Sliding charge-density waves as a dynamic critical phenomenon

Daniel S. Fisher
AT&T Bell Laboratories, Murray Hill, New Jersey 07974
(Received 20 April 1984)

The dynamic properties of sliding charge-density waves are discussed in terms of a classical description of impurity pinning, with emphasis on the behavior near threshold considered as a dynamic critical phenomenon. A mean-field model introduced previously [Phys. Rev. Lett. **50**, 1486 (1983)] is analyzed in detail, including ac response above and below threshold and hysteretic behavior below threshold. For short-range interactions, the weak-pinning limit is discussed and scaling behavior is predicted near threshold. The earlier prediction of ac noise with intensity inversely proportional to the square root of the volume with a diverging amplitude near threshold caused by a diverging correlation length is analyzed in terms of the scaling behavior; this interpretation of the noise is semiquantitatively confirmed by recent experiments of Mozurkewich and Grüner [Phys. Rev. Lett. **51**, 2206 (1983)]. Many of the ideas presented here may be applicable to other systems, especially weakly pinned flux flow in type-II superconductors.

I. INTRODUCTION

There are currently about half a dozen experimental systems[1,2] [the best known being $NbSe_3$ (Ref. 1)] which exhibit a charge-density wave at low temperatures[3] and concomitant nonlinear conductivity characterized by non-Ohmic behavior above a small threshold electric field.[1,4,5] It is commonly believed that in these materials the non-Ohmic conduction is caused by sliding of the charge-density wave which is prevented from moving below the threshold field by pinning to impurities and other lattice defects. Yet despite considerable theoretical activity, a quantitative understanding of the phenomena is lacking, especially near threshold. Some of the notable features which are common to most of the systems are the following.

(1) A charge-density wave (CDW) which is low-order commensurate along two of the directions and incommensurate or high-order (typically fourth-order) commensurate along the third direction;[3] the CDW is thus effectively a one-dimensional modulation of a low-order commensurate structure. All of the nonlinear effects (below) occur only for electric fields in the third direction.

(2) Ohmic conductivity at low fields E.[2,4,5]

(3) A relatively sharp threshold field E_T, typically on the order of 10—100 mV/cm (which is very small on scales of microscopic electric fields).[2,4,5]

(4) Nonlinear conductivity for $E>E_T$ with the excess current growing near threshold between quadratically and linearly in $E-E_T$, and linearly in E far above threshold.[5]

(5) Coherent ac noise[4] in many *small* samples for $E>E_T$ with (a) the principal frequency linear in the excess current[6] and (b) an amplitude which grows rapidly relative to the dc excess current as $E \to E_T^+$.[4,6]

(6) Hysteretic behavior on time scales much longer than characteristic microscopic times.[7–9]

(7) Interference effects between ac and dc.[6]

While there is no *direct* evidence that the charge-density waves move, strong indirect support for the sliding charge-density-wave picture is given by (a) in (5) above. Naively, if the CDW coherently moves one wavelength at a time in a periodic manner, it will produce an ac current with a frequency $\nu_N = v/\lambda$, where v is the average CDW velocity and λ its wavelength. The effective charge density n, deduced from the ratio of the noise frequency to the excess current by assuming that the CDW carries a current density $j_{CDW} = nev$, is quite reasonable.[6,10]

A drastically oversimplified "model" of the CDW consisting of a single overdamped particle (representing some effective collective coordinate of the CDW) moving in a sinusoidal potential with spatial period λ (representing the effective potential due to the impurities) and a superimposed linear ramp with slope proportional to the applied electric field has been extensively used to attempt fits to the data.[11] While this single-particle model gives a threshold field, ac current in response to a dc field with frequency proportional to the dc current, and some of the ac effects qualitatively similar to those in the experiments,[11] it suffers from obvious theoretical drawbacks and in fact fails badly quantitatively and sometimes qualitatively. In particular, (i) the single-particle model (or other similar models) yields a square-root dependence of the excess current on $E-E_T$, in striking contrast to the concave upwards experimental data; (ii) the ratio of the rms ac current at the fundamental frequency to the dc current does not depend critically on $E-E_T$ (except that it decreases well above threshold); (iii) there is only one time scale in the model (inversely proportional to v, the dc velocity), thus hysteretic effects on long time scales cannot exist.

A realistic microscopic model which probably contains the features necessary to explain most of the experimental data was proposed some time ago by Fukuyama and Lee:[12] It consists of an incommensurate single-Q CDW with wave vector $Q = 2\pi/\lambda$ and a slowly varying phase $\phi(r)$, interacting weakly with impurities at random positions $\mathbf{R}_j$ in the underlying lattice. The (slightly modified) classical Hamiltonian is

$$\mathscr{H}=\int d\mathbf{r}\left[\frac{K}{2}(\nabla\phi)^2-V_I\sum_j\delta(\mathbf{r}-\mathbf{R}_j)\cos[\mathbf{Q}\cdot\mathbf{r}+\phi(\mathbf{r})]\right], \tag{1.1}$$

where K is the phase stiffness of the CDW and V_I the impurity pinning strength. Since the CDW is assumed to be incommensurate, the preferred phase at impurity j, $\mathbf{Q}\cdot\mathbf{R}_j$, is pseudorandomly between 0 and 2π. In the strongly damped limit, the inertia of the CDW can be neglected and the equation of motion for the phase becomes

$$\frac{d\phi(\mathbf{r})}{dt}=\Gamma\left[-\frac{\delta\mathscr{H}}{\delta\phi(\mathbf{r})}+\frac{neE}{Q}\right], \tag{1.2}$$

where n is the effective charge density carried by the CDW and Γ is a dissipative coefficient parametrizing the drag due to electron-phonon interactions, etc.

The important features of this model, in addition to an infinite number of degrees of freedom, are the following: a short-range CDW stiffness, random preferred phases (mod2π) at impurity positions, a dissipative equation of motion with linear coupling to the applied electric field, and neglect of thermal fluctuations. Lee and Rice[13] argued, through what is essentially a scaling argument (see Sec. VII) that for arbitrarily weak pinning this model has a threshold field in three dimensions below which the CDW distorts but does not move and above which it slides—they did not, however, make any predictions about the dynamic behavior.

Recently, a perturbative treatment of the dynamics far above threshold (for a model similar to that of Fukuyama and Lee[12]) was shown to yield quantitative agreement with several aspects of the experiments, in particular, the high-field I-V curve.[14,15] However, for reasons discussed in Sec. III, the perturbation expansion breaks down near threshold and is thus not useful for investigating the behavior in that regime.

The remainder of the theoretical work on sliding charge-density waves falls roughly into four categories.

(i) Calculations with the single-particle model or attempted rationalizations[16] of it from the Fukuyama-Lee model, yielding the drawbacks[17] mentioned above.

(ii) Treating the CDW as a rigid array of discommensurations to try to explain the noise;[18–20] along with other drawbacks this leaves out the internal degrees of freedom of the CDW.

(iii) Microscopic or semimicroscopic quantum-mechanical calculations which are unlikely to yield the macroscopic effects of the *collective* motion of the CDW.[21–23]

(iv) Numerical simulations on one-dimensional systems[24] which (except for recent work on incommensurate, rather than randomly pinned systems which is discussed in a separate paper[25]) have been rather inconclusive both as far as the behavior near threshold and the existence (or absence) of coherent ac noise.

The numerical calculations are, however, the only nonperturbative treatments of the problem which involve the essential feature of a large number of nonlinearly interacting degrees of freedom.

In this paper we will primarily be interested in the behavior relatively near threshold, in particular, the dc I-V curves, ac noise, ac response, and hysteresis. We will assume throughout most of this paper that all of the physics necessary to describe long-wavelength, low-frequency behavior of the system is contained in the Fukuyama-Lee model[12] and in particular that (A) the CDW stiffness is short range, (B) the local preferred phases at the impurities are uncorrelated, (C) the phase of the CDW is a single valued function of position, i.e., there are no dislocations in the CDW, (D) thermal fluctuations can be ignored so that the only effects of temperature are to modify the parameters of the model, and (E) the inertia of the CDW is negligible.

In this paper we will introduce a discrete, somewhat simplified version of the Fukuyama-Lee[12] model in which the stiffness of the CDW is represented by interactions between phases of the CDW at different impurity sites and the electric field by a force F which tries to continuously increase the phase at each site. These phases each represent a region of the CDW. The current carried by the CDW is just proportional to its velocity v, which is the time derivative of the spatially averaged phase $\bar{\phi}(t)$.

It will be argued that the threshold behavior is a critical phenomenon and thus has all the difficulties and features of conventional critical phenomena, in particular, the breakdown of perturbation theory and the existence of scaling laws and critical exponents near threshold. (There are, of course, additional complications due to the dynamical aspects of the problem.) A mean-field treatment, which is valid in the limit of infinite-range stiffness of the CDW, is carried out and its consequences explored.[26] Some conclusions about systems with short-range interactions are derived or speculated.

The paper is organized as follows. The main results are summarized in the remainder of this section. The simplified model is introduced in Sec. II and the single-particle approximation[11] and its failure are discussed. In Sec. III perturbation theory in the disorder[14,15] is briefly analyzed and it is argued that there is a diverging correlation length as threshold is approached from above.[26] The bulk of the paper, Secs. IV—VI, contains the mean-field treatment[26] and its consequences. The dc I-V curve is analyzed in Sec. IV, while Secs. V and VI deal with the ac response above and below threshold, respectively. Section VI also contains a discussion of hysteretic behavior. The properties of sliding CDW's with short-range interactions, in particular, the existence of a threshold[13] and crossover from weak to strong pinning, are discussed in Sec. VII. Section VIII, which is somewhat speculative, is concerned with scaling behavior and the properties of ac noise in finite systems with intensity inversely proportional to the square root of the volume. Assumptions (C), (D), and (E) above concerning the effects of thermal fluctuations, inertia, and defects are analyzed in Sec. IX. Finally, Sec. X contains preliminary comparison with experiments on CDW's, discussion of other experimental systems and some conclusions. Various technical details of the mean-field calculations are relegated to Appendixes A and B.

The discussion in Secs. II and III motivates the consideration of the threshold behavior as a critical phenomenon and analyzes difficulties with simple approaches. Except for the introduction of the general model at the beginning of Sec. II, the later sections are not strongly dependent on these two. The rest of the paper relies heavily on the results in Sec. IV. The analysis of the ac behavior above threshold is dependent on the results of Sec. V, while discussion of the response below threshold depends on those results in Sec. VI. Section IX can be omitted by the reader who accepts the assumptions mentioned above.

Most of the quantitative results of this paper are obtained from a mean-field approximation which is strictly valid in the limit of infinite-range interactions, but should, nevertheless, yield considerable qualitative insight into the behavior of, for example, three-dimensional systems with short-range interactions. We first summarize the main results of the mean-field theory and then discuss the more speculative conclusions about short-range interactions.

A. Summary of mean-field results

Providing that the pinning is sufficiently strong, there exists a unique threshold field F_T, below which the CDW will always be stationary in steady state. Above threshold the CDW will move and there will be a unique steady state in which the average phase advances uniformly:

$$\bar{\phi}(t)=vt \ . \tag{1.3}$$

Far above threshold, the effects of the pinning can be neglected and the velocity v will just be proportional to F. In this limit the phase at each impurity site will advance quite smoothly. However, as the field is lowered towards threshold, the motion of the phases will become more and more jerky and near threshold the average velocity will behave as

$$v \sim f^{\zeta} \ , \tag{1.4}$$

where we have defined the reduced field

$$f=(F-F_T)/F_T \ , \tag{1.5}$$

which measures the deviation from threshold. In mean-field theory the exponent ζ is given by

$$\zeta_{\mathrm{MFT}}=\tfrac{3}{2} \ . \tag{1.6}$$

Because of the jerky motion of individual phases, there are *two* diverging time scales near threshold. The first of these is just the period of the motion, which is inversely proportional to the velocity and hence behaves as $f^{-\zeta}$. The second time scale measures the amount of time within each period during which a typical phase is moving much faster than the spatially averaged velocity v. This time scale is much shorter than the period near threshold and behaves as $f^{-\mu}$ with $\mu_{\mathrm{MFT}}=\frac{1}{2}$. This second characteristic time scale and the corresponding frequency

$$\Omega \sim f^{\mu} \tag{1.7}$$

will show up in a measurement of the *linear* response to a uniform ac field with frequency ω applied in addition to the dc field. The differential ac conductivity $\sigma(\omega)$ will be given at low frequencies by the derivative dv/dF of the dc current. Surprisingly, however, this behavior will persist up to frequencies on the order of Ω which is much larger than the frequency v, which characterizes the periodic motion.

Below the threshold field, the behavior of the system is rather complicated and depends on the past history: There are many metastable stationary states and considerable hysteresis. For a range of fields below threshold, there will be some phases which are almost unstable and will jump forward as the field is increased slightly. Because of these jumps, the linear ac response will be singular of low frequencies for a whole range of fields. In particular, the real part of the ac polarizability, $\chi(\omega)=\sigma(\omega)/(-i\omega)$ will have a cusp at low frequencies:

$$\mathrm{Re}\chi(\omega) \sim \chi(\omega=0)-C\,|\,\omega\,| \ . \tag{1.8}$$

Furthermore, its zero-frequency limit will differ from the dc polarizability $\chi_0=d\bar{\phi}/dF$ which will depend on the direction of change of F. The dc polarizability will exhibit a critical singularity as threshold is approached, however, the behavior is nonuniversal and history dependent.

B. Summary of results for short-range interactions

Most of the qualitative features of the infinite-range mean-field theory should persist if the CDW stiffness is short range. The primary qualitative difference will be that, in contrast to mean-field theory, there will be a nonzero threshold for arbitrarily weak pinning in *any* dimension.

For weak pinning there will be a characteristic length scale ξ_0, which is the smallest distance over which the CDW phase will vary appreciably. This length scale, which was first discussed for three-dimensional CDW's by Lee and Rice,[13] diverges in the limit of weak pinning. It plays a role quite analogous to the BCS bare coherence length ξ_0 in superconductivity and is the relevant microscopic (or semimicroscopic) length scale for the threshold phenomena. However, just as for superconductors and conventional critical phenomena, there is another length scale, ξ, which diverges at threshold.

Above threshold the correlation length ξ measures the decay rate of the local CDW velocity-velocity correlation function. At a factor of 2 or so above threshold, ξ will be comparable to ξ_0. However, as the field is decreased and the CDW slows down, larger and larger regions of the CDW will spend most of their time relatively stationary with the phases near to where they will get stuck below threshold. Once each period, the phases will jump quickly forward in a time $\sim\Omega^{-1}$ to a new position at which the pinning forces will again roughly balance the applied field. The linear dimension of these semicoherent regions is a measure of the correlation length which diverges near threshold as

$$\xi \sim \xi_0 f^{-\nu} \ . \tag{1.9}$$

The exponent ν is only known in the mean-field limit; calculations (which are not discussed in detail here) of linear fluctuations about mean-field theory yield $\nu_{\mathrm{MFT}}=\frac{1}{2}$.

The existence of anomalously large coherent regions near threshold suggests that in a finite system of volume

V there might be considerable noise at the characteristic frequency of the motion which is just the average velocity v. Naively, the relative magnitude of the ac current, $j_{\rm ac}$, at frequency v to the dc current, $j_{\rm dc}\sim v\sim f^{\zeta}$, would be expected to scale as the square root of the number of correlation volumes in the sample. However, a more careful analysis of the decay of velocity correlations within a region of volume ξ^d yields the result,

$$\frac{\langle j_{\rm ac}^2\rangle^{1/2}}{j_{\rm dc}}\sim f^{(\nu/2)(d-4+\eta)}\left[\frac{\xi^d}{V}\right]^{1/2} . \tag{1.10}$$

The exponent η is 0 in the mean-field limit. Near threshold, there will be many harmonics of the fundamental noise frequency v up to a frequency of order $\Omega\sim f^{\mu}$.

The exponents ζ, ν, μ, and η which characterize the critical behavior above threshold are not known for short-range interactions. However, they should attain their mean-field values above an (at this state unknown) upper critical dimension, d_c. Two exponent inequalities can be derived, which will be valid in any dimension: $\mu<\zeta$ and $d-4+\eta\geq 0$. The second inequality implies that d_c is at least 4.

Below threshold, we expect hysteretic behavior analogous to that in mean-field theory although the possibility of very slow long-wavelength distortions of the CDW causes additional complications. As the field is slowly increased from zero, regions of the CDW initially of typical linear dimension ξ_0 will become unstable and jump forward only to be stopped by other regions. As threshold is approached, the size of these regions is expected to diverge. Although it will depend on the past history and perhaps on the details of the pinning, the typical size, ξ, of the regions is likely to diverge with an exponent ν' which is the same as the exponent ν above threshold.

The presence of regions which are about to go unstable will, as in mean-field theory, give rise to a cusp in the real part of the polarizability at low frequencies. We conjecture that this cusp will be of the same form as in mean-field theory, i.e., $|\omega|$.

A discussion of the experiments in the light of these somewhat speculative results about the short-range model is contained in the last section.

II. MODEL AND PROBLEMS WITH THE SINGLE-PARTICLE PICTURE

The simplest model which retains all of the important physics consists of a set of impurities labeled by an index j, each of which tries to pin the local phase, ϕ_j, of the CDW at a value β_j (mod2π) with positive pinning strength h_j independently distributed with probability $\rho(h_j)$. The elasticity of the CDW is represented by effective interactions J_{ij} between the phases at the impurity sites $\mathbf{R}_i$ and $\mathbf{R}_j$. The Hamiltonian is then simply taken to be

$$\mathscr{H}_0=-\sum_j h_j\cos(\phi_j-\beta_j)+\frac{1}{2}\sum_{[ij]}J_{ij}(\phi_i-\phi_j)^2 . \tag{2.1}$$

We take purely relaxational equations of motion and include the effects of the uniform applied field F (proportional to the electric field),

$$\frac{d\phi_j}{dt}=-\frac{\delta\mathscr{H}_0}{\delta\phi_j}+F \tag{2.2}$$

(the effects of inertial terms will be discussed in Sec. IX).

The dimensionality of the CDW is reflected in the interactions J_{ij}. In a d-dimensional system the impurities will be distributed at positions $\mathbf{R}_j$ in d-dimensional space, and the effective interactions will fall off rapidly with the distance $\mathbf{R}_i-\mathbf{R}_j$ between the impurities. All of the essential features will be preserved if the $\mathbf{R}_j$ are taken to be points of a d-dimensional lattice with $J_{ij}=J$ for $[ij]$ nearest neighbors and zero otherwise. For reasons that we will discuss in detail below, even this apparently simple model is likely to be impossible to solve in any nonzero dimension except asymptotically in the limit $F\gg J,\{h_j\}$, where perturbation theory is useful. It is thus instructive to consider first the simple case of one impurity (i.e., a zero-dimensional problem) in some detail.

The equation of motion for one phase is (dropping the subscripts and choosing $\beta=0$ for convenience)

$$\frac{d\phi}{dt}=-h\sin\phi+F , \tag{2.3}$$

which is just the "single-particle" model.[11] For purpose of illustration we rederive here the results of Ref. 11 for this model.

It is clear by inspection that the solutions of this equation are of two types, depending on whether F is less than or greater than h. For $F<h$, the long-time behavior (for almost all initial conditions) is simply that ϕ decays exponentially to a constant ϕ_0 at long times with

$$\phi_0=\sin^{-1}(F/h)+2\pi n \tag{2.4}$$

and n integral. Linearizing the equation about any of these minima of the total potential $\mathscr{H}(\phi)=\mathscr{H}_0(\phi)-F\phi$, we obtain with $\phi=\phi_0+\psi$:

$$\begin{aligned}\frac{d\psi}{dt}&=-(h^2-F^2)^{1/2}\psi+O(\psi^2)\\ &\equiv-\lambda(F)\psi+O(\psi^2) .\end{aligned} \tag{2.5}$$

The solution thus becomes less and less stable as F approaches the threshold value $F_T=h$ with the characteristic relaxational frequency $\lambda(F)\sim(F_T-F)^{1/2}\sim|f|^{1/2}$ for $F\to F_T^-$ where the reduced field $f=(F-F_T)/F_T$. At exactly the threshold field, the linear term in ψ about $\phi_0=\pi/2$ vanishes and for $F>F_T$ there is no static solution. As $F\to F_T^-$ the position of the static solution ϕ_0 approaches its value at threshold with a cusp: $\phi_0(F_T)-\phi_0(F)\sim|f|^{1/2}$ and the polarizability $\chi(F)=d\phi_0/dF$ thus diverges as $|f|^{-1/2}$.

