{matrix} \right\}$$
$$\times \sum_{l=l_L}^{l_U} \pi^{-l} \times \left\{ \begin{matrix} 1 \\ (-1)^l \end{matrix} \right\} \times \sum_{r=0}^{l} \mathscr{P}_{l-r,r}^{(m,n)}(k)(k-1)^{j_{m,n,l,r}} [E(k)]^{l-r} [K(k)]^r , \tag{5}$$

where the upper and lower values of l are, respectively,

$$l_U = \max(m,n) , \tag{6a}$$

and

$$l_L = \min(m,n) , \tag{6b}$$

and

$$j_{m,n,l,r} = r + \{[(m-n)l] \text{mod}(2)\} \delta_{r,0} . \tag{7}$$

Further, $k = k_>$ ($k = k_<$) for $T > T_c$ ($T < T_c$); $D_{m,n}$ is an inverse integer denominator extracted for convenience;. $p_{m,n,\pm}$ is a positive semidefinite integer; and $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds, respectively. As indicated in the notation, for $m-n$ even, $\mathscr{P}_{l-r,r}^{(m,n,+)}(k_>)$ and $\mathscr{P}_{l-r,r}^{(m,n,-)}(k_<)$ are distinct polynomials in their respective variables, whereas for $m-n$ odd, $\mathscr{P}_{l-r,r}^{(m,n)}(k)$ is the same function of $k = k_>$ and $k = k_<$. Several features of these general formulas (4) and (5) with (6) and (7) are easy to see from the properties of the relevant Pfaffians. For example, the fact that the maximum degree of the homogeneous polynomials in $K(k)$ and $E(k)$ is $\max(m,n)$ follows, since this is precisely the length of the minimal path linking the two spins in the optimal method discussed above. If one used the original method[4] involving only horizontal and vertical steps across the lattice, one would naively and erroneously expect that the maximum degree of the homogeneous polynomials in $K(k)$ and $E(k)$ in $S_{m,n}$ would be $(m+n)$ rather than $\max(m,n)$, and in explicit calculations, this result would appear only after massive (and, within the context of the method, apparently miraculous) cancellations of terms from the $2(m+n) \times 2(m+n)$ Pfaffians.

We proceed to list our explicit results for the correlation functions. The $D_{m,n}$ are $D_{2,1}=1$, $D_{3,1}=3^{-1}$, $D_{3,2}=D_{4,1}=3^{-2}$, $D_{4,2}=(3^3 \times 5)^{-1}$, $D_{4,3}=(3^4 \times 5^2)^{-1}$; and the powers $p_{m,n,\pm}$ are $p_{2,1,\pm}=1$, $p_{3,2,\pm}=3$, $p_{4,1,\pm}=4$, $p_{4,2,\pm}=5$, $p_{4,3,\pm}=6$, $p_{3,1,-}=p_{3,1,+}-1=2$.

The polynomials $\mathscr{P}_{l-r,r}^{(m,n)}(k)$, $m=2,3,4$, for $m-n$ odd, are listed below:

$$\mathscr{P}_{2,0}^{(2,1)}(k)=2^2, \quad \mathscr{P}_{1,1}^{(2,1)}(k)=2(k+3), \quad \mathscr{P}_{0,2}^{(2,1)}(k)=2(k+1), \quad \mathscr{P}_{1,0}^{(2,1)}(k)=-1, \quad \mathscr{P}_{0,1}^{(2,1)}(k)=k+1 .$$

$$\mathscr{P}_{3,0}^{(3,2)}(k)=-2^3[3(k^2+6k+1)], \quad \mathscr{P}_{2,1}^{(3,2)}(k)=2^3(-k^4-15k^3+k^2+39k+8) ,$$
$$\mathscr{P}_{1,2}^{(3,2)}(k)=2^3(k+1)(-3k^3+k^2+27k+7), \quad \mathscr{P}_{0,3}^{(3,2)}(k)=2^4(k+1)^2(3k+1) ,$$
$$\mathscr{P}_{2,0}^{(3,2)}(k)=-2^2(k^2+1)(k^2-6k+1), \quad \mathscr{P}_{1,1}^{(3,2)}(k)=2^2(k+1)(3k^3-2k^2+9k-2) ,$$
$$\mathscr{P}_{0,2}^{(3,2)}(k)=2^2(k+1)^2(3k-1) .$$