For fields greater than threshold, there is a unique solution (up to a trivial shift of the origin of time) which can easily be obtained analytically:

$$\phi=2\tan^{-1}\left[\left[\frac{-h+F}{F+h}\right]^{1/2}\tan[(F^2-h^2)^{1/2}t/2]\right]+\frac{\pi}{2} . \tag{2.6}$$

This solution is periodic with period $P=2\pi/(F^2-h^2)^{1/2}$ and hence the average velocity is $v=(F^2-h^2)^{1/2}$, which goes to zero at threshold as f^{ζ} with $\zeta=\frac{1}{2}$. This critical behavior of the velocity can be simply derived by noting that on physical grounds (and in the actual solution), the phase spends most of each period near a sticking point at $\phi_s=(\pi/2)+2\pi n$ at which the potential $\mathscr{H}(\phi)$ is very flat. Near these sticking points, the equation of motion is

$$\frac{d\phi}{dt}=(F-h)+\frac{h}{2}(\phi-\phi_s)^2+O((\phi-\phi_s)^4) . \tag{2.7}$$

This equation can be scaled to yield

$$\frac{d[(\phi-\phi_s)/\sqrt{f}\,]}{d(th\sqrt{f}\,)}=1+\frac{1}{2}\left[\frac{\phi-\phi_s}{\sqrt{f}}\right]^2 \tag{2.8}$$

and we thus simply see that the phase will remain within $\sqrt{f}$ of ϕ_s for a transit time, $t_T\sim 1/h\sqrt{f}$. The solution to Eq. (2.8) diverges at a finite time corresponding to $\phi-\phi_s$ becoming of order 1 where the higher-order terms in Eq. (2.7) become important. During the remainder of each period, when ϕ is not near any sticking point the velocity is of order 1. The average velocity in the critical region near threshold thus scales simply as the inverse of the time t_T, to get through the region near the sticking point, whence $v\sim f^{1/2}$.

For this simple single-phase model, we see that several quantities exhibit critical behavior below threshold: The characteristic relaxation frequency $\lambda(F)$ goes to zero at threshold as $|f|^{\mu}$ with $\mu=\frac{1}{2}$ and the polarizability $\chi(F)$ diverges as $|f|^{-\gamma}$ with $\gamma=\frac{1}{2}$. Above threshold the mean velocity v goes to zero as f^{ζ} with $\zeta=\frac{1}{2}$. Furthermore, it is clear from the above discussion that these exponents will not depend on the details of the potential.

One of the primary questions we will address in this paper is how the critical behavior near threshold, in particular, the exponents, differ for systems with an infinite number of degrees of freedom from those of the simple one-phase model.

Before going on to infinite systems, we first consider arbitrary *finite* collections of N coupled phases. For sufficiently small values of the field F, the total potential

$$\mathscr{H}\{\phi_j\}=\mathscr{H}_0\{\phi_j\}-F\sum_j\phi_j \tag{2.9}$$

will always have at least some local minima. These minima will form a finite number of infinite classes of equivalent minima. Equivalent minima will differ only by an *overall* phase which is an integral multiple of 2π. We can thus restrict our attention to the finite set of inequivalent minima with the overall phase $\bar{\phi}=N^{-1}\sum_j\phi_j$ lying between, say, $-\pi$ and $+\pi$.

For small fields there will generally be a number of inequivalent metastable states that are static, locally stable solutions of the equations of motion. As the field is increased some of these states will disappear until there is only one left which will itself disappear at a finite threshold field. For determining properties of the system sufficiently near threshold we can focus attention on this last minimum.

The principal curvatures of the potential, λ_α, about this minimum, are just the spectrum of relaxational frequencies. They will all be positive for $F<F_T$ and, generically, exactly *one* of them, say, λ_1, will go to zero as $F\to F_T^-$. Sufficiently close to threshold the long-time behavior will be dominated by the last minimum and its lowest frequency λ_1. These properties can be obtained by an expansion of the equations of motion about the state $\{\phi_{sj}\}$ which is the limiting position of the last minimum as $F\to F_T^-$. If we diagonalize the linearized equations of motion about this point, we will find relaxational frequencies $\lambda_{s\alpha}$ with one, λ_{s1} for example, zero, and the others positive. Thus, there is one special slow direction at threshold. Straightforward analysis[27] of the nonlinear terms shows that the motion in this direction dominates, and the system behaves similarly to the single-particle case. As $F\to F_T^-$ the polarizability thus diverges with $\gamma=\frac{1}{2}$ and the lowest relaxational frequency goes to zero with $\mu=\frac{1}{2}$. Above threshold the transit time t_T through the slow sticking point $\{\phi_{sj}\}$ behaves as $f^{-1/2}$ and hence the velocity again goes to zero with $\zeta=\frac{1}{2}$.

The exponents for any finite collection of N coupled phases are generically the same as for one phase; they represent zero-dimensional behavior. However, it is natural to expect that when N increases, the size of the zero-dimensional critical region will decrease. (Note that this will not always be the case. For example, if all the β_j's are equal, the size of the $d=0$ critical region will remain of order 1 as $N\to\infty$.) This will occur in several ways. First, the neglect of the motion transverse to the slow direction in determining the transit time through the sticking point is only justified if $t_T^{-1}\ll\lambda_{s2}$. In the limit $N\to\infty$, λ_{s2} (and some other higher $\lambda_{s\alpha}$'s) will tend to zero, restricting the $d=0$ critical regime. In addition, the solution may pass through other slow regions during each period. The contributions to the total period of the transit times through these regions can only be neglected if they are much less than t_T. As N grows this will also restrict the $d=0$ critical region.

In this paper we will not be primarily concerned with the crossover from infinite N to finite $N,d=0$ behavior. (A detailed discussion of this question will be given elsewhere.[28]) However, it should be clear from the above discussion that an infinite system is unlikely to exhibit the same critical behavior as a small finite one.

III. PERTURBATION THEORY

We have argued in the preceding section that, for a large system, the asymptotic behavior near threshold cannot be obtained from a simple analysis since it involves crucially a large number of degrees of freedom. The problems created by these large number of degrees of freedom can also be seen, as in conventional critical phenomena, by examining perturbation theory in the nonlinearities—in this case the $\{h_j\}$.

Let us consider the model given by Eqs. (2.1) and (2.2) with the $\mathbf{R}_j$ (for definiteness) a d-dimensional hypercubic lattice with lattice spacing 1 and the J_{ij} constant nearest-neighbor interactions J. The static properties of the system in zero field are similar to those of a zero-temperature X-Y magnet in a random magnetic field (h_j)

with a constraint that there are no vortices [i.e., the phase is a single-valued function of position on $(-\infty,\infty)$].

We can attempt perturbation theory about the trivial ordered ground state $\phi_j=\bar{\phi}$ for all j in powers of the mean-square random field $\langle h^2\rangle=\int\rho(h)h^2dh$. To lowest order, the mean-square deviation of a given phase from the average $\bar{\phi}$ is

$$\langle\psi_j^2\rangle=\tfrac{1}{2}\langle h^2\rangle\int_{\mathbf{q}}\frac{1}{[J\Delta(\mathbf{q})]^2}\ , \tag{3.1}$$

where we have defined $\psi_j=\phi_j-\bar{\phi}$ and $\Delta(\mathbf{q})$ is the Fourier transform of the lattice Laplacian: $\Delta(\mathbf{q})\approx-2q^2d$ for small q. The integral in Eq. (3.1) is thus divergent at long wavelengths in less than four dimensions. This divergence is generally believed to destroy the long-range order of an X-Y magnet in a random field for $d\le 4$. Except perhaps for d near 4, little can be learned about the static properties of the system for $d<4$ from perturbation theory, except, as will be seen later, the characteristic magnitude of the threshold field for weak pinning, i.e., $\langle h^2\rangle\ll J^2$. For $d>4$, the integral in Eq. (3.1) (and all others occurring in perturbation theory) is perfectly finite which suggests that the ground state of the system at $F=0$ will have long-range phase order if the pinning strength is sufficiently weak.

If we concern ourselves instead with *moving* solutions with average phase $\bar{\phi}=vt$, then it is straightforward to expand in powers of the pinning strength. With no pinning, we have $v=F$. If at each order the velocity v is adjusted appropriately as a function of F, this expansion can be made uniformly valid at all times. The first correction due to the pinning[14,15] is

$$v=F\left[1-\tfrac{1}{2}\langle h^2\rangle\int_{\mathbf{q}}\frac{1}{F^2+J^2\Delta^2(\mathbf{q})}\right]. \tag{3.2}$$

The integral in Eq. (3.2) is cut off at long wavelengths by time averaging of the long-wavelength components of the pinning potential. This is due to the inability of the long-wavelength components of the deviations, ψ_j, of the phases from $\bar{\phi}$ to adjust to the potential on the time scale on which $\bar{\phi}$, and hence the effective force on ψ_j, is varying. To lowest order (quadratic in h) the equal-time fluctuations of the phases away from their average are given by

$$\langle\psi_i(t)\psi_j(t)\rangle=\frac{\langle h^2\rangle}{2}\int_{\mathbf{q}}\frac{e^{i\mathbf{q}\cdot(\mathbf{R}_i-\mathbf{R}_j)}}{F^2+J^2\Delta^2(\mathbf{q})} \tag{3.3}$$

and hence $\langle\psi_j^2\rangle$ appears to be finite at this order. Higher-order corrections will, however, yield divergent contributions to the *static* part of the Fourier-transform correlation function yielding, along with the dynamic parts, a correlation function of the form

$$\langle\psi(\mathbf{q},\omega)\psi(\mathbf{q}',\omega')\rangle=\delta(\mathbf{q}+\mathbf{q}')\delta(\omega+\omega')\left[\sum_{n=-\infty}^{\infty}a_n\frac{\delta(\omega-vn)}{\omega^2+J^2\Delta^2(\mathbf{q})}\right], \tag{3.4}$$

where $a_{\pm1}$ shows up at first order in perturbation theory and the other a_n shows up at higher order. The static part, $n=0$, will cause $\langle\psi_j^2(t)\rangle$ to diverge in less than four dimensions. Thus, while the time averaging suppresses somewhat the effect of the long-wavelength components of the disorder, there are still divergent contributions to *static* correlation functions when the CDW is moving (in contrast to the speculation in Ref. 26). These divergences apparently do not show up in the dynamic (i.e., $\omega\neq0$) correlation functions or in the perturbation expansion for the velocity, at least not in dimensions greater than two. (A detailed analysis of the high-field perturbation theory is beyond the scope of this paper—a more detailed discussion will be contained in a future paper.[28]) From Eq. (3.3) and examination of the higher-order terms it can be seen that the local velocity—local velocity correlation function [which is not affected by the $n=0$ terms of Eq. (3.4)] falls off exponentially at high fields with a correlation length which to leading order is just

$$\xi=2Jd/F\ . \tag{3.5}$$

Thus, the local *velocities* exhibit long-range order although the *phases* do not. Since the finite velocity correlation length in the moving state is caused by temporal averaging of the disorder, it is likely to diverge as the average velocity goes to zero at the threshold field. Higher-order terms in perturbation theory will also contain integrals over momenta cutoff at ξ^{-1}. The threshold behavior should thus exhibit many of the features of conventional critical phenomena and hence exact calculations of the behavior near threshold are unlikely to be feasible, except in certain limits. In the following sections we discuss a mean-field approximation which, by analogy with critical phenomena, might be expected to be valid in sufficiently high dimensions.

The behavior in low dimensions, in particular, $d<4$, will be discussed in Secs. VII and VIII and some of the issues raised here concerning perturbation theory, correlation lengths, etc., will be discussed further.

IV. MEAN-FIELD THEORY

In the presence of long-range interactions, the fluctuations due to the disorder of the force on one phase from the others is expected to be small. Mean-field theory is formally valid in the limit of infinite-range interactions with the total interaction strength $\sum_j J_{ij}$ held fixed. In this limit, the effects of all the other phases can be considered as just a nonfluctuating mean field which can be determined self-consistently.[26]

We thus consider a large number, N, of phases each coupled to all the others with a strength $J_{ij}=J/N$ (where the normalization is chosen to keep the total coupling finite). The total energy is given by

$$\mathscr{H}=(J/2N)\sum_{[ij]}(\phi_i-\phi_j)^2-\sum_j h_j\cos(\phi_j-\beta_j)-F\sum_j\phi_j$$
$$=\frac{J}{2}\sum_j(\phi_j-\bar{\phi})^2-\sum_j h_j\cos(\phi_j-\beta_j)-F\sum_j\phi_j\ , \tag{4.1}$$

where each phase is only coupled to the mean field

$$\bar{\phi}(t)=\frac{1}{N}\sum_j\phi_j(t)\ . \tag{4.2}$$

The equation of motion for a single phase is simply

$$\frac{d\phi_j}{dt} = -h_j \sin(\phi_j - \beta_j) + J[\bar{\phi}(t) - \phi_j] + F \ . \tag{4.3}$$

In the thermodynamic, infinite-range limit $N \to \infty$, $\bar{\phi}(t)$ will not fluctuate and we can impose a given $\bar{\phi}(t)$, find solutions to Eqs. (4.3) for all the $\phi_j(t)$, and then require that the self-consistency condition, Eq. (4.2), be satisfied at all times. It is convenient to measure all fields, frequencies, and pinning strengths in units of the coupling J, which we henceforth set equal to 1.

As $N \to \infty$ all β_j will occur with equal probability and the h_j with probability $\rho(h_j)$ which we take to be zero for h_j larger than a maximum value $\hat{h}$. We can thus label each ϕ_j by its associated β_j and h_j (since the j dependence of the equation of motion is entirely through β_j and h_j) and replace $(1/N)\sum_j$ by $(1/2\pi)\int d\beta \int \rho(h)dh$ dropping the j subscripts for the time being.

We will particularly be concerned with two types of steady-state solutions of the mean-field equations of motion: *static* with $\bar{\phi}$=constant, and *uniformly moving* with $\bar{\phi} = vt$. The possible existence of other solutions is discussed at the end of this section.

The equation of motion for a single phase $\phi_j \to \phi(\beta_j, h_j)$ can be written usefully as the gradient of an effective time-dependent potential:

$$\frac{d\phi}{dt} = -\frac{d}{d\phi} W(\phi; \bar{\phi}(t), \beta, h, F)$$

$$= -h \sin(\phi - \beta) + [\bar{\phi}(t) - \phi] + F \ , \tag{4.4}$$

where

$$W = \tfrac{1}{2}[\phi - \bar{\phi}(t)]^2 - h \cos(\phi - \beta) - F\phi \ . \tag{4.5}$$

Two regimes can now be distinguished. *Weak pinning,* $\hat{h} < 1$: If the maximum value of h_j is less than 1, then W will have a unique minimum as a function of ϕ for *all* β, h, and all times. *Strong pinning,* $\hat{h} > 1$: If some h_j are greater than 1, W will have several minima for at least some β, h for a given $\bar{\phi}, F$. The local potential W is plotted in Fig. 1 for the strong-pinning case. We first discuss the simpler weak-pinning case in detail.

A. Weak pinning: $\hat{h} < 1$

Since for this case W always has a unique minimum, a unique static solution, $\phi_0(\beta, h)$, to the equations of motion Eq. (4.4) exists for each $\bar{\phi}$ and F. The effective potential W is the sum of a parabolic part from the elastic plus field energies and a cosine part from the pinning. It follows that the displacement of the minimum of W from the minimum $(\bar{\phi} + F)$ of the parabolic part by itself depends only on the relative positions of the minima of the parabolic and the cosine parts. Therefore, $\phi_0(\beta, h) - \bar{\phi} - F$ is an odd 2π periodic function of $-\beta + \bar{\phi} + F$. It follows that $(1/2\pi)\int_0^{2\pi} \phi_0(\beta, h) d\beta = \bar{\phi} + F$ for each h, whence

$$\langle \phi_0 \rangle \equiv (1/2\pi) \int_0^{2\pi} d\beta \int_0^{\hat{h}} \phi_0(\beta, h)\rho(h) dh = \bar{\phi} + F \ . \tag{4.6}$$

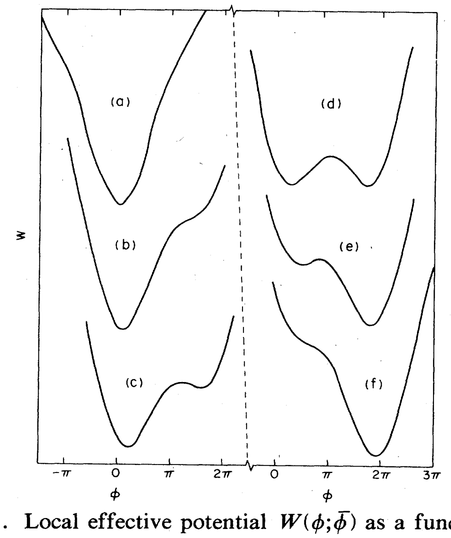

FIG. 1. Local effective potential $W(\phi; \bar{\phi})$ as a function of ϕ for a strongly-pinned phase with pinning strength $h > 1$ for various values of $\bar{\phi}$ incremented by amounts of $\pi/3$. In a moving state with velocity v, $\bar{\phi}$ advances as vt so the various curves represent W after successive time intervals $\pi/3v$. In steady state the phase will lag slightly behind (to the left) of the leftmost minimum of W and at time t_s between (e) and (f) it will quickly jump out of the disappearing minimum to the next one.

There are thus *no* self-consistent static solutions to the mean-field equations for *any* F except for $F = 0$. In zero-field, static solutions exist for any $\bar{\phi}$; these can be adiabatically changed into one another as $\bar{\phi}$ is increased without any phases "jumping:" i.e., $d\phi_0(\beta, h, F=0, \bar{\phi})/d\bar{\phi}$ is finite for all β, h.

In the presence of a *small* field, F, it is natural to expect from the above discussion, that there will be a uniformly moving solution $\bar{\phi} = vt$ with v small in which each phase follows almost adiabatically the minimum in its local time-dependent effective potential,

$$\phi(\beta, h, t) \approx \phi_0(\beta, h, F, \bar{\phi} = vt) \equiv \phi_A(\beta, h, t) \ , \tag{4.7}$$

where we have defined the *adiabatic solution,* $\phi_A(\beta, h, t)$ to be at the minimum of the local potential at all times. This adiabatic approximate solution satisfies the equation of motion for a single phase

$$\frac{d\phi}{dt} = -h \sin(\phi - \beta) + vt - \phi + F \tag{4.8}$$

with the left-hand side equal to zero. As will be the full steady-state solution, $\phi_A(t) - vt$ is periodic in time with period $P = 2\pi/v$.

It is useful to consider a mean-field solution with a given velocity and calculate the field needed to make it satisfy the self-consistency condition. We can find the steady-state solution with a given small v by perturbation theory in powers of v about the adiabatic approximation ϕ_A:

$$\phi(t) = \phi_A(t) + v\theta_1(t) + v^2\theta_2(t) + \cdots \ , \tag{4.9}$$

where the $\theta_i(t)$ are of order 1 and periodic in time with period P.

Since $d\phi_0/d\bar{\phi}$ is bounded, $d\phi_A/dt$ is of order v at all times and we can thus substitute Eq. (4.9) for ϕ into the equation of motion and, by collecting terms of each order in v, can solve iteratively for θ_1, θ_2, etc. From terms of order v, we have

$$\frac{d\phi_A}{dt} = -hv\theta_1\cos(\phi_A-\beta)-v\theta_1 \quad (4.10)$$

from which θ_1 can be obtained in terms of the implicitly defined ϕ_A. At this point it is convenient to note that the solutions to Eq. (4.8) have a trivial dependence on the phase β,

$$\phi(\beta,h,t)=\beta+\phi(\beta=0,\, h,t-\beta/v) \quad (4.11)$$

and hence the deviations $\psi=\phi-vt$ of the phases from $\bar{\phi}$ only depend on β through a temporal phase shift

$$\psi(\beta,h,t)=\psi(\beta=0,\, h,t-\beta/v)\ . \quad (4.12)$$

This is a consequence of the fact that the motion of all the phases with the same h *relative* to their preferred values, β, is the same up to temporal shifts which depend only on β. The self-consistency condition can thus be rewritten in terms of averaging over time for one period:

$$\begin{aligned} vt=\bar{\phi}(t) &= (1/2\pi)\int d\beta \int \rho(h)\phi(\beta,h,t)dh \\ &= vt+(1/2\pi)\int d\beta\int \rho(h)\psi(\beta=0,\, h,t-\beta/v)dh \\ &= vt+\int dh\,\rho(h)\int_0^P dt'\frac{v}{2\pi}\psi(\beta=0,\, h,t')\ , \end{aligned} \quad (4.13)$$

i.e., we require that the average over one period and over h of $\psi(\beta=0,\, h,t)$ (or equivalently any other β) must be zero. In addition, the field F enters the single-phase equation of motion in a simple way, so that with v fixed, the average of the phase deviation over one period satisfies

$$\langle \psi(F,t)\rangle_{\rm period}=F+\langle\psi(F=0,t)\rangle_{\rm period}\ . \quad (4.14)$$

Similarly,

$$\langle \psi_A(F,t)\rangle_{\rm period}=\langle \phi_A(F,t)-vt\rangle_{\rm period}=F$$

and since $\psi=\psi_A+v\theta_1+O(v^2)$ it follows that the self-consistency condition will be satisfied with

$$v=\sigma F+O(F^2)\ , \quad (4.15)$$

i.e., response to the field is *linear*. A detailed calculation (see Appendix A) yields the linear conductivity

$$\sigma=\left[\int_0^{\hat{h}}\frac{\rho(h)}{(1-h^2)^{1/2}}dh\right]^{-1} \quad (4.16)$$

which is generally less than 1. At high fields, as for the finite-dimensional case, $v\approx F$ so that the response $v(F)$ will be linear in both the high- and low-field limits (but with different slopes) as shown in Fig. 2.

From Eq. (4.16) we see that as the maximum pinning strength $\hat{h}$ approaches 1, σ tends to zero. As $\hat{h}$ increases past 1, the local effective potentials W, for $h>\hat{h}$ have several minima and the adiabatic approximation for the solutions breaks down. This brings us to the strong-pinning case.

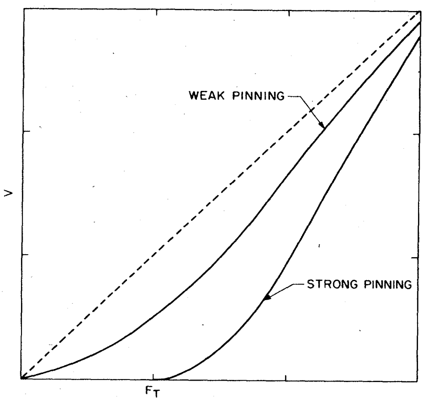

FIG. 2. Schematic plots of the velocity v, as a function of the applied field F, for weak and strong pinning in mean-field theory. The dashed line, $v=F$, is the asymptotic high-field limit for both curves. Note the linear response at small fields for weak pinning and the threshold F_T for strong pinning.

B. Strong pinning: $\hat{h}>1$

We first examine static solutions which we require to be locally stable. This implies that each phase lies at a local minimum of its local potential. Since for $h>1$, W can have several minima, there will generally be a large number of metastable solutions for each $\bar{\phi}$. Thus, in contrast to the weak-pinning case, the solution function $\phi(\beta,h,\bar{\phi},F)$ will be a *multivalued* function of its arguments. However, it will still have the property that for a given h, $\phi_0-\bar{\phi}-F\equiv\tilde{\Upsilon}$ is a periodic (although multivalued) function only of $\alpha=-\beta+\bar{\phi}+F$. This function is plotted for $h\approx 2$ in Fig. 3. The self-consistency condition is just that the average over α and h of the chosen branches of $\tilde{\Upsilon}(\alpha,h)$ is equal to $-F$.

For small F, this condition can clearly be satisfied with several choices of the branch of $\tilde{\Upsilon}$ for many α,h. However, as F increases, it becomes imperative that for most α the lowest branch of $\tilde{\Upsilon}$ (for each h) is chosen; we denote this branch, represented by a heavy solid line in the figure, $\Upsilon(\alpha,h)$. The function Υ is the smallest solution of

$$-h\sin(\Upsilon+\alpha)-\Upsilon=0\ . \quad (4.17)$$

As F increases the number of possible self-consistent metastable states decreases until at a maximum value of F, which is the threshold field F_T, there is a unique self-consistent solution (up to the overall phase $\bar{\phi}$) corresponding to Υ and for $F>F_T$ no self-consistent solutions exist. From the above discussion, it follows that the threshold field will be given by

$$F_T=-\frac{1}{2\pi}\int_0^{2\pi}d\alpha\int_0^{\hat{h}}dh\,\rho(h)\Upsilon(\alpha,h) \quad (4.18)$$

which will be strictly positive as long as $\hat{h}>1$.[29] In Sec. VI we will discuss some of the properties of the system in the pinned phase when $F<F_T$. However, we first consider uniformly moving solutions for the strong-pinning case with $F>F_T$.

We note that, in contrast to the weak-pinning case, the solution at $F=F_T$ does *not* vary smoothly as $\bar{\phi}$ is in-

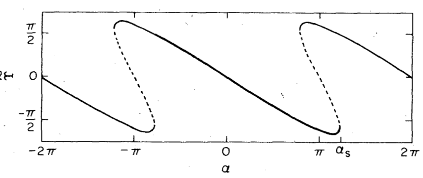

FIG. 3. Multivalued function $\tilde{\Upsilon}(\alpha)$ (solid line) which represents the difference between the minima of the local effective potential W and the average phase, $\bar{\phi}$, as a function of $\alpha=-\beta+\bar{\phi}+F$ for a strongly-pinned phase with $h>1$, and preferred phase β. The dashed line represents the maxima in W and the heavy solid line the special branch Υ which is important at threshold and ends at the singular point α_s.

creased. In particular, because of the discontinuity in Υ as a function of α, there will always be some phase which will jump as $\bar{\phi}$ is increased adiabatically. This is due to the disappearance of a minimum of the local potential, $W(\phi)$ for these phases, as $\bar{\phi}$ is increased. For small velocities, v, it is still reasonable to expect that the motion of each phase will almost follow a minimum in its W and move rapidly to the next one when that minimum disappears. We thus expect that for small v, the solution will be approximately given by the adiabatic solution ϕ_A which satisfies the equation of motion, Eq. (4.8), with the $d\phi/dt$ term ignored:

$$\phi_A(\beta,h,t)\equiv vt+F+\Upsilon(vt-\beta+F,h) \ . \tag{4.19}$$

The preferred phase, β, again enters this expression in a simple way; $\psi_A=\phi_A(\beta,h,t)-vt$ is a function of the combination $t-\beta/v$ rather than β and t independently. By inspection of the equation of motion, it is clear that the actual solution will have this same property. We will thus focus our attention on a single β (for simplicity $\beta=0$) and replace averages over β of $\psi(\beta,h,t)$ by temporal averages. We will drop the β's and just consider $\beta=0$.

Since the steady-state ϕ will be periodic we need consider only one period. If, as in the weak-pinning case, we try to expand the actual solution with small velocity, v, about the adiabatic solution,

$$\phi(t)=\phi_A(t)+\theta(t) \ , \tag{4.20}$$

we will encounter difficulties due to the discontinuities in ϕ_A for $h>1$. The self-consistency condition can be simply expressed in terms of $\theta(h,t)$. We require that

$$\begin{aligned} 0&=\langle\phi(\beta,h,t)-vt\rangle\\ &=\int dh\,\rho(h)\frac{1}{P}\int_0^P dt[\phi(\beta=0,h,t)-vt]\\ &=\int dh\,\rho(h)\frac{1}{P}\int_0^P dt[F+\Upsilon(vt+F,h)+\theta(h,t)]\\ &=F-F_T+\langle\theta(h,t)\rangle_{h,\text{period}} \ , \end{aligned} \tag{4.21}$$

where the last equality follows from Eq. (4.18) for the threshold field. Thus, we are particularly interested in the average of θ.[30] Anomalous contributions will come from the h's which are greater than 1; we thus focus our attention on a particular $h>1$. For most of the period, ϕ will lag only slightly behind the slowly moving minimum of the potential $W(\phi)$ (see Fig. 1) at ϕ_A and θ will be of order v as in the weak-pinning case. However, when the minimum of W near which the phase is disappears at time t_s, the phase will take some time to catch up with the next minimum and will lag behind ϕ_A by an amount of order 1 until it does so. Far away from t_s, we can solve the equation of motion perturbatively; $\theta(t)$ will be of order v. However, near t_s this is clearly not possible and we must examine the nonlinear behavior in detail. The detailed asymptotic analysis is presented in Appendix A; the method and results are outlined here.

We first examine the behavior of the position of the lowest ϕ minimum of $W(\phi)$ (i.e., ϕ_A). Since $\phi_A(t)$ (for $\beta=0$) is a solution to the equation

$$\begin{aligned} \frac{dW}{d\phi}(\phi_A(t),t)&=0\\ &=h\sin\phi_A+\phi_A-vt-F \ , \end{aligned} \tag{4.22}$$

the condition that ϕ_A a *minimum* of the potential is just that

$$\frac{d^2W}{d\phi^2}(\phi_A(t),t)>0 \ , \tag{4.23}$$

i.e.,

$$h\cos\phi_A+1>0 \ . \tag{4.24}$$

The minimum will disappear at a time t_s and position $\phi_s(h)\equiv\phi_A(t_s,h)$, where

$$h\cos\phi_s+1=0 \ . \tag{4.25}$$

From the plot of ϕ_A in Fig. 4, it can be seen that the point at which the lowest ϕ minimum disappears (as against when a new minimum appears) is given by the solution to Eq. (4.25) with

$$\sin\phi_s>0 \ . \tag{4.26}$$

The behavior near to ϕ_s can be found by expanding Eq. (4.22) about the singular point ϕ_s,t_s:

$$\phi_s-\phi_A\approx\left[\frac{2v(t_s-t)}{h\sin\phi_s}\right]^{1/2} \ , \tag{4.27}$$

i.e., ϕ_A has an upward square-root cusp as is evident from the figure. The time derivative of ϕ_A can be obtained by differentiating Eq. (4.22), yielding

$$\frac{d\phi_A}{dt}=\frac{v}{1+h\cos\phi_A(t)}>0 \ , \tag{4.28}$$

where the positivity follows from condition Eq. (4.24). The time derivative of ϕ_A is of order v except near t_s. At t_s, ϕ_A jumps from ϕ_s to a value ϕ_r.

Away from t_s, we can obtain θ perturbatively by substituting Eq. (4.20) in the equation of motion Eq. (4.8) and keeping the lowest-order terms in powers of v. This yields

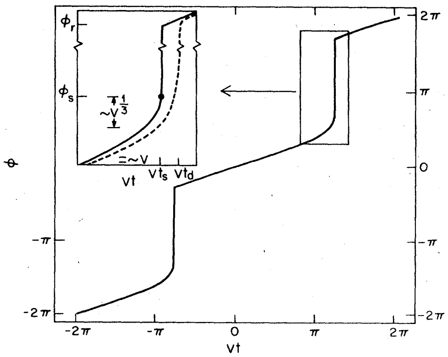

FIG. 4. Adiabatic approximation $\phi_A(t)$ to a steady-state solution for strong pinning $\phi(t)$, moving with a small average velocity v. The actual solution lies below ϕ_A by an amount of order v except near the jump in ϕ_A. The expanded region in the inset shows ϕ_A (solid line) and ϕ (dashed line) near the singular time t_s at which ϕ_A jumps from ϕ_s to ϕ_r. Note that the time scale $t_d - t_s$ for the jump of the phase out of a disappearing minimum of its local potential into a new one is of order $v^{-1/3}$.

$$\theta(t) \simeq \frac{-v}{[1+h\cos\phi_A(t)]^2} + O(v^2) = -\frac{1}{v}\left[\frac{d\phi_A}{dt}\right]^2 \tag{4.29}$$

which is strictly negative; i.e., the phase lags behind the minimum in W as expected physically. In deriving Eq. (4.29) we have ignored terms of order θ^2 and $d\theta/dt$. Near to t_s,

$$\theta(t) \propto (t_s - t)^{-1} . \tag{4.30}$$

It can be seen (see Appendix A) that θ^2 and $d\theta/dt$ will become non-negligible only when $t_s - t \sim v^{-1/3} \ll P$. In the limit $v \to 0$, the perturbative result Eq. (4.29) for θ will thus be valid closer and closer to t_s on the scale of the period P. The contribution to $\langle \theta \rangle$ from the region far from t_s will be of order v but there will be an anomalous contribution from times $t_s - t$ small but positive. This region yields a contribution to $\langle \theta \rangle$ of the form

$$\frac{1}{P}\int_{v^{-1/3}}^{1/v} \frac{1}{\tilde{t}} d\tilde{t} \sim v \ln v , \tag{4.31}$$

where $\tilde{t} = t_s - t$ and the lower and upper cutoffs come, respectively, from where the θ^2 and $d\theta/dt$ terms in the equation of motion cannot be neglected and from where $\phi_s - \phi_A(t)$ deviates from its asymptotic form near t_s, Eq. (4.27).

Within a time of order $v^{-1/3}$ of t_s, the equation of motion becomes fully nonlinear. However, an expansion can be made of the nonlinear equation about the point ϕ_s, t_s at which the right-hand side of Eq. (4.8) and its derivative with respect to ϕ vanish (corresponding to the potential W being *cubic* at t_s, ϕ_s). Ignoring terms of order $(\phi - \phi_s)^3$ we have

$$\frac{d\phi}{dt} = \left[\frac{h}{2}\sin\phi_s\right](\phi - \phi_s)^2 + v(t - t_s) \tag{4.32}$$

which can be scaled by

$$\phi - \phi_s = C_h^2 v^{1/3} \chi , \quad t - t_s = C_h v^{-1/3} \tau \tag{4.33}$$

with

$$C_h = \left[\frac{h}{2}\sin\phi_s(h)\right]^{-1/3} \tag{4.34}$$

independent of v to give

$$\frac{d\chi}{d\tau} = \chi^2 + \tau . \tag{4.35}$$

In these scaled variables the $(\phi - \phi_s)^3$ terms become of order $v^{1/3}$, their neglect can thus be justified for small v.

The nonlinear equation [Eq. (4.35)] can be solved analytically and a uniform approximation to ϕ obtained by matching asymptotic expansions; the details are contained in Appendix A. It is straightforward, however, to obtain the asymptotic form of the desired $\langle \theta(t) \rangle$. From the characteristic scales near ϕ_s, t_s given by Eq. (4.33) we expect ϕ to remain within order $v^{1/3}$ of ϕ_s for a time of order $v^{-1/3}$. However, at a fixed positive value of $\tau = \tau_d$ corresponding to a time t_d, the desired solution to the scaled equation [with $\chi(\tau \to -\infty) \simeq -\sqrt{-\tau}$] will diverge. Near to this point $\phi - \phi_s$ will become of order 1 and we can no longer neglect terms of order $(\phi - \phi_s)^3$. However, by this time the instantaneous velocity $d\phi/dt$ will also be of order 1 and we can then neglect the explicit time dependence in the equation of motion. The solution will then relax to a value near $\phi_A(t_d) \approx \phi_r$ in a time of order 1.

The dominant contribution to $\langle \theta \rangle$ will come from the time from t_s to t_d when θ will be of order 1. Since this occurs for a fraction of the period of order $v^{2/3}$, this will yield from each $h > 1$, a $v^{2/3}$ contribution to $\langle \theta \rangle$ which will dominate the v and $v \ln v$ contributions from $t \lesssim t_s$ (and from $h < 1$). From the form of the self-consistency condition Eq. (4.21) we conclude that near threshold

$$v \approx B(F - F_T)^{3/2} , \tag{4.36}$$

i.e., $\zeta = \frac{1}{2}$. The coefficient B will depend on the distribution $\rho(h)$ of the pinning strengths; it is given by Eq. (A43).

For strong pinning, we have found that below a nonzero threshold field F_T there are only static solutions while above the threshold field there are uniformly moving solutions but no static solutions. Furthermore, the velocity appears to be a unique function of the field F and goes to zero continuously at threshold with an exponent $\zeta = \frac{3}{2}$. It is plotted schematically in Fig. 2. At the borderline between weak and strong pinning, $\hat{h} = \hat{h}_M = 1$ there is a multicritical point at $F = 0$. As mentioned already, as $\hat{h} \to 1^-$ the linear conductivity in the weak-pinning regime goes to zero. Similarly, as $\hat{h} \to 1^+$, the threshold field will go to zero and the coefficient B of the $(F - F_T)^{3/2}$ dependence of the velocity [Eq. (4.36)] will diverge. The detailed behavior near this multicritical point, which is nonuniversal, is easy to derive from the results in Appendix A.

In the next section, we will show that the uniformly moving mean-field solution is stable to small perturbations. However, it has not so far been possible to rule out either above or below F_T the existence of solutions to the mean-field equations of motion which, even at long times, are neither uniformly moving nor stationary. It appears unlikely that such solutions exist since, if we assume a solution $\bar{\phi}(t)$ exists with a nonuniform but periodic $d\bar{\phi}/dt$, the resulting $d\langle\phi(t)\rangle_{\beta,h}/dt$ will generally have less harmonic content than $d\bar{\phi}/dt$ because the different $\phi(\beta,t)$ will tend to be out of phase, and hence the self-consistency condition cannot be satisfied. We will therefore assume for the remainder of this paper, that *below threshold only stationary solutions exist* at long times and *above the same threshold only uniformly moving solutions exist* at long times. For the mean-field case, it can be shown that the uniformly moving solution discussed here is unique up to an overall shift in the temporal origin. The finite-dimensional case is discussed in Sec. VII.