$$\mathscr{P}_{4,0}^{(4,1)}(k)=-2^4(k^2-18k+1)(5k^2+6k+5) ,$$
$$\mathscr{P}_{3,1}^{(4,1)}(k)=2^4(-k^5+81k^4+294k^3+434k^2+235k-19) ,$$
$$\mathscr{P}_{2,2}^{(4,1)}(k)=2^4(k+1)(29k^4+148k^3+358k^2+260k-27) ,$$
$$\mathscr{P}_{1,3}^{(4,1)}(k)=2^4(k+1)^2(3k^4+24k^3+118k^2+128k-17) ,$$
$$\mathscr{P}_{0,4}^{(4,1)}(k)=2^6(k+1)^3(3k^2+6k-1), \quad \mathscr{P}_{3,0}^{(4,1)}(k)=2^3(k^4+48k^3+110k^2+48k+1) ,$$
$$\mathscr{P}_{2,1}^{(4,1)}(k)=2^3(k+1)(26k^4+59k^3-127k^2-115k-3) ,$$
$$\mathscr{P}_{1,2}^{(4,1)}(k)=2^3(k+1)^3(3k^3+9k^2-89k-3) ,$$
$$\mathscr{P}_{0,3}^{(4,1)}(k)=-2^3(k+1)^3(15k^2+24k+1), \quad \mathscr{P}_{2,0}^{(4,1)}=-2^2(3k)(k+1)^2(k^2+6k+1) ,$$

$$\mathscr{P}_{1,1}^{(4,1)}(k)=-2^2(3k)(k+1)^2(k^3+3k^2+11k+1),\quad \mathscr{P}_{0,2}^{(4,1)}(k)=-2^4(3k^2)(k+1)^3\ ,$$
$$\mathscr{P}_{1,0}^{(4,1)}(k)=-2(3k)(k+1)^2(k^2+6k+1),\quad \mathscr{P}_{0,1}^{(4,1)}(k)=2(3k)(k+1)^3(5k+1)\ .$$

$$\mathscr{P}_{(4,0)}^{(4,3)}(k)=-2^6(k^4+15k^3-16k^2+15k+1)(11k^4-86k^2+11)\ ,$$
$$\mathscr{P}_{(3,1)}^{(4,3)}(k)=-2^6(3k^9+114k^8-67k^7-1939k^6+267k^5+825k^4-3609k^3-321k^2+590k+41)\ ,$$
$$\mathscr{P}_{(2,2)}^{(4,3)}(k)=-2^6[3(k+1)](6k^8-6k^7-315k^6+16k^5+295k^4-1150k^3-133k^2+244k+19)\ ,$$
$$\mathscr{P}_{(1,3)}^{(4,3)}(k)=2^6(k+1)^2(153k^6+18k^5-405k^4+1468k^3+223k^2-398k-35)\ ,$$
$$\mathscr{P}_{(0,4)}^{(4,3)}(k)=-2^9(k+1)^3(3k^2-1)(3k^2-10k-1)\ ,$$
$$\mathscr{P}_{(3,0)}^{(4,3)}(k)=2^5[3(k^4+14k^2+1)](k^4-15k^3-16k^2-15k+1)\ ,$$
$$\mathscr{P}_{(2,1)}^{(4,3)}(k)=-2^5[3^2(k+1)](2k^8-3k^7+64k^6-16k^5-13k^4-159k^3+10k^2-14k+1)\ ,$$
$$\mathscr{P}_{(1,2)}^{(4,3)}(k)=-2^5[3^2(k+1)^2](13k^6-8k^5-21k^4-108k^3+7k^2-12k+1)\ ,$$
$$\mathscr{P}_{(0,3)}^{(4,3)}(k)=2^5[3(k+1)^3](7k^2+1)(3k^2+10k-1)\ .$$