V. STABILITY AND ac RESPONSE IN MEAN-FIELD THEORY

In the preceding section, static and uniformly moving solutions of the mean-field equations of motion were discussed. The static solutions were stable by construction, but it is necessary to show that the uniformly moving solutions are stable, at least locally. In this section we consider linear response of the mean-field system to a small uniform ac applied field, $A(t)$ (in particular, $A_0e^{-i\omega t}$) in addition to the dc field, F. We consider self-consistent solutions to the equations of motion for $\phi(\beta,h,t)$:

$$\frac{d\phi}{dt}=-h\sin(\phi-\beta)-\phi+\bar{\phi}(t)+F+A(t) \tag{5.1}$$

and expand about the uniformly moving solution $\phi_v(\beta,h,t)$ with $A(t)=0$ and $\bar{\phi}_v(t)=vt$ discussed in the preceding section. We thus write

$$\phi(\beta,h,t)=\phi_v(\beta,h,t)+\eta(\beta,h,t) \tag{5.2}$$

and

$$\bar{\phi}(t)=vt+\bar{\eta}(t) \tag{5.3}$$

with

$$\bar{\eta}(t)=\langle\eta(\beta,h,t)\rangle_{\beta,h} \ . \tag{5.4}$$

As before, it is convenient to focus on a given h. The linearized equations of motion for the perturbations $\eta(\beta,t)$ for a given h are

$$\frac{d\eta}{dt}=-\Lambda(h,t-\beta/v)\eta+\tilde{A}(t) \ , \tag{5.5}$$

where

$$\Lambda(h,t-\beta/v)=h\cos[\phi_v(\beta,t)-\beta]+1 \tag{5.6}$$

is a periodic function of $t-\beta/v$ and

$$\tilde{A}(t)=\bar{\eta}(t)+A(t) \tag{5.7}$$

is the total time-dependent force on η. The fact that Λ depends on only the combination $t-\beta/v$ follows from the discussion in the preceding section. Since Λ contains all frequencies which are harmonics of v, if $\tilde{A}(t)=\tilde{A}_0e^{-i\omega t}$, η will contain all frequencies of the form $\omega+nv$. We can thus write

$$\eta(\beta,t)=\sum_n \eta_n(\beta)e^{(-i\omega-ivn)t} \ . \tag{5.8}$$

However, from the β dependence of Eq. (5.5), it can be seen that

$$\eta_n(\beta)=e^{in\beta}\eta_n \ , \tag{5.9}$$

where $\eta_n\equiv\eta_n(\beta=0)$. Thus, as before the behavior for all β can be obtained from a single $\beta=0$. From Eqs. (5.8) and (5.9) it follows that

$$\begin{aligned}(1/2\pi)\int d\beta\eta(\beta,t)&=\eta_0e^{-i\omega t}\\ &=e^{-i\omega t}\langle\eta(\beta=0,t)e^{i\omega t}\rangle_t\end{aligned} \tag{5.10}$$

so that $\bar{\eta}(t)$ will (as should be expected) contain only the driving frequency ω. We define the "local" response function $K(h,\omega)$ by

$$\eta_0(h,\omega)=K(h,\omega)\tilde{A}(\omega) \ . \tag{5.11}$$

The self-consistency condition then gives for the total polarizability $\chi(\omega)$ defined by

$$\bar{\eta}(\omega)=\chi(\omega)A(\omega) \ , \tag{5.12}$$

$$\chi(\omega)=\frac{\bar{K}(\omega)}{1-\bar{K}(\omega)} \ , \tag{5.13}$$

with

$$\bar{K}(\omega)=\int dh\,\rho(h)K(h,\omega) \ . \tag{5.14}$$

From the equation of motion Eq. (5.5) for η it can be readily shown that

$$K(h,\omega)=\int_0^\infty dt\,e^{i\omega t}\int\frac{d\beta}{2\pi}\exp\left[-\int_0^t\Lambda\left[h,t'-\frac{\beta}{v}\right]dt'\right] \ . \tag{5.15}$$

Note that the response, $\bar{K}(\omega=0)$, of $\bar{\eta}$ to a zero frequency $\tilde{A}$ can be immediately obtained since it is equivalent to a change of time $t\to t+\tilde{A}_0/v$. This results in $\bar{\phi}(t)$ changing from vt to $v(t+\tilde{A}_0/v)$ and hence yields $\bar{\eta}=\tilde{A}_0$. This translation mode in time implies that $\bar{K}(\omega=0)=1$. In the limit of low frequency, we expect that

$$-i\omega\chi(\omega)\approx\frac{dv}{dF} \ , \tag{5.16}$$

i.e., the differential conductivity. This yields for small ω,

$$\bar{K}(\omega)=1+i\omega\left[\frac{dv}{dF}\right]^{-1}+O(\omega^2) \ . \tag{5.17}$$

A. Stability

Stability of the uniformly moving solution is equivalent to the absence of poles in $\chi(\omega)$ in the upper half-plane. Poles in χ could come either from $\bar{K}$ being equal to 1 (as at $\omega=0$) or from poles in $\bar{K}$. Both possibilities can be excluded by examining Eq. (5.15) for $K(h,\omega)$. If ω is in the upper half-plane then $|e^{i\omega t}|<1$ for all $t>0$. This implies that for all ω in the upper half-plane (and also on the real axis away from $\omega=0$) that

$$|K(h,\omega)| < |K(h,\omega=0)| \tag{5.18}$$

and hence

$$|\bar{K}(\omega)| < |\bar{K}(0)| = 1 . \tag{5.19}$$

To prove that $\bar{K}$ has no poles in the upper half-plane, we need only show that $\langle \Lambda(t) \rangle$ is positive since this implies that the time integral in Eq. (5.15) is convergent for all ω with positive imaginary part. In fact, since the pinning potential on one phase is $-h\cos(\phi-\beta)$ one expects that $\langle h\cos(\phi_v-\beta)\rangle_t$ will be positive and hence that $\langle \Lambda(t)\rangle_t > 1$ for all β,h. This can be proven in a few lines from the equation of motion. Hence $\bar{K}(\omega)$ has no poles with $\mathrm{Im}\omega > -1$ and a single disturbed phase will decay to its steady-state solution with an average rate *faster* than in the absence of pinning.

Thus far we have strictly only shown that the uniformly moving solution is stable to *uniform* perturbations. However, it is straightforward to extend the conclusion to all small perturbations. Consider a perturbation of field A_j on the jth phase. Then, with the local response defined by

$$K_j(\omega) = \int_0^\infty dt\, e^{i\omega t} \exp\left[-\int_0^t \Lambda\left[h_j, t' - \frac{\beta_j}{v}\right] dt'\right] , \tag{5.20}$$

the total response function is

$$\chi_{ij}(\omega) = \frac{d\phi_i(\omega)}{dA_j(\omega)} = \frac{K_i(\omega)K_j(\omega)}{N[1-\bar{K}(\omega)]} + K_i(\omega)\delta_{ij} . \tag{5.21}$$

The Fourier transform of $\chi_{ij}(\omega)$ cannot have any parts growing with time by the arguments given above. Therefore, the uniformly moving solution is locally stable to all perturbations.

B. Critical behavior of polarizability

We now consider the form of the uniform polarizability $\chi(\omega)$ in the critical region at low frequencies near to threshold. It is convenient, as above, to focus on the response, $\eta(t)=\eta(\beta=0,h,t)$ of a single phase to the combined force, $\tilde{A}_0 e^{-i\omega t}$, of the applied field and the mean field. As long as the frequency is small compared to 1, for most of each period the $d\eta/dt$ term in the linearized equation of motion for η will be small, and an adiabatic approximation can be made as in the preceding section. When the time is far from the singular point t_s, $\Lambda(t)$ will be of order 1 and varying on a time scale v^{-1}. Thus, for most of each period, η can be approximated by an adiabatic term plus corrections of order ω and v:

$$\eta(t) \approx \frac{\tilde{A}_0 e^{-i\omega t}}{\Lambda(t)} + O(\omega, v) . \tag{5.22}$$

As in the case of dc response, this adiabatic approximation breaks down near to t_s. In this regime we can again rescale the equations of motion by Eq. (4.33) to give [with X in Eq. (4.33) here called $\tilde{X}$ to avoid confusion with the polarizability]

$$\frac{d\tilde{X}}{d\tau} = \tilde{X}^2 + \tau + \tilde{a}_0 e^{-i\Omega\tau} , \tag{5.23}$$

where the (h-dependent) scaled total ac force $\tilde{a}_0$ and frequency Ω are given by

$$e^{-i\omega t_s}\tilde{A}_0 = v^{2/3} C_h \tilde{a}_0 \tag{5.24}$$

and

$$\omega = \frac{v^{1/3}}{C_h}\Omega \tag{5.25}$$

with C_h given by Eq. (4.34). The dominant effect of a small $\tilde{a}_0$ in Eq. (5.23) is to shift the scaled time at which $\tilde{X}$ diverges, from τ_d by an amount proportional to $\tilde{a}_0$. This will give rise to a contribution to η_0 proportional to $\tilde{a}_0$. From the scale of the variables, this will be of the form

$$[\eta_0(h)]_{\mathrm{singular}} = \frac{v}{2\pi}[\phi_r(h) - \phi_s(h)] \left[\frac{C_h}{v^{1/3}}\right] \left[\frac{\tilde{A}_0}{v^{2/3}C_h}\right] Y\left[\frac{C_h\omega}{v^{1/3}}\right] , \tag{5.26}$$

where the factor of $v/2\pi$ comes from averaging η over one period and Y is a function only of Ω (see detailed derivation in Appendix B). This singular part of the response will dominate at low frequencies near threshold (except for a term equal to $\tilde{A}_0$ from the adiabatic part) and yield a scaling form for the polarizability

$$\chi(\omega,f) \approx f^{\zeta-1-\mu}\,\Xi\left[\frac{\omega}{f^\mu}\right] , \tag{5.27}$$

where $v \sim f^\zeta$ with $\zeta=\frac{3}{2}$ has been used. In terms of f, the characteristic frequency scale is $f^{1/2}$, i.e., $\mu=\frac{1}{2}$.

The scaling function Ξ depends on the distribution $\rho(h)$, of pinning strengths and has a simple form in the limit of small values of its argument:

$$\Xi(y) \approx \frac{\frac{3}{2}BF_T^{1/2}}{-iy} \tag{5.28}$$

since $-i\omega\chi(\omega) = dv/dF$ for small frequencies.

The behavior of the scaling function for large values of its argument is more subtle. At sufficiently high frequencies (i.e., large compared to 1), the time derivative term in Eq. (5.5) will dominate and hence $K(h,\omega)$ and $\chi(\omega)$ will behave as $1/-i\omega$ independent of f. It is clear that this is not consistent with the scaling form Eq. (5.27). For very large ω, the singular part of $\chi(\omega)$ thus does not dominate. A discussion of the large y dependence of $\Xi(y)$ and a derivation of the scaling form is contained in Appendix B.

The results of this Appendix B Eq. (B10), imply that the scaling function Ξ is extremely nonuniversal, depending on the details of the pinning strength distribution, $\rho(h)$. This surprising lack of universality is discussed briefly at the end of the next section.

From the scaled form of $\tilde{A}_0$ in Eq. (5.24), it is apparent that the total response will be linear only if $\tilde{A}_0 \ll v^{2/3} \sim f^1$. This is equivalent to the condition that the applied ac field A_0 be much smaller than the distance of the dc field from threshold, $F - F_T$. It is important to note that the linear ac response near threshold has frequency dependence on a scale $\Omega \sim f^{1/2}$ which is much larger than the intrinsic frequency of the motion which is just $v \sim f^{3/2}$. This appearance of two singular frequency scales near threshold is in striking contrast to the behavior for the single-phase case discussed in Sec. II.

Far above threshold, it can be seen from perturbation theory that the characteristic frequency scale of the ac response will become of order v since the motion at high velocities involves only a few harmonics of v.

VI. RESPONSE BELOW THRESHOLD IN MEAN-FIELD THEORY

In Sec. IV, it was shown that in the strong-pinning case below threshold there exist a large number of metastable states. Thus, one should expect that many properties of the system below threshold will be dependent on the past history of the system. In this section we study some examples of this hysteretic behavior and the concomitant breakdown of naive linear-response theory. In particular, we show that there will generally be regimes in which the zero-frequency limit of the ac polarizability, $\chi(\omega)$ will *not* equal the dc polarizability $\chi_0 = d\bar{\phi}/dF$.

A. dc response

Since many of the results will depend on details of the distribution, $\rho(h)$, of pinning strengths, we will, for definiteness, concentrate on two examples which exhibit most of the interesting features.

Case 1:

$$\rho(h') = \delta(h' - h), \quad h > 1 \tag{6.1}$$

and *case 2:*

$$\rho(h) = \begin{cases} (\hat{h} - h_0)^{-1} & \text{for } h_0 < h < \hat{h} \\ 0 & \text{otherwise} \end{cases} \tag{6.2}$$

with $\hat{h} > h_0 > 1$. In addition, we will primarily restrict the discussion to states which can be reached by adiabatically increasing or decreasing the field from the absolute ground state at $F = 0$ which is unique up to the overall phase $\bar{\phi}$.

Case 1. Constant h. We first discuss the single h case. As introduced in Sec. IV, there will be a multivalued function $\tilde{\Upsilon}$ of $\alpha = -\beta + \bar{\phi} + F$ which describes the possible static locally stable solutions $\phi_0(\beta, \bar{\phi}, F) = \tilde{\Upsilon}(\alpha) + \bar{\phi} + F$. The absolute ground state at $F = 0$ can be found for a given $\bar{\phi}$ (which we take to be zero) by choosing for each β, ϕ_0 equal to the absolute minimum of the local potential $W(\phi)$ since the total energy is just given by

$$\mathscr{H} = \sum_j W_j(\phi_0(\beta_j, h_j); \bar{\phi}, F) \ . \tag{6.3}$$

This ground state corresponds to the choice of the branch of $\tilde{\Upsilon}$ shown in Fig. 3 which is a continuous odd function of α on $(\alpha_M - 2\pi, \alpha_M) = (-\pi, \pi)$. The phase $\phi_0(\beta)$ in this state will be a continuous function of β on $(-\pi, \pi)$. As the field is increased adiabatically, we expect that at least initially ϕ_0 will be a smooth function on this same domain since a small field will not cause what were the absolute minima of W to disappear. The relevant branch of $\tilde{\Upsilon}$ in a small field is thus the same branch as at $F = 0$ but with α now in the range $(\alpha_M - 2\pi, \alpha_M)$ with

$$\alpha_M = \pi + \bar{\phi}(F) + F \ . \tag{6.4}$$

For the constant-h case, this will be true all the way up to the threshold field at which point α_M will be equal to the singular point α_s at which this branch of $\tilde{\Upsilon}$ disappears [at this point the chosen branch will be exactly $\Upsilon(\alpha)$ as defined in Sec. IV; see the heavy solid line in Fig. 3].

The polarizability $\chi_0 = d\bar{\phi}/dF$ can be obtained straightforwardly by differentiating the expression

$$\phi_0(\beta, \bar{\phi}, F) = \bar{\phi} + F + \tilde{\Upsilon}(\bar{\phi} + F - \beta) \tag{6.5}$$

with respect to F and then averaging with respect to β or, equivalently, α:

$$K_0 = \frac{\chi_0}{1+\chi_0} = \frac{1}{2\pi} \int_{\alpha_M - 2\pi}^{\alpha_M(F)} \left[1 + \frac{d\tilde{\Upsilon}}{d\alpha} \right]$$
$$= 1 + \frac{1}{2\pi} [\tilde{\Upsilon}(\alpha_M) - \tilde{\Upsilon}(\alpha_M - 2\pi)] \tag{6.6}$$

with the branch of $\tilde{\Upsilon}$ chosen as in the above discussion.

Even at threshold, when $\alpha_M = \alpha_s$, $(1/2\pi)[\tilde{\Upsilon}(\alpha_M) - \tilde{\Upsilon}(\alpha_s)]$ will be strictly negative and hence $\chi_0(F \to F_T)$ is a *finite* constant. There will be a singularity, however, in the second derivative of $\bar{\phi}$,

$$\frac{d^2\bar{\phi}}{dF^2} = \frac{d\chi_0}{dF} = \frac{(1+\chi_0)^3}{2\pi} \left[\frac{d\tilde{\Upsilon}}{d\alpha} \right]_{\alpha_M(F)-2\pi}^{\alpha_M(F)} , \tag{6.7}$$

coming from the upper limit of the integral in Eq. (6.6). Near to α_s and $\Upsilon_s = \Upsilon(\alpha_s) < 0$, Υ behaves as

$$\Upsilon(\alpha) - \Upsilon_s \approx -\left[\frac{2(\alpha - \alpha_s)}{\Upsilon_s} \right]^{1/2} \tag{6.8}$$

and

$$\alpha_M(F) \approx \alpha_M(F_T) - (F_T - F)[\chi_0(F_T) + 1] \ , \tag{6.9}$$

so that

$$\frac{d\chi_0}{dF} \sim |f|^{-1/2} \tag{6.10}$$

as $F \to F_T^-$.

As F is decreased, the phases will follow the same branch as for F increasing. Thus, for case 1, while there are many metastable states, a particular special state (up to overall $\bar{\phi}$) can be reached from *any* of them by first increasing F to F_T, which forces the system into the unique state which exists at threshold and then decreasing F to the desired value. Once this has been done, none of the other metastable states can be reached by adiabatically changing F (they can of course be reached by introducing a nonuniform field) and all phases with the same β will have the same ϕ_0. This property and the finiteness of $\chi_0(F_T)$ are both pathologies of the system with all pinning strengths the same, essentially arising from the fact that the local threshold field at which a single phase first jumps out of a disappearing minimum into another is the same for all of the phases.

Case 2. Flat distribution of h's. More generic behavior is found by considering a distribution of pinning strengths. In this case there will be a series of functions $\tilde{\Upsilon}(\alpha,h)$ yielding the static phases for each h. The important lowest branch $\Upsilon(\alpha,h)$ of each $\tilde{\Upsilon}$ will have a singularity at a point $\alpha_s(h)$ which is an increasing function of h.

We again consider states which can be adiabatically reached from the ground state at $F=0$. As F increases from zero, the relevant branches will be the same as for the single h case except that now α_M can be larger than $\alpha_s(h)$. The β averaged local static response:

$$K_0(h)=\left\langle \frac{d\phi_0(\beta,h,f)}{d(\bar{\phi}+F)} \right\rangle_\beta = 1+\frac{1}{2\pi}[\tilde{\Upsilon}(\alpha_M,h)-\tilde{\Upsilon}(\alpha_M-2\pi,h)] \tag{6.11}$$

can again be obtained by differentiating Eq. (6.5) and the full static response from

$$\chi_0=\frac{\bar{K}_0}{1-\bar{K}_0} \tag{6.12}$$

with

$$\bar{K}_0=\int dh\,\rho(h)K_0(h)\ . \tag{6.13}$$

However, we now have $K_0(h)=1$ for all h,F satisfying $\alpha_M(F)\ =\bar{\phi}+\pi+F>\alpha_s(h)$. The threshold is determined by

$$\alpha_M(F_T)=\alpha_s(\hat{h})\ , \tag{6.14}$$

i.e., when the branch of $\tilde{\Upsilon}(\alpha,h)$ exactly corresponds to $\Upsilon(\alpha,h)$ for all h's for which $\rho(h)>0$.

Once $\alpha_M(F)$ is greater than $\alpha_s(h)$ for a particular h, as F is increased some of the phases with this h will jump by an amount $\Upsilon(\alpha_s-2\pi)-\Upsilon(\alpha_s)$ and the local response averaged over β will be equal to 1. The pinning strength $h_1(F)$ for which the phases are just about to start jumping as F is increased, is given implicitly by

$$\alpha_s[h_1(F)]=\alpha_M(F)\ . \tag{6.15}$$

Near to threshold, $h_1(F)$ will approach $\hat{h}$. The only h's contributing nonzero amounts to $1-\bar{K}_0$ are $h_1<h<\hat{h}$. For $h_1\rightarrow\hat{h}$,

$$1-\bar{K}_0=\frac{1}{2\pi}\int_{h_1(F)}^{\hat{h}} dh\,\rho(h)\Delta_h\ , \tag{6.16}$$

where we have defined the jump, Δ_h in $\Upsilon(h)$ at $\alpha_s(h)$ by

$$\Delta_h=\Upsilon(\alpha_s^+)-\Upsilon(\alpha_s^-)\ . \tag{6.17}$$

For the case 2 under consideration, Eq. (6.16) yields (since $\Delta_{\hat{h}}>0$)

$$\frac{1}{1+\chi_0(F)}=1-\bar{K}_0(F)\sim\hat{h}-h_1(F)\ . \tag{6.18}$$

Differentiating Eq. (6.15) with respect to F, using Eqs. (6.14) and (6.18) and noting that $d\alpha_s(h)/dh$ goes to a constant at $h=\hat{h}$, we obtain

$$\frac{dh_1}{dF}\sim\frac{1}{\hat{h}-h_1(F)} \tag{6.19}$$

which can be integrated to yield that

$$[(\hat{h}-h_1(F)]^2\sim(F_T-F) \tag{6.20}$$

and hence from Eq. (6.18),

$$\chi_0\sim|f|^{-1/2}\ . \tag{6.21}$$

It is tempting to associate a critical exponent γ with the singular part of the polarizability near threshold:

$$\chi_0^{\text{singular}}\sim|f|^{-\gamma}\ . \tag{6.22}$$

As threshold is approached from the $F=0$ ground state, $\gamma=0$ for case 1 (Ref. 31) and $\gamma=+\frac{1}{2}$ for case 2. It can be shown that for this past history, γ can be any value between 0 and 1 for $\hat{h}<\infty$ and any value from 1 to ∞ for $\hat{h}=\infty$. For all cases the behavior near threshold is controlled by the form of $\rho(h)$ for $h\rightarrow\hat{h}$. Thus, the *singularity* in χ_0 as $F\rightarrow F_T^-$ is nonuniversal in contrast to the behavior in the preceding section above but near threshold. We note, however, that the nonuniversality of the *scaling function* for the frequency- and field-dependent polarizability *above* threshold may be related to the inherent nonuniversality *below* threshold.

B. Hysteretic behavior

We now briefly discuss some of the possible hysteretic behavior. For case 1, almost all choices of the initial distribution of the phases in the possible minima of their local potentials for $F<F_T$ will result in $\chi_0\rightarrow$const at F_T. However, the *magnitude* of the singularity, i.e., $\chi_0(F_T)$ will depend on the past history, although only for the first approach to threshold as mentioned above. For case 2, and in general, the behavior will be much more strongly history dependent.

For definiteness, we consider decreasing F slightly from F_T. The phases (for a given h) with $\alpha\gtrsim\alpha_s(h)$ will *not* go back to the lower branch of $\tilde{\Upsilon}$ from which they came as F was increased to F_T from the $F=0$ ground state. Since we have chosen the smallest h, h_0, to be greater than 1, $\tilde{\Upsilon}$ will have at least two branches for each h. For a finite range of F, $F_l<F<F_T$ about F_T, the field can thus be decreased without any phases jumping to the lower branch: i.e., the lower limit of the range of α, $\alpha_M-2\pi$ will be greater than $-\alpha_s(h)$ for all the h's. From Eq. (6.11) it follows that $K_0^{\downarrow}(h)$ (where the arrow denotes the direction of charge of F) will be less than 1 for each h and hence that $\chi_0^{\downarrow}(F)$ will be finite in this range of F as F is decreased from F_T. If F is increased back to F_T again, χ_0 will remain finite, in contrast to the divergence of $\chi_0^{\uparrow}$ when coming from the $F=0$ ground state. With this past history χ_0 will not depend on whether F is being increased or decreased. Another history for which χ_0 is reversible in some range will occur for case 2 for F increasing and

decreasing starting from the $F=0$ ground state but remaining less than some value F_u. The relevant criteria for reversibility at a given value of F and with a given past history is that $d\widetilde{\Upsilon}(h,\alpha)/d\alpha$ has no δ-function contributions (for any h or α) for the branches on which the phases lie. This is just the statement that no phases jump on either increasing or decreasing F.

If in case 2 the field is decreased from threshold below F_l or increased from the $F=0$ ground state above F_u, then χ_0 will depend on the direction of change of F but will always be finite, i.e., there will be a $\chi_0^{\uparrow}$ and $\chi_0^{\downarrow}$ which are different (they may be accidentally equal at isolated points). From Eq. (6.11) it can be seen that the increasing (decreasing) polarizability $\chi_0^{\uparrow}$ ($\chi_0^{\downarrow}$) will only diverge if *all* the phases with $\alpha=\alpha_s(h)$ [$\alpha=-\alpha_s(h)$] lie on the $\Upsilon(-\Upsilon)$ branch of $\widetilde{\Upsilon}$. This will occur only right at threshold and $\chi_0^{\uparrow}$ will only diverge as $F\to F_T$ with certain special past histories. In particular, if F follows the path $F_T\to F_{\min}\to F_T$ it will diverge only if $F_{\min}=-F_T$. The divergence of $\chi_0^{\uparrow}$ on this path with $F_{\min}=-F_T$ will have the same exponent but generally a *larger* amplitude than the divergence of $\chi_0^{\uparrow}$ as $F\to F_T$ from the $F=0$ ground state. The behavior for case 2 with F increasing to F_T from the $F=0$ ground state and then decreasing below F_l and back to F_T is sketched in Fig. 5.

The hysteretic behavior for case 2 is reasonably generic, however, if the minimum value of the field h_0, had been *smaller* than 1, then there would be hysteresis at all fields with all past histories and $\chi_0^{\uparrow}(F)$ and $\chi_0^{\downarrow}(F)$ would generally be different everywhere.

C. ac response

We now turn to the linear ac response below threshold. In contrast to the dc response discussed above, at finite-frequency phases cannot jump back and forth from one minimum to another. Thus, the ac response will not include δ-function contributions to $d\phi/dt$.

The linearized equation of motion for a deviation of the phases $\eta(\beta,h,t)$ about $\phi_0(\beta,h)$ driven by an ac field $A(t)$ are

$$\frac{d\eta}{dt}=[-1-h\cos(\phi_0-\beta)]\eta+\widetilde{A}(t)\ , \tag{6.23}$$

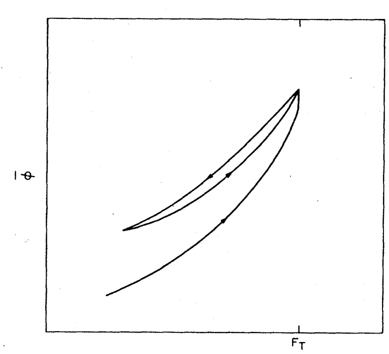

FIG. 5. Possible hysteretic behavior for the average phase $\bar{\phi}$ as the field F is increased adiabatically from the $F=0$ ground state to F_T and then down and back up again as indicated by the arrows.

where $\widetilde{A}(t)=A(t)+\overline{\eta}(t)$ as in Sec. V. The β-averaged response $K(h,\omega)$ to $\widetilde{A}(\omega)$ is given by

$$K(h,\omega)=\int_{\alpha_M-2\pi}^{\alpha_M}\frac{d\alpha}{2\pi}\frac{1}{-i\omega+\{1+h\cos[\widetilde{\Upsilon}(\alpha,h)+\alpha]\}}\ , \tag{6.24}$$

where we have again parametrized β by α (with maximum value α_M) and assumed for simplicity that all of the phases with the same β and h have equal values. We can take the zero-frequency limit immediately, and using the equilibrium condition which gives

$$1+h\cos[\widetilde{\Upsilon}(\alpha)+\alpha]=\frac{1}{1+d\widetilde{\Upsilon}/d\alpha}\ , \tag{6.25}$$

we have

$$K(h,\omega=0)=\int_{\alpha_M-2\pi}^{\alpha_M}\frac{d\alpha}{2\pi}\left[1+\frac{d\widetilde{\Upsilon}}{d\alpha}\right]. \tag{6.26}$$

Naively, this looks like it will immediately yield the result $K_0(h)$ from Eq. (6.11). However, if the range of integration of α on the chosen branch of $\widetilde{\Upsilon}$ includes a point α_s (or similarly $-\alpha_s$) at which the branch changes, then the δ-function contribution to $d\widetilde{\Upsilon}/d\alpha$ at that point should *not* be included, in contrast to the dc case. The integral over α in Eq. (6.26) will yield

$$K(h,\omega=0)=1+\widetilde{\Upsilon}(\alpha_M)-\widetilde{\Upsilon}(\alpha_s^+)+\widetilde{\Upsilon}(\alpha_s^-)-\widetilde{\Upsilon}(\alpha_M-2\pi)\ . \tag{6.27}$$

If the branches of $\widetilde{\Upsilon}$ corresponding to the state under consideration change at α_s, then $\widetilde{\Upsilon}(\alpha_s^+)-\widetilde{\Upsilon}(\alpha_s^-)=\Delta_h>0$ and $K(h,\omega=0)<K_0^{\uparrow}(h)$.

For any state, $K(h,\omega=0)$ will be less than 1 for all $h>1$. The *zero-frequency limit* of the ac polarizability $\chi(\omega=0)$ will then always be *finite*. For case 1, $\chi(\omega=0)$ will be equal to the static χ_0 along the paths coming from F_T or the $F=0$ ground state, since $\overline{K}_0$ in that case did not include contributions from phase jumps. For case 2 (and generically) χ_0 will equal $\chi(\omega=0)$ only along paths with no hysteresis. Generally, $K(h,\omega=0)\le K_0^{\uparrow}(h),\ K_0^{\downarrow}(h)$ and since all K's are between 0 and 1, it follows that

$$\chi(\omega=0)\le\chi_0^{\uparrow},\chi_0^{\downarrow}\ . \tag{6.28}$$

Even though $\chi(\omega\to 0)$ is finite, it is nevertheless singular near threshold and at any point at which there is hysteretic behavior.

We first examine case 1 near threshold, at low frequencies. The maximum α, α_M, will be near to α_s:

$$\alpha_s-\alpha_M\sim|f| \tag{6.29}$$

and

$$1+h\cos[\Upsilon(\alpha)+\alpha]\sim(\alpha_s-\alpha)^{1/2}\ . \tag{6.30}$$

Thus, the form of the singular part of the integral in Eq. (6.24) will be for $|\omega|,\ |f|\ll 1$,

$$K^{\text{singular}}(\omega,f)\sim\int_{|f|}^{1}\frac{d\alpha'}{-i\omega+\sqrt{\alpha'}}\ , \tag{6.31}$$

where $\alpha'=\alpha_s-\alpha$. This integral will yield a scaling function for K (ignoring all constants):

$$\chi^{\text{singular}}(\omega,f)\sim K^{\text{singular}}(\omega,f)\sim |f|^{1/2}\left[-1-\frac{i\omega}{|f|^{1/2}}\ln(|f|^{1/2}-i\omega)\right]. \tag{6.32}$$

Since the nonsingular part of K goes to a constant as $\omega,f\to 0$, the singular part of χ will be a constant multiple of K^{singular}. From Eq. (6.32) we conclude that for case 1,

$$\chi^{\text{singular}}(\omega=0,f)\sim |f|^{1/2} \tag{6.33}$$

and $(d\chi/dF)(\omega=0,f)\sim |f|^{-1/2}$ [which should be expected since for this case $\chi(\omega=0)=\chi_0$]. Exactly *at* threshold,

$$\chi^{\text{singular}}(\omega,f=0)\sim -i\omega\ln(-i\omega) \tag{6.34}$$

so that the conductivity at threshold has the form

$$\sigma(\omega)=-\omega\,\text{Im}\chi^{\text{singular}}\sim\omega^2\ln|\omega|\ . \tag{6.35}$$

[Note that the nonsingular part of $\sigma(\omega)$ will go as $0+C\omega^2$, hence the singular part dominates.] The real part of χ will have a $|\omega|$ cusp.

We now consider what happens more generally when there is a range of h's as in case 2. At any point, with $|F|\neq F_T$ at which there is hysteresis, i.e., $\chi_0^{\uparrow}\neq\chi_0^{\downarrow}$, there will be phases which are just about to jump to another minimum as F is increased or decreased. Thus, for some *finite* fraction of h's there will be phases distributed up to $\alpha_s(h)$ (or $-\alpha_s$) on the branch of $\widetilde{\Upsilon}(\alpha,h)$ which ends at α_s. For these $\hat{h}$, $K_h(\omega)$ will behave the same way at low frequencies as in case 1 *at* threshold. We can therefore conclude after averaging over h that in *any* state with hysteresis, the conductivity will be singular as $\omega>0$ with

$$\sigma(\omega)\sim\omega^2\ln|\omega| \tag{6.36}$$

and the real part of χ will have a $|\omega|$ cusp.

For case 2, if we increase F from the $F=0$ ground state, the conductivity will be analytic at $\omega=0$ [i.e., $\sigma(\omega)\sim\omega^2$ and $\chi_0^{\uparrow}=\chi_0^{\downarrow}-\chi(\omega=0)$] until F reaches a critical value F_u at which $\alpha_M=\alpha_s(h_0)$. As F is further increased, σ will behave nonanalytically as in Eq. (6.36) and $\chi_0^{\uparrow}>\chi_0^{\downarrow}=\chi(\omega=0)$. The equality of $\chi_0^{\downarrow}$ and $\chi(\omega=0)$ along this path is simply due to the *absence* of phase jumps as F is decreased. As threshold is approached $\chi_0^{\uparrow}$ will diverge while $\chi_0^{\downarrow}$ and $\chi(0)$ remain finite.

D. Nonlinear response

In regimes in which the low-frequency linear response is pathological (as discussed above) one should expect that the nonlinear response will also be singular at low frequencies. This will arise physically from the large response of phases which lie in very flat minima of their potentials, i.e., those that would jump if F were changed slightly. For definiteness we consider the quadratic dc response to a uniform $A\cos\omega t$ force at low frequencies.

From the leading nonlinear terms in the equation of motion Eq. (6.23) for the phase deviations η, it can be readily shown that the quadratic dc response of a phase with $\alpha\lesssim\alpha_s(h)$ on the lowest branch of $\widetilde{\Upsilon}$ has the schematic form

$$\eta(\alpha,\omega'=0)\sim A^2\frac{2}{(\alpha_s-\alpha)^{1/2}}\,\frac{1}{-i\omega+(\alpha_s-\alpha)^{1/2}}\,\frac{1}{i\omega+(\alpha_s-\alpha)^{1/2}}\ . \tag{6.37}$$

Integration of this over α up to α_s will yield the form of $\overline{\eta}(\omega'=0)$:

$$\overline{\eta}(\omega'=0)\sim A^2\frac{1}{|\omega|}\ , \tag{6.38}$$

i.e., a divergent nonlinear polarizability.

In a regime with nonsingular linear response [i.e., $\chi_0^{\uparrow}=\chi_0^{\downarrow}=\chi(\omega=0)$], the quadratic response at zero frequency will just be proportional to $d\chi_0/dF$. We see, however, that in the hysteretic regime, the quadratic response is infinite as $\omega\to 0$ and hence definitely larger than the finite, although history and definition dependent $d\chi_0/dF$.

E. Two-sided scaling and universality

In conventional critical phenomena, scaling functions can usually be found which describe the behavior on *both* sides of the transition. This is primarily due to the existence of ordering fields (e.g., the magnetic field in a ferromagnet) which destroy the transition. An exception is spin glasses for which, because of the existence of many order parameters, scaling functions above and below T_c behave quite differently.

The threshold problem of interest here has some features in common with mean-field theory of spin glasses—universal exponents on one side of the transition and hysteresis and nonuniversal behavior on the other. In our case, however, there is an "ordering field" which destroys the pinned phase at least in mean-field theory. This is just the temperature, the effects of which will be discussed briefly in Sec. IX. It is possible that some two-sided scaling functions can be found by considering the effects of small but nonzero temperature near threshold. This possibility merits future study.

An additional unusual feature of the scaling near threshold, is the apparent complete lack of universality of scaling functions such as Ξ. Although the fundamental reason for this is not clear, it may be related to the absence of two-sided scaling and the large degeneracy on the pinned side. However, the presence of two different critical frequency scales seems to play a role. Both this and the nonuniversality arise in mean-field theory from the presence of *two* singularities in the functions $\Upsilon(\alpha,h)$ which determine the local response: they have jumps *and* square-root cusps with independent magnitudes.

VII. SHORT-RANGE INTERACTIONS: THRESHOLD FIELD AND WEAK PINNING

Until now, all of the detailed discussion has concerned the mean-field limit in which we have seen that analytic calculations can be performed. In the next two sections we attempt to use the intuition gleaned from the mean-field results along with scaling arguments to draw some conclusions about the behavior of systems with short-range interactions. The discussion will be much more qualitative, and in some parts, quite speculative.

Some of the features of mean-field theory, in particular the existence of a threshold field with nontrivial critical behavior for sufficiently strong pinning, should persist with short-range interactions. In this section we discuss the behavior of the threshold field in various dimensions as a function of the strength of the pinning. As mentioned previously, we will assume that below a threshold field (when it is nonzero) there exist only stationary solutions at long times and above the *same* threshold field there exist only uniformly moving solutions at long times (in an infinite system). In contrast to mean-field theory, the coexistence of uniformly moving and stationary solutions has not been explicitly ruled out.

There are two main issues concerning the extent to which mean-field theory can be carried over to systems with short-range interactions. The first, and eventually perhaps the most interesting, concerns the upper critical dimension, d_c, above which, by analogy with critical phenomena, the exponents (e.g., ζ) describing the threshold will obtain their mean-field values which should be valid in the limit of infinite dimensionality. Unfortunately, we will have little to say about this issue here except to note that it is independent of the second important issue, which is the question of how much the *phase diagram* in various dimensions as a function of F and $\hat{h}$ resembles the mean-field phase diagram. This latter issue primarily concerns the lower critical dimension of the multicritical point which separates the regimes with and without a threshold in mean-field theory.

We thus return to the short-range model introduced in Sec. III. It is convenient to work with a continuum version of the model with a length cutoff of order 1 which is the microscopic length scale—roughly the distance between impurities. The equation of motion is

$$\frac{\partial \phi}{\partial t} = \nabla^2 \phi - h(\mathbf{r}) \sin[\phi(\mathbf{r}) - \beta(\mathbf{r})] + F , \tag{7.1}$$

where we have normalized the stiffness to 1. We are interested in the limit of weak pinning, i.e., where the characteristic $h(\mathbf{r})$ is small.

From the weak-pinning perturbation theory discussed in Sec. III, it appears that four dimensions plays a special role. In particular, the infrared singularities that make the zero-field perturbation theory divergent [Eq. (3.1)] do not appear in $d > 4$. Moreover, the perturbative corrections to the high-field behavior of the velocity given by Eq. (3.2) are not dominated by long wavelengths in $d \geq 4$. By examining the high-field perturbation theory, it can be seen that the general form of an nth-order term for the velocity will be schematically

$$h^{2n} \int_{\mathbf{q}_1} \cdots \int_{\mathbf{q}_n} \mathrm{Im} \left[\prod_{j=1}^{2n-1} \frac{1}{iFm_j + \mathbf{p}_j^2} \right] , \tag{7.2}$$

where the $\mathbf{p}_j$ are linear combinations of the $\mathbf{q}_i$, the m_j are nonzero integers, and the h^{2n} includes various combinations of moments of $\rho(h)$ of total order $2n$. (Note, the terms with *any* $m_j = 0$ cancel each other. This can be shown by methods similar to those of Efetov and Larkin.[32]) Because of the imaginary part, each term will be F times an even function of F. In more than four dimensions, all the integrals should be convergent by power counting and, at least naively, the limit $F \to 0$ should be nonsingular. This suggests that for $d > 4$, a critical strength of the pinning is needed before perturbation theory breaks down, even as $F \to 0$. This would imply linear response (i.e., $v \sim \sigma F$) for weak pinning in accordance with mean-field theory. As we will see later, however, this naive argument is almost certainly incorrect.

In less than four dimensions, the integrals appearing in nth-order perturbation theory will generally diverge[33] for small F as $F|F|^{n(d/2-2)}$ and the velocity in this limit will have a power series of the form

$$v = F \left[1 + \sum_h a_n (h^2 |F|^{d/2-2})^n \right] , \tag{7.3}$$

with h, a characteristic value of the $h(\mathbf{r})$. The term in large parentheses, which is the natural expansion parameter, can be written in terms of F over a characteristic field $h^{4/(4-d)}$, which is small for h small. Since the velocity will presumably go to zero at a finite value of the expansion parameter, this suggests that the threshold field will scale as

$$F_T \sim h^{4/(4-d)} \tag{7.4}$$

for weak pinning in $d < 4$ in agreement with the arguments of Lee and Rice[13] and Larkin and Ovchinikov.[34] Since F appears in perturbation theory as an inverse length squared, there will be a characteristic length associated with this threshold field

$$\xi_0 \sim h^{2/(d-4)} \tag{7.5}$$

which is just the Lee-Rice pinning length.[13] From the zero-field perturbation theory, this will be the characteristic length over which the phase deviations vary by order 1, and in a sense to be made more precise below, is the relevant microscopic (or semimicroscopic) length for the threshold behavior. Arguments based on perturbation theory are potentially dangerous, so it is useful to consider more general arguments which lead to Eq. (7.5).

We now show how the weak-pinning behavior discussed above for both $d < 4$ and $d > 4$ can be derived from a renormalization-group argument. We are interested in the behavior of the model, Eq. (7.1), for weak pinning at small fields. Thus, it is natural to consider the behavior under rescaling of lengths by a factor L near to the trivial zero-pinning static fixed point at $h = 0$ and $F = 0$. If we rescale so that the form of the equation of motion remains the same then we find that time scales as L^{-2} and the renormalized field and pinning strength scale as

$$F_R(L) \sim FL^2 \tag{7.6}$$

and

$$h_R(L) \sim hL^{2-d/2} , \tag{7.7}$$

while the stiffness is kept fixed. The pinning strength rescaling comes from the fact that typically the $h(\mathbf{r})$ will act incoherently over a volume L^d yielding an effective h reduced by the square root of the volume.