The polynomials $\mathscr{P}_{l-r,l}^{(m,n,+)}(k_>)$ (for $T>T_c$) and $\mathscr{P}_{l-r,l}^{(m,n,-1)}(k_<)$ (for $T<T_c$), for $m-n$ even and $m=3,4$, are listed below:

$$\mathscr{P}_{3,0}^{(3,1,+)}(k_>)=2^3(11k_>^2+6k_>-1),\quad \mathscr{P}_{2,1}^{(3,1,+)}(k_>)=2^3(k_>+1)(7k_>^2+12k_>-3)\ ,$$
$$\mathscr{P}_{1,2}^{(3,1,+)}(k_>)=2^3(k_>+1)(k_>+3)(k_>^2+2k_>-1),\quad \mathscr{P}_{0,3}^{(3,1,+)}(k_>)=2^3(k_>-1)(k_>+1)^2\ ,$$
$$\mathscr{P}_{1,0}^{(3,1,+)}(k_>)=-2k_>(k_>+1)(k_>^2+3k_>-2),\quad \mathscr{P}_{0,1}^{(3,1,+)}(k_>)=2^2k_>(k_>+1)^2\ .$$

$$\mathscr{P}_{3,0}^{(3,1,-)}(k_<)=2^3(-k_<^2+6k_<+11),\quad \mathscr{P}_{2,1}^{(3,1,-)}(k_<)=2^4(k_<+1)(3k_<+13)\ ,$$
$$\mathscr{P}_{1,2}^{(3,1,-)}(k_<)=2^4(k_<+1)(k_<^2+9k_<+10),\quad \mathscr{P}_{0,3}^{(3,1,-)}(k_<)=2^3(k_<+1)^2(3k_<+5)\ ,$$
$$\mathscr{P}_{1,0}^{(3,1,-)}(k_<)=2(k_<+1)(2k_<^2-3k_<-1),\quad \mathscr{P}_{0,1}^{(3,1,-)}(k_<)=-2(k_<+1)^2(3k_<+1)\ .$$

$$\mathscr{P}_{(4,0)}^{(4,2,+)}(k_>)=2^5(2k_>^6-57k_>^5-699k_>^4-258k_>^3+324k_>^2+123k_>+53)\ ,$$
$$\mathscr{P}_{(3,1)}^{(4,2,+)}(k_>)=-2^6[3(k_>+1)](7k_>^4+140k_>^3+242k_>^2+92k_>+31)\ ,$$
$$\mathscr{P}_{(2,2)}^{(4,2,+)}(k_>)=-2^5[3(k_>+1)](3k_>^6+102k_>^5+225k_>^4-116k_>^3-435k_>^2-210k_>-81)\ ,$$
$$\mathscr{P}_{(1,3)}^{(4,2,+)}(k_>)=-2^7(k_>+1)^2(9k_>^5+27k_>^4-36k_>^3-144k_>^2-77k_>-35)\ ,$$
$$\mathscr{P}_{(0,4)}^{(4,2,+)}(k_>)=2^6[3(k_>+1)^3](3k_>^3+15k_>^2+9k_>+5)\ ,$$
$$\mathscr{P}_{(2,0)}^{(4,2,+)}(k_>)=-2^3[3(k_>+1)](3k_>^6-22k_>^5-47k_>^4+22k_>^3-13k_>^2+8k_>+1)\ ,$$
$$\mathscr{P}_{(1,1)}^{(4,2,+)}(k_>)=2^4[3(k_>+1)^2](6k_>^4+15k_>^3+k_>^2+9k_>+1)\ ,$$
$$\mathscr{P}_{(0,2)}^{(4,2,+)}(k_>)=-2^3[3(k_>+1)^3](6k_>^3-3k_>^2+8k_>+1)\ .$$