Equation (7.7) implies that in $d>4$, the pinning is irrelevant, i.e., becomes weaker and weaker on long length scales relative to the stiffness, suggesting that the zero-pinning behavior (with $v \sim F$) will be valid for weak pinning in agreement with the above discussion.

In $d<4$, on the other hand, the pinning is relevant and will scale to be of the order of the stiffness at a length scale, ξ_0, which is given by Eq. (7.5). At this scale, where both the pinning strength and the stiffness are of order 1, the threshold field, *if there is one,* will also be of order 1, hence the bare threshold field will be related to the bare pinning strength by $F_T \sim h^{4/(4-d)}$ in agreement with Eq. (7.4). The scaling behavior near the trivial zero-pinning fixed point, implies that as a function of the strength of the disorder and the field, the velocity for *weak pinning* and *small fields* will, in $d<4$, have a *crossover scaling form:*

$$v \sim FD(F/h^{4/(4-d)}) \,. \tag{7.8}$$

The full scaling function $D(Z)$ in principle contains the behavior both above (Z large), below (Z small), and near ($Z \sim 1$) to the threshold field if it exists. However, *no information* on the scaling function can be obtained by studying the behavior *near* the *trivial fixed point.* The properties of this fixed point only give the *form* of the scaling function in Eq. (7.8) *not* the scaling function itself. In particular, the existence of a threshold does *not* follow from such a scaling argument, other arguments need to be given for the limits of the scaling function.

One limit of the scaling function, $Z \gg 1$ corresponding to $F \gg F_T$, can be derived from the perturbation theory discussed above. However, the behavior near threshold, which we assume occurs, will depend on the properties of the *threshold fixed point* with F^*, h^* both of order 1. A schematic renormalization-group flow diagram is shown in Fig. 6. Note the crossover away from the trivial fixed point and the separatrix which determines $F_T(h)$ running from the trivial to the threshold fixed point. The threshold critical behavior *for any h,* in particular, the exponent ζ, will be determined by the properties of the *threshold* fixed point. The pinning length ξ_0 is just the length scale above which the behavior will be dominated by this fixed point. Thus, it serves as the appropriate microscope length (analogous to the BCS superconducting coherence length ξ_0) for analyzing the threshold behavior, in particular, for determination of the upper critical dimension d_c above which the threshold fixed point yields mean-field critical behavior.

Since it is easy to convince oneself that for sufficiently strong pinning there will exist a nonzero threshold, and since the pinning in $d<4$ tends to grow with increasing length scale, the conclusions from the naive renormalization-group arguments given above are probably valid: there always exists a threshold in $d<4$ with F_T scaling as $h^{4/(4-d)}$ for weak pinning. (This is in contrast to the incommensurate pinning problem discussed in Ref. 25, for which there is no threshold for weak pinning.)

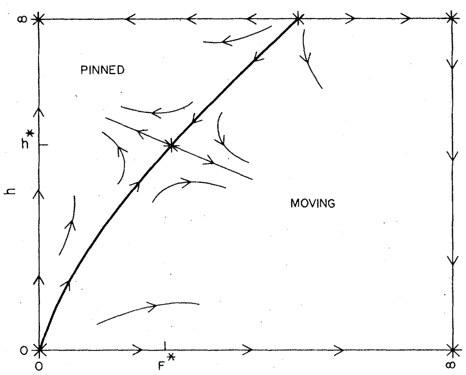

FIG. 6. Schematic renormalization-group flow diagram for short-range interactions as a function of the typical pinning strength h and applied field F. The thick line represents the threshold field $F_T(h)$. Note at its lower end the weak-pinning fixed point at $h=0$, $F=0$ which determines the threshold field for small h, at the other end the uncoupled single-phase fixed point at $h=\infty$, and in between the threshold fixed point with h^* and f^* both finite.

It might be expected that the existence of a long characteristic length scale, ξ_0, would imply that by analogy with critical phenomena (e.g., superconductivity), the critical region near F_T would be very small. However, this is probably *not* the case here, since, in a certain sense, the threshold field *itself* is a fluctuation effect controlled by the same parameter as the crossover. This is in contrast to the usual case in critical phenomena where one parameter determines the transition temperature, T_c, and another (small) parameter the crossover from the noncritical to the critical regime, with T_c finite even when the crossover parameter is arbitrarily small.

From the form of the crossover scaling, Eq. (7.8), it is apparent that the weakness of the pinning, h, only sets the overall scale for the threshold. Physically, the absence of a narrow critical region for ξ_0 large can be understood by observing that the threshold fields of isolated regions of size ξ_0^d will deviate from their average by amounts of the same order as their average. It is plausible that there may be more subtle crossover effects which *do* tend to reduce the critical region, however, it is clear that, *a priori,* one should not expect it to be particularly small.

In $d>4$, we must examine the above arguments on the irrelevancy of the pinning for small h rather more carefully. (Note, these arguments were the basis of the claim made by the author in Ref. 26 for the absence of a threshold in $d>4$ for weak pinning. I now believe these arguments to be invalid for the reasons discussed below.) As long as the renormalized probability distribution of the $h(\mathbf{r})$ is well behaved, it is probably reasonable to characterize $\rho(h)$ by a typical value $\tilde{h}$ and consider the renormalization-group flows for $\rho(h)$ to be roughly determined by those for $\tilde{h}$. However, if the behavior of $\rho(h)$ for $h \gg \tilde{h}$ becomes important, then this approximation may become invalid. To see if this occurs, we consider a *bounded* distribution of h with maximum value $\bar{h} \sim \tilde{h}$ and examine the effects of the rare configurations of $h(\mathbf{r})$

which result in large values of the renormalized h_R at some length scale. In particular, we consider a region, R, of volume l^d in which all of the preferred phases are close to a particular value, for example $-\pi/4<\beta(\mathbf{r})<\pi/4$ for all $\mathbf{r}\in R$, and all the $h(\mathbf{r})$ are of order of the characteristic value $\tilde{h}$. A region of this type will occur with probability, p, of order e^{-l^d} (we ignore all constants). At $F=0$ in the ground state (and other low-energy metastable states), the phase $\phi(\mathbf{r})$ will be near zero in all of R. If the phase at the boundary of R is pulled away from zero by a small amount by an applied field (or the effects of other regions), the effects of the boundary will decay exponentially inside R with a characteristic length $\lambda\sim\tilde{h}^{-1/2}$. If l is of order of a few times λ, this region will act as a strong-pinning site, i.e., if there is a finite concentration of such regions, then for sufficiently small fields there will be static solutions with the phase in the interior of R near its preferred value (here zero) in zero field. [If the distribution of h is *unbounded,* then the rare large h's will also act as strong-pinning sites although the estimate for the threshold given below will change if $\rho(h)$ falls off more slowly than $e^{-h^{d/2}}$.] The existence of static solutions for small F can be seen as follows.

The typical spacing between regions like R will be $L\sim p^{-1/d}$. If we assume that between these regions the pinning has a negligible effect, then the phase in a static solution will obey

$$\nabla^2\phi=-F \tag{7.9}$$

between these regions. By considering a surface integral of $d\mathbf{s}\cdot\nabla\phi$ over a region of size $\sim L$ surrounding R, we can conclude that at the surface of R, ∂R,

$$\nabla\phi\mid_{\partial R}\sim\frac{FL^d}{l^{d-1}} \,. \tag{7.10}$$

Provided $\nabla\phi\mid_{\partial R}$ is small compared to $1/\lambda$, which we take to be the same order as $1/l$, a static solution can be constructed with $\phi\sim0$ in the interior of R and $\phi\mid_{\partial R}\sim\lambda\nabla\phi\mid_{\partial R}$ with ϕ and $\nabla\phi$ continuous across the surface ∂R (similarly for the other regions like R). This implies that static solutions will exist up to a field of order $L^{-d}l^{d-2}$. For fields larger than this, the phase on the surface ∂R would have to be of order 2π and a 2π phase wall of width $\sim\lambda$ would be nucleated and propagate into R destroying the static solution.

If we pick $\lambda\sim\tilde{h}^{-1/2}$, the above argument implies that in $d>4$ the threshold field will be at least

$$F_T>e^{-\tilde{h}^{-d/2}} \,, \tag{7.11}$$

i.e., that, in contrast to the naive arguments given above (and to mean-field theory) there will be a threshold field for *all* strengths of the pinning, even if the pinning strengths are bounded. This finite threshold is caused by rare regions with anomalously coherent pinning, with spacing $L\sim e^{\tilde{h}^{-d/2}}$ which we again identify as a semimicroscopic length ξ_0. Since the threshold field is essentially singular at $\tilde{h}\to0$, it will never show up in perturbation theory in $\tilde{h}$. [We note that effects of such rare coherent regions in random thermodynamic systems are believed to cause essential singularities in the free energy, originally shown to exist by Griffiths.[35] However, this may be the first example of Griffiths's singularities crucially affecting the presence (or absence) of a transition at which some quantities have power-law singularities.]

It is unclear, *a priori,* whether the bound Eq. (7.11) on the threshold field is reasonably close to the actual threshold for $d>4$; it clearly cannot be for $d<4$, where *typical* configurations of the pinning will determine the threshold. To see if a better bound is possible, we consider a less-atypical region, R, of size l^d with a total effective field

$$\left|\int_R d^dr\,h(\mathbf{r})e^{i\beta(\mathbf{r})}\right| \,,$$

of size $h_a l^d$ with

$$l^{-d/2}\ll h_a\ll\tilde{h} \,. \tag{7.12}$$

There is again a characteristic length $\lambda_a\sim h_a^{-1/2}$ for variation of distortions of the phase within a typical R satisfying Eq. (7.12). If we again estimate the threshold field by the point at which $\nabla\theta\mid_{\partial R}\sim1/\lambda_a$ and we again choose $l\sim\lambda_a$, then we find that $F_T>L^{-d}l^{d-2}$, where again the separation between regions like R is given by $\sim p^{-1/d}$ with the probability p given by

$$p\sim\exp\left[-\frac{h_a^2l^d}{\tilde{h}^2}\right]$$
$$\sim\exp\left[-\frac{h_a^{2-d/2}}{\tilde{h}^2}\right] \,. \tag{7.13}$$

Since the bound on the threshold field scales as p, the best estimate is from the largest p. For $d>4$, this will occur for h_a as large as possible. If the maximum h is of order $\tilde{h}$, then the best estimate for F_T is obtained for $h_a\sim\tilde{h}$ yielding the same result as above, with the threshold controlled by the most coherent regions and scaling as Eq. (7.11). On the other hand, if the distribution of h is unbounded then the threshold can be controlled by individual strong-pinning sites. In either case, for $d>4$, the pinning will be dominated by rare, widely spaced regions, which pin strongly. In less than four dimensions, by contrast, the probability p in Eq. (7.13) will be largest for h_a *small.* In this case, p can become of order 1 for $h_a\sim\tilde{h}^{4/(4-d)}$, i.e., $l\sim\tilde{h}^{2/(d-4)}$ which is just the Lee-Rice length ξ_0. At this point, the arguments given clearly break down since L and l will be the same order and the renormalization-group argument given earlier is certainly better—however, the latter is now supported by the *absence* in $d<4$, of domination by rare configurations of the pinning. It is comforting to note that the domination of the pinning in $d<4$ by typical rather than atypical configurations can be justified by an argument based on considering the atypical configurations.

VIII. SHORT-RANGE INTERACTIONS: THRESHOLD BEHAVIOR, LINEAR RESPONSE, AND SCALING

In the preceding section, it was argued that with short-range interactions in any dimension, there will always be a nonzero threshold field which goes to zero for weak pinning. In this section we speculate on the critical behavior near threshold and discuss the appearance of ac noise in finite systems.

We first consider the role of the length scale ξ_0 discussed above. In the strong-pinning limit, the characteristic microscopic scale for phase deviations will be on the order of the separation between the impurities, and within a region containing a few impurities there will generally be several metastable configurations of the phases in zero field. For weak pinning, on the other hand, the phase deviations will only become of order 1 on a length scale ξ_0 and in regions larger than ξ_0^d, there will generally be metastable configurations of the phases. However, in regions *smaller* than ξ_0, there will typically be a *unique* (modulo an overall 2π phase shift) metastable configuration. At fields a few times the threshold field, the high-field perturbation theory should be qualitatively valid, and the velocity will be on the order of F_T implying that the velocity-velocity correlation functions decay with a correlation length on the order of ξ_0. It is thus apparent that the essential effects of the infinite number of degrees of freedom show up *only on length scales larger than* ξ_0. Except for far above threshold, the time dependence of the phases in a region of size less than ξ_0^d will follow the time dependence of the average phase in that region relatively closely. It is thus useful to renormalize the system up to a scale ξ_0 and treat each Lee-Rice domain as a single phase—this picture should preserve all of the universal features near to threshold. (In contrast to hopes in the literature, calculations[16] for one Lee-Rice domain[13] of size ξ_0^d cannot possibly yield appreciable hysteretic behavior and are unlikely even to give a reasonable approximation to the velocity at fields closer than a factor of 2 or so to threshold.[17] Note that the typical threshold field of an *isolated* region of size ξ_0^d will be comparable to the threshold field of an infinite system, although the fluctuations in the threshold field from sample to sample will be large; probably the same order of magnitude as F_T.)

In many ways, the Lee-Rice length ξ_0 is analogous to the BCS coherence length ξ_0 in superconductors. It determines the characteristic length scale for correlations far (i.e., a factor of 2) from threshold and is the scale above which collective effects start to manifest themselves in an essential way.[17]

In this section we will discuss scaling behavior near threshold; we first consider the moving state above threshold. While in mean-field theory there is a unique state above threshold (up to a shift in the origin of time), it is not clear whether this is true with short-range interactions. Perturbative arguments suggest that it is true at least at very high velocities. Motivated by this, we will assume for the remainder of this paper that *above threshold* there is a *unique moving state* (up to an overall temporal shift) in any dimension. In addition, we will assume that, as in mean-field theory, *moving* and *static states cannot coexist,* i.e., for a given field, the steady-state solutions are either always moving or always static, independent of the initial conditions. Since there is no apparent reason why different moving states, if they could exist, would have the same threshold field as F is decreased, it is likely that the two assumptions above are either both true or both false. If they are false, then much of the following discussion is unfortunately moot.

A. Moving state

Above threshold, the local velocities, although not the phases themselves, will have long-range order. As suggested by arguments in earlier sections, there will be a correlation length $\xi(F)$ in the moving state whose "bare" value at the order of twice threshold is ξ_0 and which diverges at threshold. By analogy with conventional critical phenomena it is natural to expect that ξ diverges as

$$\xi(F) \sim \xi_0 f^{-\nu} . \tag{8.1}$$

This correlation length will determine the long-distance falloff of the velocity-velocity correlation function to its long-range value of v^2. It will be shown in a future paper[28] by considering small fluctuations about the mean-field solution in high dimensions, that the mean-field value of ν is $\frac{1}{2}$.

There is a simple (although rough) physical interpretation of the correlation length. If we let $v_L(t)$ be the velocity at time t averaged over a region of size L^d, then $v_L(t)$ will be only weakly time dependent (i.e., small harmonic content) for L much larger than ξ, but $v_L(t)$ will be jerky (large harmonic content) for $L \lesssim \xi$. This interpretation is important in considering ac noise (see below). A region of size much smaller than ξ will spend most of each period near local minima of the energy. The correlations in the velocity may, however, fall off as a power law of distance for $r \ll \xi$, but still large, (see below) implying that this simple picture is not quite right.

By analogy with mean-field theory, it is natural to speculate that the velocity in a region of size $\sim \xi^d$ will exhibit harmonics at frequencies $nv \approx nf^{\zeta}$ up to some cutoff frequency Ω above which the amplitude of the harmonics will rapidly decrease. This frequency, Ω, will also go to zero at threshold as

$$\Omega \sim f^{\mu} \tag{8.2}$$

with $\mu < \zeta$. The linear ac response, χ, to an extra *uniform* ac field above threshold will behave as $f^{\zeta-1}/(-i\omega)$ out to a frequency of order Ω and exhibit a scaling form for ω, $v \ll 1$,

$$\chi(\omega, f) \sim f^{\zeta-\mu-1}\Xi(\omega/f^{\mu}) . \tag{8.3}$$

However, since the function Ξ is nonuniversal in mean-field theory, the degree of universality with short-range interactions, or even whether such a scaling form will be valid, is somewhat questionable.

It is not clear how nonuniversality of the form occurring in mean-field theory could be understood in a renormalization-group framework. It might perhaps be expected that in low dimensions the threshold behavior will be less controlled by local singularities than it is in mean-field theory, and hence it will be more universal. This question is left for future investigation and although the answer may invalidate some of the results of this section, will not be discussed further here.

In addition to scaling for the uniform response, one can also consider the spatially dependent polarizability, $\chi(q,\omega)$. At least in more than four dimensions, the response at long wavelengths and low frequencies with f *fixed* will be diffusive, i.e.,

$$\chi(q,\omega) \sim \frac{f^{\zeta-1}}{-i\omega+Dq^2} \tag{8.4}$$

with a phase diffusivity D which will go to zero at threshold. By scaling, one expects that with $\omega=v\sim f^{\zeta}$ the characteristic length scale of the response will be ξ, so that

$$D\sim f^{\zeta-2\nu} . \tag{8.5}$$

In $d\le 4$, the system in the moving state may still respond diffusively, although it is harder to produce a convincing argument for this because of divergences in the high-field perturbation theory for the response at low frequencies.[28] If it does *not* respond diffusively, it is almost certainly *subdiffusive,* i.e., at a given low frequency, the response will fall off *more* rapidly than the

$$\chi(r,\omega) \sim \frac{e^{-(-i\omega/D)^{1/2}r}}{r^{(d-1)/2}} \tag{8.6}$$

fall off with distance given from Eq. (8.4) [e.g., as $\exp(-|\omega|^{\lambda}r)$ with $\lambda<\frac{1}{2}$].

The falloff of the response of one part of the system to *finite-frequency* perturbations far away causes the finite-frequency correlation functions to fall off exponentially. [Note that in contrast, the *static* (zero-frequency) fluctuations, will decay (or grow) as a power of the separation, as discussed in Sec. III.] In steady state the local velocity will be periodic in time so that the velocity-velocity correlation function will contain only harmonics of v:

$$\langle[-i\omega\phi(r,\omega)-v][-i\omega'\phi(r',\omega')-v]\rangle = \delta(\omega+\omega')\sum_{n\,(\neq 0)}\delta(\omega-vn)C_n(r-r') . \tag{8.7}$$

In particular, the correlation function of the principal harmonic $\omega=v$ will fall off exponentially with the correlation length ξ (higher harmonics will be likely to fall off with higher multiples of ξ). We hypothesize that for ξ and r both large, C_1 will obey a scaling form (with a factor of v^2 pulled out for convenience)

$$C_1(r,f) \sim \frac{f^{2\zeta}}{\xi^{d-4+\eta}}\Gamma_1(r/\xi) . \tag{8.8}$$

There will be similar scaling functions Γ_n for the higher harmonics.

B. ac noise

The coherent ac noise in a finite system of volume V can be simply obtained from the velocity-velocity correlation function. The mean-square ac current density at frequency $\omega=nv$ is just proportional to

$$P_n = V^{-1}\int_V C_n(r)d^dr , \tag{8.9}$$

where the volume is measured in units of ξ_0^d. By scaling we have

$$P_n \sim V^{-1}\xi^{4-\eta}f^{2\zeta} \sim V^{-1}f^{2\zeta-\nu(4-\eta)} \tag{8.10}$$

and so the magnitude of the rms noise at the principal frequency $\omega=v$ relative to the dc current, $v\sim f^{\zeta}$, will diverge as

$$\frac{\langle j_{ac}^2\rangle^{1/2}}{j_{dc}} \sim f^{(\nu/2)(d-4+\eta)}\left[\frac{\xi^d}{V}\right]^{1/2} . \tag{8.11}$$

This enhancement of the square root of volume noise near threshold is caused by the diverging correlation length. The number of semicoherent domains in a volume V is V/ξ^d *not* V/ξ_0^d.[17] In a volume of (linear) size the correlation length, ξ, the ratio of the rms ac velocity to the dc velocity will be

$$\frac{\langle v_\xi^2(\omega=v)\rangle^{1/2}}{v} \sim f^{(\nu/2)(d-4+\eta)} . \tag{8.12}$$

The incoherence within a correlation volume thus introduces a factor $(\xi/\xi_0)^{(4-\eta-d)/2}$ into the rms noise amplitude. As long as the instantaneous time derivative of any phase is positive, which it should be in steady state, the $\omega=v$ Fourier components of the local velocity will be bounded by v and hence the $\omega=v$ correlation function $C_1(r)$ remains bounded by v^2 at all distances implying that

$$d-4+\eta\ge 0 . \tag{8.13}$$

Hence, the effective coherence volume determining the ac noise in volume V is $\xi^{4-\eta}$ which is generally *less* than or equal to the correlation volume ξ^d. The harmonics of the principal noise peak will persist out to frequencies of order Ω—the harmonic content thus diverges near threshold.

It is possible that in low dimensions (possibly even any $d\le 4$) some of these scaling relations break down or are modified if the polarizability $\chi(q,\omega)$ is not diffusive. This and related subtle questions will be left for future investigation.

C. Stationary states

There will generally be a large number of metastable states for fields below threshold and the response of the system will depend on the details of the past history. Near threshold (and possibly for all $F<F_T$) the zero-frequency limit of ac response functions will differ from the static response and, for example, the real part of the conductivity will be singular at low frequencies and the real part of the polarizability will have a cusp at $\omega=0$. Both of these effects arise from regions of the system which are almost unstable, i.e., which will move by a finite amount if the field is increased infinitesimally.

Recall that in mean-field theory, the real part of the conductivity behaves as $\omega^2\ln|\omega|$ and the real part of the polarizability as const$-|\omega|$ for low frequencies in the regime below threshold with phase jumps. These singularities arise from singularities in the *local* response: i.e., the almost flat regions in the local effective potentials, and the form of the singularity is not changed by the response of the mean field. One might thus expect that the singularities in the response of a system with short-range interactions due to localized regions which are about to become unstable will have the *same* form as in mean-field theory. There are, however, several subtleties which arise in a careful consideration of the response in the regime of interest. We leave as an intriguing open question whether or not the form of the singularities for short-range interactions will be the same as in mean-field theory.

By analogy to mean-field theory, one might expect that in some sense the number of metastable states decreases as F approaches F_T. At low fields, ξ_0^d (the Lee-Rice domain size in $d<4$) is the typical smallest volume of a region which can be moved to create a different metastable state which only differs from the original one in that region and its vicinity. As the field is increased, some regions of size ξ_0^d will reach a "local threshold" and jump by a finite amount only to be stopped by neighboring regions, leaving the system in a new metastable state. However, as the threshold field is approached, the neighboring regions will themselves be close to instabilities and will often also jump. The characteristic size of a region which must be moved to create a different metastable state will thus grow as F approaches F_T. *At* threshold, there will no longer be other metastable states if F is further increased, hence one expects that the size of these regions will diverge at F_T.

A natural definition of the correlation length, $\xi(F)$, below threshold is thus a typical linear dimension of the smallest regions which can be moved to create different metastable states. The singular response of the system below threshold will show up primarily on scales larger than ξ, it is thus possible that ξ could be defined in terms of wave-vector dependent response functions. A precise definition of ξ is difficult because of the hysteretic behavior; however, it is possible that for a typical approach to threshold (e.g., from the $F=0$ ground state) the correlation length will diverge as $\xi_0|f|^{-\nu}$ with ν universal and equal to the exponent of the velocity-velocity correlation length above threshold.

Although there are many inherent ambiguities, the behavior below threshold is definitely interesting and should be pursued further especially since the hysteretic behavior and singular response may be prototypical of other more complicated systems which exhibit similar effects (e.g., spin glasses and random-field magnets). One should note that most of the "glassy" behavior in the system of interest here will disappear at finite temperatures; this may, however, also be strictly true of three-dimensional spin glasses.

IX. THERMAL FLUCTUATIONS, DEFECTS, AND INERTIA

From the beginning of this paper, we have assumed [assumptions (C), (D), and (E) of Sec. I] that thermal fluctuations, defects, and inertia can all be ignored. In this section we briefly discuss the effects which occur if they are not ignored and the conditions under which they can be neglected.

A. Thermal effects

We first consider the effects of small thermal fluctuations on the depinning transition. In order to take into account the effects of nonzero temperature, the equation of motion for a phase ϕ_j must be modified to be

$$\frac{d\phi_j}{dt}=-\frac{\delta\mathscr{H}}{\delta\phi_j}+F+\epsilon_j(t) \tag{9.1}$$

with $\epsilon_j(t)$ a Gaussian correlated Langevin force with correlations

$$\langle\epsilon_j(t)\epsilon_i(t')\rangle=2T\delta_{ij}\delta(t-t')\ . \tag{9.2}$$

We first discuss mean-field results for strong pinning.

Thermal fluctuations will provide a mechanism for selecting between different metastable states which is absent at zero temperature. In particular, in zero applied field, small thermal fluctuations will drive the system out of the metastable states towards the overall ground state. By examining the form of the metastable states discussed in Sec. IV, it is apparent that the energy can always be lowered by moving an individual phase from one minimum of its local potential W_j to another lower one. Thus, at long times, the system will decay towards the overall ground state (with small fluctuations around it) with a rate, $1/\tau_{\max}$ limited by activation over the highest barriers

$$\frac{1}{\tau_{\max}}\sim e^{-c\hat{h}/T}\ . \tag{9.3}$$

In the presence of a nonzero applied field below threshold, *all* states are metastable. However, at finite temperature the system can always lower its total energy, $\mathscr{H}=\mathscr{H}_0-F\phi$ by jumping individual phases forward. Thus, in any nonzero field the average phase, $\bar{\phi}$ will creep forward and there will *not* be a sharp threshold field.

At a given low temperature, the small field behavior of the average velocity can be obtained by a method analogous to that for the weak-pinning limit at $T=0$. The probability distribution $P_j(\phi_j,t)$ for each phase will be strongly peaked at all times near minima of the local potential $W_j(\phi_j,t)$. In the limit of zero velocity, P_j will approach an adiabatic form (analogous to the adiabatic solution at $T=0$)

$$P_j^A(\phi_j,t)=\frac{e^{-W_j/T}}{Z_j}\ , \tag{9.4}$$

where $Z_j(t)$ is a normalization factor. Once each period, the position of the lowest minimum of W_j at which P_j^A is peaked will jump as the relative energy of two minima changes sign. For a time interval of order T/v (ignoring all coefficients of order 1 such as h's, etc.), the weight of P_j^A will be shared between the two minima. By symmetry, it is straightforward to show that the average over j of ϕ_j-vt weighted by the $\{P_j^A\}$ is just $+F$. The primary effect of the nonadiabaticity is to shift to slightly later times the transfer of weight of P_j from one minimum to the next. This shift will be by a hopping time of order $e^{+E_{Bj}/T}$ (where the barrier heights E_{Bj} are of order 1) which is much less than the transition time T/v for v sufficiently small. The effect of this delay is to make $\langle \phi_j - vt \rangle$ less than its adiabatic value by an amount of order $v\langle e^{E_{Bj}/T}\rangle_j$. Thus, the self-consistency condition $\langle \phi_j - vt \rangle = 0$ will be satisfied in the limit of small v by

$$v \sim \frac{F}{\langle e^{E_{Bj}/T}\rangle_j} , \tag{9.5}$$

i.e., linear response for small F.

As might be expected, the creep rate is dominated by the large barriers, arising from large pinning strengths, h_j. As the field is increased at low temperatures, there will be a rapid (but smooth) crossover near the $T=0$ threshold field from slow creep dominated by thermal hopping over barriers to rapid motion with the thermal fluctuations playing a smaller role.

Near the $T=0$ threshold field the dominant effects of small thermal fluctuations will be to speed up the rate at which the phases move out of disappearing minima of their local potentials. The equation of motion for a phase near to the singular point ϕ_s, t_s at which the minimum in which its probability is concentrated disappears is

$$\frac{d(\phi-\phi_s)}{dt} = C_h^{-3}(\phi-\phi_s)^2 + v(t-t_s) + \epsilon(t) . \tag{9.6}$$

After rescaling by powers of v as in Eq. (4.32), it is apparent that the effective temperature (given by the correlation of the scaled noise as a function of the scaled time) is just T/v. This suggests that the properties near threshold, in particular, the velocity as a function of the field, should exhibit scaling behavior, in particular,

$$v(F,T) \sim T^{\zeta/\tau}\overline{B}(f/T^{1/\tau}) , \tag{9.7}$$

with $\tau=\frac{3}{2}$ the thermal crossover exponent and $f=F-F_T$. The nonuniversal (at least in mean-field theory) scaling function, $\overline{B}(b)$, will be a monotonically increasing function of b which is exponentially small for $b \to -\infty$, and proportional to $b^{3/2}$ for $b \to +\infty$. At the $T=0$ threshold field, the velocity will go as

$$v(F_T,T) \sim T^{\zeta/\tau} . \tag{9.8}$$

As can be seen from this discussion, temperature acts in many ways as a symmetry-breaking field for the depinning transition, destroying the pinned phase.

We next consider the effects of thermal fluctuations on depinning with *short*-range interactions. As in mean-field theory, thermal fluctuations in zero field will drive the system towards lower-energy states although the process can be considerably slower due to the slowness of the motion of long-wavelength distortions. For strong pinning, the thermally activated phase jumps will consist of motion of only a few phases, however, for weak pinning, the jumps are likely to occur primarily on the scale ξ_0 with $\sim \xi_0^d$ phases moving together (or in close succession).

At nonzero fields less than F_T, the average phase will gradually creep forwards by jumps of regions of size ξ_0^d. The barrier heights determining the rate for this creep can be estimated for weak pinning. The characteristic range of the energy of a region of size ξ_0^d is of order ξ_0^{d-2} (with, by definition of ξ_0, roughly equal portions coming from gradient and pinning energies). Since in dimensions $d<4$, $\xi_0 \sim h^{2/(d-4)}$, the typical barrier heights between local minima of the energy will behave as

$$E_B \sim h^{-2(d-2)/(4-d)} \tag{9.9}$$

which implies that in three dimensions, the effects of temperature will be *suppressed* for weak pinning by a factor of $h^2 \propto \xi_0$. This will, however, *not* be the case in one and two dimensions.

At any nonzero temperature, we expect that, as in mean-field theory, the velocity will be a linear function of F for small fields and the sharp threshold will be destroyed. Near threshold a scaling form analogous to Eq. (9.7) is likely to govern the effects of small thermal fluctuations. However, in the limit of weak pinning in three dimensions the thermal effects become negligible yielding a sharp threshold as the pinning strength tends to zero.

B. Defects

Due to large defects in the underlying lattice (e.g., dislocation lines), finite cooling rates, or the rare collective effects of many weak impurities, there are always likely to be lines or loops of 2π phase dislocations in the CDW (for the single wave-vector CDW's of interest here, the dislocations are simply vortex lines). These dislocation lines will tend to be pinned in space by the disorder. If they are so strongly pinned that they do not move (except perhaps for small bounded motion), then their effect is just to introduce a *fixed* background (non-single-valued) phase configuration about which single-valued phase distortions can be defined. In this case, the effects of the phase dislocations will be small: they will only slightly modify the energy of the single-valued phase distortions.

If the dislocations can be moved, on the other hand, they may cause additional metastability and hysteresis. In particular, at high fields when the phase is moving rapidly the pinning of the dislocations will tend to be reduced by temporal averaging and they may move and annihilate. If this occurs, then when the field is reduced rapidly, the dislocations may not be able to reform at low CDW velocities, and as a result there could be many possible steady-state velocities at a given field (and correspondingly a nonunique threshold) each corresponding to a different number or metastable configuration of the dislocations.

If the pinning is very weak (as appears to be the case in the experiments) there are likely to be very few dislocations and their effect will be small. However, it is possible (as has been suggested by Ong *et al.*,[36] that large electric field gradients near contacts can create dislocations which might then play an important role in some of the "phase memory" effects observed as the field is increased and decreased through threshold. The possible effects of defects on hysteretic behavior are clearly interesting and should be pursued.

C. Inertia

We now turn to the effects of inertia. It is instructive to first briefly consider the one-particle case discussed in Sec. II. We add a mass, m, and consider the equation of motion scaled for convenience to make $h=1$,

$$\frac{d\phi}{dt}+m\frac{d^2\phi}{dt^2}=-\sin\phi+F\ . \tag{9.10}$$

With m sufficiently large, it is apparent that stationary *and* moving solutions can coexist for $F \lesssim F_T^s = h$ where the s denotes the *static* threshold field. On the other hand, since the behavior near threshold is dominated by the *slow* motion through the sticking points $\phi_s = 2\pi n + \pi/2$ a small mass will be formally irrelevant near the critical point at $F = h$. (The oscillations about a static solution will be strongly overdamped as $F \to F_T^s$.) However, the instantaneous velocity $d\phi/dt$ always attains a value of order 1 for ϕ far from any ϕ_s. But it can be seen that the loss of kinetic energy due to the damping will be sufficient so that for $m \ll 1$, the velocity in the region near ϕ_s will be small in any steady-state solution. In this limit the irrelevancy of m about ϕ_s near threshold and the rapid energy dissipation away from ϕ_s can be used to show that for m less than a critical value of order 1, stationary and steady-state moving solutions cannot coexist, as is the case for $m = 0$.

In mean-field theory, power counting from the equation of motion Eq. (4.32) near to the important singular point suggests that in this case also, a small mass will be irrelevant near threshold. A detailed argument is necessary in order to show that for m sufficiently small, the velocity of a phase just before the singular point when a minimum of its potential disappears will retain only a negligible inertial memory of the large velocity it had in falling from one minima to the next after the previous minimum disappeared. We speculate that the result will be that for m less than a critical value, steady-state moving solutions cannot coexist with stationary solutions in mean-field theory.

What about short-range interactions? Similar considerations again suggest that the memory of the rapid local velocity during a jump of one region will again be negligible at the time of its next jump. We hence speculate that quite generally, *neither the critical behavior nor the lack of coexistence of steady-state moving and stationary solutions* will be affected by a *sufficiently small mass*. A careful study of this question and the onset of hysteretic nonunique behavior as the mass is increased should be interesting.

X. EXPERIMENTS AND CONCLUSIONS

In this section the theoretical assumptions and predictions discussed in this paper are briefly compared with experimental results on sliding CDW's along with suggestions of future tests of the theory. At the end of this section other experimental systems are discussed and general open questions reviewed.

A. Experiments on sliding CDW's

As discussed in the introduction, there are quite a few experimental systems which appear to exhibit sliding CDW motion with many features in common. Before comparing the experimental results with the theory we first consider how valid the assumptions (A)—(E) discussed in the introduction are likely to be for the experimental systems.

Assumption (A), that the CDW stiffness is short range, is reasonable as long as the number of uncondensed electrons is sufficiently large so that their conductivity can screen out charge buildup on the time and length scales important for the threshold behavior; this is the case for all of the systems at least in the range of temperatures studied. (For a detailed discussion in the high velocity limit, see Ref. 15.)

If the impurities are positioned on the underlying lattice with only short-ranged correlations in their positions, then as long as the CDW is incommensurate, the preferred phases at the impurity sites will not be significantly correlated [assumption (B)]. However, rather surprisingly, the systems in which the CDW wave vector becomes locked (in a range of temperature) to a *commensurate* value (e.g., TaS_3) exhibit qualitatively very similar behavior to the incommensurate systems.[37] Naively, *all* commensurate systems are expected to exhibit a much steeper break in the I-V curve at threshold with the $\zeta = \frac{1}{2}$ behavior characteristic of a zero-dimensional system. A possible resolution of this puzzle is to argue that the commensurate pinning potentials are very weak relative to the impurity pinning potential (consistent with the apparent lack of dependence of the threshold field on whether or not the CDW is commensurate) and that the unit cells of the CDW are so large that the number of possible impurity sites per cell is very large.[38] In this case, the number of possible values of the preferred phases, β_j, while not infinite as in the incommensurate case, could be sufficiently large so that there is a large region of effectively randomly pinned behavior with only a narrow (and perhaps unobservable due to thermal or other rounding of the threshold) commensurate critical region very near to E_T. This explanation is not very satisfactory and this problem is potentially rather serious; however, we will not discuss it further here.

In all of the systems studied, the threshold fields are extremely small on scales of microscopic fields.[1,2] This, and the direct measurement of long-range coherence of the CDW in some of the materials [greater than one micron along the incommensurate direction in $NbSe_3$ (Ref. 3)] strongly suggest that all the CDW's so far investigated are in the weak-pinning limit with ξ_0 very large. From the discussion in Sec. IX, this implies that there are probably few dislocations in the *bulk* of the CDW (except perhaps some immobile strongly-pinned ones which will not affect the behavior much) justifying assumption (C). (We will comment later on the possible effects of dislocations near to the contacts.) Similarly, since the CDW's all exhibit three-dimensional coherence, the existence of a long Lee-Rice length ξ_0 implies that the effects of thermal depinning are negligible [assumption (D)].

Lastly, direct measurement of the frequency-dependent linear response at zero applied dc field shows strongly overdamped behavior[39] justifying the neglect of inertia [assumption (E)]. Even if the speculation in Sec. IX that there is a critical mass for onset of hysteresis is incorrect, the inertial effects are likely to be sufficiently small in the experiments to generate at worst a small amount of hysteresis in the I-V curve.

Thus, we have at least tentatively justified all of the important assumptions, and we now turn to comparison with the experiments. Observation (1) (Ref. 3) (the numbers in parentheses refer to the experimental features discussed in the introduction), yields no useful tests of theory unless (as can hopefully be done via Mössbauer x rays or neutrons) the predicted moving CDW Bragg peaks can be directly observed which would provide striking confirmation of (at least) the general sliding CDW picture.

The ohmic conductivity below threshold (2) is believed to be due to the uncondensed electrons, the apparent absence of significant precursor nonlinearities near threshold[4,5] (3) is consistent with the predicted smallness of thermal depinning effects.

As noted in the introduction, a dramatic failure of the simple single-particle picture[11] is the dependence (4) of the excess current, $j_{\rm CDW}$ on $E-E_T$ near threshold.[4,5] The mean-field theory predicts that $j_{\rm CDW}\sim(E-E_T)^{\zeta}$ with $\zeta=\frac{3}{2}$. This mean-field value for the exponent ζ is unlikely to be correct in three dimensions. However, it is quite likely that ζ tends to *increase* from its zero-dimensional value of $\frac{1}{2}$ towards its mean-field value as the dimension is raised; by three dimensions ζ might be relatively near to $\frac{3}{2}$. Experimentally, while power-law fits have not been carried out, a value of ζ somewhere between 1 and 2 appears likely—certainly the upwards concavity of the *I-V* curve is encouragingly (if perhaps fortuitously) similar to the mean-field result. In addition, it should be recalled that far above threshold the *I-V* curve is quantitatively fit by the perturbation results discussed in Sec. III and Refs. 14 and 15.

The feature of the experiments which seems to have attracted the most attention (although this author has always considered it to be a red herring) is the apparently coherent ac noise in response to a dc applied field.[4] As mentioned previously, the linear dependence of the principal noise frequency, ν_N, on the nonlinear current $j_{\rm CDW}$ [(a) in (5)][6] arises naturally in most theories, including the present one. However, the predictions in Sec. VIII of the dependence of the rms noise intensity on the volume of the system and the distance from threshold provide much more stringest tests of the theory.