$$\mathscr{P}_{(4,0)}^{(4,2,-)}(k_<)=2^5(53k_<^6+123k_<^5+324k_<^4-258k_<^3-699k_<^2-57k_<+2)\ ,$$
$$\mathscr{P}_{(3,1)}^{(4,2,-)}(k_<)=2^6(k_<+1)(13k_<^6+63k_<^5+198k_<^4-210k_<^3-999k_<^2-93k_<+4)\ ,$$
$$\mathscr{P}_{(2,2)}^{(4,2,-)}(k_<)=2^5[3(k_<+1)](k_<^7+10k_<^6+63k_<^5-40k_<^4-729k_<^3-774k_<^2-71k_<+4)\ ,$$
$$\mathscr{P}_{(1,3)}^{(4,2-)}(k_<)=2^8(k_<+1)^2(3k_<^5-6k_<^4-105k_<^3-135k_<^2-14k_<+1)\ ,$$
$$\mathscr{P}_{(0,4)}^{(4,2,-)}(k_<)=-2^6(k_<+1)^3(57k_<^3+93k_<^2+11k_<-1)\ ,$$
$$\mathscr{P}_{(2,0)}^{(4,2,-)}(k_<)=-2^3[3k_<(k_<+1)](k_<^6+8k_<^5-13k_<^4+22k_<^3-47k_<^2-22k_<+3)\ ,$$
$$\mathscr{P}_{(1,1)}^{(4,2,-)}(k_<)=2^4[3k_<(k_<+1)^2](5k_<^4-8k_<^3+38k_<^2+16k_<-3)\ ,$$
$$\mathscr{P}_{(0,2)}^{(4,2,-)}(k_<)=2^3[3k_<(k_<+1)^3](29k_<^2+10k_<-3)\ .$$

The critical behavior of these correlation functions is interesting to note. First, as is clear from the general formulas (4)—(7), no terms involving $K(k)$ contribute as $T \to T_c$, or equivalently, as $k \to 1$ (where k is understood to denote $k_>$ for $T > T_c$ and $k_<$ for $T < T_c$). This is necessary, since $K(k)$ diverges (logarithmically) as $k \to 1$, and, since, because of the different powers of π^{-l}, no cancellations of these divergences could occur among different terms involving $K(k)$. In contrast, since $E(k=1)=1$, the finiteness of the correlations function by itself would allow any terms of the form $[E(k)]^l$ to remain as $k \to 1$. However, as can be seen from (4)—(7), only a certain subset of these terms actually contributes. For $m-n$ even, the alternate l levels in the hierarchy of the form $l = l_U - 1, l_U - 3, \ldots, l_L + 1$ are completely absent. For $m-n$ odd and $\max(m,n)$ odd, the pure $[E(k)]^{l_U}$, $[E(k)]^{l_U-2}, \ldots, [E(k)]^{l_L+1}$ terms are annihilated as $k \to 1$ by $(k-1)$ factors. Finally, for $m-n$ odd and $\max(m,n)$ even, the pure $[E(k)]^{l_U-1}$, $[E(k)]^{l_U-3}, \ldots, [E(k)]^{l_L}$ terms are annihilated by $(k-1)$ factors. Thus formulas (4)—(7) imply that the general structure of the correlation functions at $T = T_c$ is

$$S_{m,n}(T=T_c) = \begin{Bmatrix} \pi^{-l_L} \\ \pi^{-2[(l_L+1)/2]} \end{Bmatrix} \sum_{r=0}^{[(m-n)/2]} c_{m,n,r}\pi^{-2r} \quad \text{for } m-n \begin{cases} \text{even} \\ \text{odd} , \end{cases} \tag{8}$$

where the $c_{m,n,r}$ are calculable constants, and $[\nu]$ denotes the integral part of ν. All of the cases of $m-n$ and $\max(m,n)$ noted above are illustrated by the correlation functions which have been given here.

In conclusion, this paper presents exact, explicit expressions for the off-axis correlation functions $S_{m,n}$ with $(m,n)=(2,1)$, (3,1), (3,2), (4,1), (4,2), and (4,3) in the isotropic $d=2$ Ising model. It is valuable to know these functions since, together with the diagonal and row column correlation functions, they constitute a precise description of the short- to intermediate-range spin-spin interactions in the model. Furthermore, through these studies, we have discovered the general structural formula (4)—(7) for $S_{m,n}$ with m and n arbitrary.

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