Unfortunately, the noise measurements have been rather irreproducible with marked differences between experiments even among those on the same material. There does, however, seem to be a general tendency for small samples to exhibit more noise (relative to the dc current) than larger samples[40–42] (which often exhibit none[43]) and for the intensity of, for example, the principal harmonic to grow near threshold [(b) of (5)].[4,44] In fact, some samples seem to have an almost completely pulsed response extremely near to threshold.[8]

Very recent experiments[44] on very small samples of $NbSe_3$ lend some support to the suggestion (made initially in Ref. 26) that the noise (at least in some of the experiments) is a $(\text{volume})^{-1/2}$ finite-size effect with a diverging amplitude near threshold. Mozurkewich and Gruner[44] observe noise with intensity consistent with a $(\text{volume})^{-1/2}$ behavior over a range of sample lengths and cross sectional areas, although the scatter in the data is rather large. In addition, their data exhibit a dramatic increase in the intensity of the first harmonic relative to the dc nonlinear current as threshold is approached.[44] Both of these observations are consistent with the prediction in Sec. VIII that (in the absence of large phase slip regions) the rms noise should scale as $(\text{volume})^{-1/2}$ with a coefficient coming from the diverging correlation length; i.e., a relative intensity of the principal harmonic growing as

$$\frac{j_{\rm rms}(\omega=2\pi\nu_N)}{j^{\rm dc}_{\rm CDW}}\sim\left[\frac{E-E_T}{E_T}\right]^{-(\nu/2)(4-\eta)}\left[\frac{\xi_0^d}{(\text{volume})}\right]^{1/2}. \tag{10.1}$$

Furthermore, far above threshold the absolute noise intensity appears to go to a constant,[44] i.e., its relative magnitude decreases inversely with the CDW dc velocity v. This is in agreement with the theoretically expected behavior which can easily be deduced from the perturbative correlation function Eq. (3.4). A detailed study of the behavior in small samples *near* to threshold would be very interesting.

After this apparent verification of some of the theoretical predictions concerning the noise intensity, it is important to make several observations. Firstly, there has been no viable mechanism proposed for establishing ac phase coherence of the CDW current density over truly macroscopic regions (except for the behavior close to threshold discussed here where ξ can be very large). Thus, *any* mechanism for the noise should be expected to yield an intensity decreasing at least as $(\text{volume})^{-1/2}$ (in striking contrast to a one-particle picture). Secondly, it is quite possible that several different mechanisms for noise generation appear under different experimental conditions. An interesting suggestion has been made by Ong *et al.*[36] that large electric field inhomogeneities near contacts could cause tearing of the CDW with an array of moving dislocation lines present near threshold. Although there are considerable difficulties with their picture involving coherence of and barriers to dislocation motion, it certainly merits further study, in particular, as there are experimental indications that the sources of the noise are localized at least in some samples.[40,41] Lastly, we note that $(\text{volume})^{-1/2}$ noise *cannot* yield a completely pulsed response. Even sufficiently close to threshold so that ξ becomes of order of the size of the system, L, and the CDW current behaves as the single-phase result $(F-F_T)^{1/2}$, the noise amplitude of the principal harmonic will (by scaling) be reduced relative to the dc current by $(L/\xi_0)^{-(d-4+\eta)/2}$.

The existence of many metastable states, some of them differing in large regions, can naturally give rise to many hysteretic and phase memory effects. While there are many experiments showing various forms of hysteretic behavior,[7–9] detailed comparison with theoretical predictions are not particularly instructive at this stage, at least in part because the mean-field model may exhibit only some of the phenomena. This is because the extremely slow long-wavelength distortions which might be expected to play a role in finite dimensions do not have a natural counterpart in an infinite-range mean-field theory. Progress in analyzing fluctuations about mean-field theory in high dimensions should be helpful. In addition, it is possible that dislocations in the CDW may play an important part in hysteretic behavior; this also bears investigating.

At this stage, one definite prediction of mean-field theory, which we have speculated is true also with short-range interactions, is the lack of hysteresis in the dc I-V curve. This is consistent with almost all the experiments and suggests that dislocations do *not* play too large a role.

Lastly, we recall that the interference effects[6] between ac and dc currents (7), in particular, the quadratic response of the dc current to an applied ac voltage, have been quantitatively fitted at high velocities to perturbative results;[15] however, there has not been a detailed analysis of the behavior near threshold. It is important to note that the existence of ac-dc interference effects only depends on the *local* ac motion; in contrast to the noise, large scale ac coherence is not necessary. Detailed calculations can be made in the mean-field approximation.

After this paper was completed, two works were made available which reported measurements of the ac polarizability, $\chi(\omega)$, below threshold. The data of Cava *et al.*[42] on $K_{0.3}MO_3$ appear to exhibit a downward cusp in the real part of χ at zero frequency, perhaps slightly sharper than the $|\omega|$ behavior found in mean-field theory, which might have been expected to be valid with short-range interactions (see discussion in Sec. VIII). Since the data are not very extensive, it is not clear whether they are inconsistent with an $|\omega|$ cusp, but we note that the presence of a divergent nonlinear response at low frequencies could make the cusp appear sharper. These data do, however, provide at least qualitative support for the picture of local regions becoming unstable as the field is increased below threshold. In a recent theoretical work, Littlewood and Varma[45] have predicted from a mode-coupling-like analysis a sharper $|\omega|^{1/2}$ cusp in the real part of χ below threshold.

The other experimental paper, by Wu *et al.*,[42] reports conductivity measurements on TaS_3 which yield a power-law *divergent* polarizability at low frequencies in striking contrast to the work of Cava *et al.* However, it is possible that the relevant frequency scales for the two systems are sufficiently different that the experiments are in very different regimes. Clearly, more theoretical and experimental work on this question should be done.

So far we have seen that many of the features of the experiments can be semiquantitatively explained in terms of the current theoretical analysis. However, there are more stringent tests of the theory which should be possible; some of these are discussed in the conclusions.

B. Other experimental systems

While the CDW systems provide a fruitful testing ground for general questions of the behavior of dynamic depinning transitions (and, of course, are interesting in their own right), it is worth considering other more easily controllable systems which might have advantages. The most likely candidate is the workhorse for investigations of collective effects; a superconductor, in particular, a type-II superconductor in a magnetic field between the lower and upper critical fields. In the absence of disorder, the flux lines will form a hexagonal Abrikosov lattice which can be moved by application of a current perpendicular to the magnetic field. However, in the presence of inhomogeneities, the flux lattice will deform and will not move unless the applied current density, j, exceeds a critical current density j_c. For obvious technological reasons, most of the experimental and theoretical work has been concerned with the strong-pinning limit where j_c is large. However, in the opposite weak-pinning limit collective effects are important (Larkin and Ovchinikov,[34] in this context, produced the first version of the weak-pinning scaling argument for the threshold force). This limit is perhaps best achieved in amorphous materials such as a-In, thin films of which have recently proved very useful in testing Kosterlitz-Thouless theories.[46] The choices of materials, the understanding of the underlying forces, and the controllability of a large number of parameters (pinning strength, flux line separation, temperature, etc.) might make these materials ideal for careful studies of collective depinning near to the critical current.

There are, however, several (hopefully minor) drawbacks. Interpretation of experiments on truly three-dimensional systems will be hindered by inhomogeneities in the current (which provides the force on the flux lines) which will tend to flow primarily near the edges. Experiments on two-dimensional films, for which H_{c1} is zero, may thus be favorable. However, in contrast to three dimensions, the neglect of temperature in the weak-pinning limit in two dimensions is not justified: the effects of thermal depinning will be roughly independent of the pinning strength for weak pinning. This drawback can be partially avoided either indirectly by making the films sufficiently thick so that thermal effects are reduced or directly by working at very low temperatures.

Note that in flux lattice films, the critical behavior will be *different* from three-dimensional CDW's for two reasons: The first, and presumably the dominant, is the difference in dimensionality and the second is the fact that the flux lattice order parameter is more complicated than a single wave-vector CDW, i.e., the flux lines can move transverse to their average direction of motion. An additional complication is the presence of only weakly screened long-range forces in flux lattice films which suppress longitudinal distortions, although these should not have a drastic effect since the shear modulus is finite.[47]

The last drawback we will mention is the potential difficulty relative to the CDW's of dislocation annihilation due to the suppression of climb by conservation of flux lines. Problems caused by this can most likely be avoided by appropriate annealing.

After some amount of consideration of the best materials and parameter ranges, studies of weakly-pinned flux lattice motion in films should prove to be extremely interesting especially since thermal and defect effects can probably be investigated as well as more ideal systems of almost perfect lattices at very low temperatures.

C. Conclusions and questions

In this paper we have shown that much of the data on sliding charge density waves can be understood semiquantitatively in terms of a model in which an infinite number of internal degrees of freedom collectively cause a sharp threshold with entirely new nontrivial critical behavior. In particular, the I-V curve, the ac noise, ac-dc interference effects and some of the hysteretic effects appear to be consistent with the theory. However, the scaling behavior near threshold, in particular, the existence of a second critical frequency scale which goes to zero at threshold more slowly than the noise frequency, has not yet been observed. Detailed studies of the low-frequency response near to threshold on both sides should provide tests of some of the theoretical calculations and speculations presented here.

As discussed above, flux lattices in type-II superconductors may also provide a promising testing ground. In addition, some of the ideas here may be useful for understanding other phenomena, such as dynamic friction, which exhibit similar features.

Finally, there are many open theoretical questions of a critical phenomena nature alluded to in the text, for example, the upper critical dimension for mean-field theory (if any), and the apparent nonuniversality of scaling functions. Formulating a renormalization-group analysis of the threshold behavior should be a challenging task which may bear fruit in other dynamic problems.

Note added in proof. A recent paper [L. Sneddon, Phys. Rev. B **30**, 2974 (1984)] on an incommensurate model with infinite-range interactions which is equivalent to the mean-field model discussed here with fixed pinning strengths, rederives the critical behavior of the velocity and presents results on the ac conductivity below threshold similar to those of Sec. VI for case 1.

ACKNOWLEDGMENTS

The author has profited greatly from discussions on sliding CDW's with many colleagues. I would like especially to thank Michael Cross, Susan Coppersmith, Leigh Sneddon, and Peter Littlewood for contributing to my theoretical understanding and Robert Fleming and Nai-Phuan Ong for clarifying many experimental points. Extensive comments and suggestions on the manuscript from Pierre Hohenberg are also gratefully acknowledged. Finally, I would like to thank Albert Schmid and Tony Fiory whose papers on flux flow in superconductors originally sparked my interest in these problems and John Bardeen whose talk at Bell Laboratories on sliding CDW's stimulated this work.

APPENDIX A: MEAN-FIELD THEORY ASYMPTOTICS

This appendix contains a detailed derivation of the dominant and subdominant critical behavior for strong-pinning mean-field theory near to threshold and, at the end, a brief derivation of the weak-pinning low-velocity conductivity.

1. Strong pinning

We are interested in the difference $\theta(t)$ (and particularly its integral) between the actual periodic solution $\phi(t)$ and the adiabatic approximation $\phi_A(t)$ to the equation of motion for a single phase with $\beta=0$,

$$\frac{d\phi}{dt}=-h\sin\phi-\phi+F+vt\equiv R_0(\phi)+vt\ .$$

To obtain the leading and subdominant behavior near threshold, we will require the average $\langle\theta\rangle_t$ to order $v\ln v$ for v small.[48] The properties of R_0 which determine the critical behavior are (a) $R_0(\phi)-\phi$ is a smooth 2π periodic function of ϕ; (b) the adiabatic approximation $\phi_A(t)$ which is the smallest ϕ solution to $R_0(\phi)=-vt$ is smooth except for one discontinuity each period occurring at a time t_s where ϕ_A jumps from $\phi_s=\phi_A(t_s^-)$ to a larger value $\phi_r=\phi_A(t_s^+)$; and (c) $[dR_0(\phi)/d\phi]_{\phi=\phi_s}=0$ but $[d^2R_0(\phi)/d\phi^2]_{\phi=\phi_s}\equiv 2M>0$ and $[dR_0(\phi)/d\phi]_{\phi=\phi_r}<0$.

It is convenient to immediately change variables to $x=\phi-\phi_s$ and measure time from t_s so that we have

$$\frac{dx}{dt}=R(x)+vt \tag{A1}$$

with for x small,

$$R(x)=Mx^2+M_3x^3+\dots \tag{A2}$$

and $\Delta=\phi_r-\phi_s$ so that for $(x-\Delta)$ small,

$$R(x)=+\Gamma(\Delta-x)+O((\Delta-x)^2)\ . \tag{A3}$$

As will be shown below, the dominant and leading subdominant critical behavior for $\langle\theta(t)\rangle_t$ will depend *only* on Γ, M, M_3, and Δ.

In the small v limit of interest, there are three possible approximations to Eq. (A1) each of which will be valid in a certain regime. These are (i) the adiabatic approximation in which dx/dt is neglected. This will be valid for all times far from 0. (ii) The leading nonlinear behavior near $x=t=0$ where $R(x)$ is approximated by Mx^2. (iii) The time-independent force approximation in which the vt [or $v(t-t_0)$ for some t_0] is neglected, which is valid provided the phase is moving fast compared to v.

The full asymptotic solution for small v will be obtained by matching asymptotic expansions in three regimes.

(i) *Adiabatic regime.* For the adiabatic approximation the natural scaled variables are $X=x$ and $T=vt$ in terms of which

$$v\frac{dX}{dT}=R(X)+T \tag{A4}$$

and we have

$$X(T)=X_A(T)+vX_1(T)+v^2X_2(T)\ , \tag{A5}$$

where $X_A(T)$ is the adiabatic solution, i.e., the minimum X solution to

$$R(X)=-T\ . \tag{A6}$$

By expanding in powers of v we obtain

$$X_1(T)=-\left[\frac{dX_A}{dT}\right]^2$$

and (A7)

$$X_2(T)=\frac{5}{2}\left[\frac{dX_A}{dT}\right]^2\left[\frac{d^2X_A}{dT^2}\right]\ .$$

This expansion in powers of v will be uniformly valid except when dX_A/dT becomes large which will happen as $T \to 0^-$ for which

$$X_A \approx -\left[\frac{-T}{M}\right]^{1/2} + \frac{1}{2}\frac{M_3 T}{M^2} + \cdots \quad \text{(A8)}$$

Near to $T=0$ we thus have

$$X_A \sim (-T)^{1/2}, \quad X_1 \sim (-T)^{-1}, \quad X_2 \sim (-T)^{-5/2}, \quad \text{(A9)}$$

etc., so that the expansion Eq. (A5) is valid only so long as $(-T) \gg v^{2/3}$.

(ii) *Singular regime.* For T near zero, we expect Eq. (A5) to be a very bad expansion. It will only become valid again once dX/dT becomes $\ll 1/v$, i.e., $dx/dt \ll 1$. Near to the singular point at $x=t=0$ we can rescale the x,t by

$$x = v^{1/3} M^{-2/3} \chi \quad \text{and} \quad t = v^{-1/3} M^{-1/3} \tau \quad \text{(A10)}$$

to yield

$$\frac{d\chi}{d\tau} = \chi^2 + \tau + (M_3 M^{-5/3}) v^{1/3} \chi^3 + O(v^{2/3}\chi^4) . \quad \text{(A11)}$$

We can then expand the solution $\chi(\tau)$ as

$$\chi(\tau) = \chi_0(\tau) + v^{1/3}\chi_1(\tau) + v^{2/3}\chi_2(\tau) + \cdots \quad \text{(A12)}$$

with $\chi_0(\tau)$ satisfying the nonlinear Riccati equation

$$\frac{d\chi_0}{d\tau} = \chi_0^2 + \tau \quad \text{(A13)}$$

which has the general solution

$$\chi_0(\tau) = \frac{\text{Ai}'(-\tau) + b_1 \text{Bi}'(-\tau)}{\text{Ai}(-\tau) + b_1 \text{Bi}(-\tau)} , \quad \text{(A14)}$$

where Ai (Bi) and Ai′ (Bi′) are Airy functions and their derivatives and b_1 is a constant of integration. Since for large negative τ, $\text{Bi}(-\tau) \gg \text{Ai}(-\tau)$, $\chi_0(\tau \to -\infty)$ will be dominated by the Bi terms unless $b_1 = 0$. For b_1 nonzero we have $\chi_0(\tau) \sim +\sqrt{-\tau}$, corresponding to the solution which comes from the unstable maximum of the potential at large negative times. We thus must clearly choose $b_1 = 0$ whence

$$\chi_0(\tau) = \frac{\text{Ai}'(-\tau)}{\text{Ai}(-\tau)} \approx -\sqrt{-\tau} \quad \text{(A15)}$$

for $\tau \to -\infty$ as desired. The first correction χ_1 to χ_0 can be readily shown to be

$$\begin{aligned}\chi_1(\tau) &= M_3 M^{-5/3} \int_{\tau_1}^{\tau} d\tau' \chi_0^3(\tau') \exp\left[\int_{\tau'}^{\tau} d\tau'' 2\chi_0(\tau'')\right] \\ &= \frac{M_3 M^{-5/3}}{[\text{Ai}(-\tau)]^2} \int_{\tau_1}^{\tau} d\tau' \frac{[\text{Ai}'(-\tau')]^3}{\text{Ai}(-\tau')} \end{aligned} \quad \text{(A16)}$$

with an as yet undetermined constant of integration τ_1. We can fix τ_1 by requiring that at large negative τ, the solution match the solution from Eq. (A5). The correct choice is $\tau_1 = -\infty$ so that χ_1 is independent of τ_1. The large positive z form of the Airy function is

$$\text{Ai}(z) = z^{-1/4} e^{-(2/3)z^{3/2}} \left[1 - \frac{5}{72}\frac{1}{(\frac{2}{3}z^{3/2})} + O(z^{-3})\right] . \quad \text{(A17)}$$

Using this form, the integral in Eq. (A16) can be evaluated asymptotically for large negative τ yielding

$$\chi_1(\tau) \approx \frac{M_3 M^{-5/3}\tau}{2} \quad \text{(A18)}$$

for $\tau \to -\infty$ which, with the rescalings taken into account exactly agrees with the second term in Eq. (A8). Since for $\tau \to -\infty$, $\chi_0 \sim \sqrt{-\tau}$, and $\chi_1 \sim \tau$, the expansion in regime (ii) will be uniformly valid only as long as $(-\tau) \ll v^{-2/3}$. Comparing this with the condition for validity for expansion (i), we conclude that *both* expansions will be valid provided that

$$v^{-1/3} \ll (-t) \ll v^{-1} \quad \text{(A19)}$$

and hence the asymptotic expansions Eqs. (A5) and (A12) can be matched at any $t_m^{(i),(ii)}$ satisfying Eq. (A19), say, $-t_m^{(i),(ii)}$ of order $v^{-1/2}$.

We now consider the behavior of the solution in region (ii) as τ increases. The zeroth-order solution $\chi_0(\tau)$ will diverge at a scaled time τ_d such that $-\tau_d$ is the least negative zero of Ai(z), $\tau_d \simeq 2.338$. Near to this point,

$$\chi_0(\tau) \approx \frac{1}{\tau_d - \tau} . \quad \text{(A20)}$$

From Eq. (A16) it can be seen that the leading correction will diverge as

$$\chi_1(\tau) \approx \frac{-M_3 M^{-5/3} \ln(\tau_d - \tau)}{(\tau_d - \tau)^2} \quad \text{(A21)}$$

for $\tau \to \tau_d$, since the integral taken up to $\tau \lesssim \tau_d$ diverges logarithmically. Thus the region (ii) expansion in Eq. (A12) will be uniformly valid only if

$$(\tau_d - \tau) \gg v^{1/3} |\ln v| . \quad \text{(A22)}$$

However, by this time, $d\chi/d\tau$ will be of order $v^{-2/3}$ hence dx/dt will be of order 1 (both up to logarithmic corrections) and approximation (iii) should be valid.

(iii) *Fast regime.* In regime (iii), the scaled variables are just the original variables x and t and we may write

$$x(t) = x_0(t) + v x_1(t) + \cdots , \quad \text{(A23)}$$

where $x_0(t)$ satisfies

$$\frac{dx_0}{dt} = R(x_0) \quad \text{(A24)}$$

which upon integrating yields up to a constant of integration $x_m = x_0(t_m)$,

$$t - t_m = \int_{x_m}^{x_0(t)} \frac{dx'}{R(x')} , \quad \text{(A25)}$$

with the matching time t_m to be chosen for convenience and x_m determined by matching to the solution in region (ii) $x_{(ii)}(t_m)$.

To match to region (ii), we would like to choose t_m so that t_m is small compared to 1. In that case, for x_0 also small we can expand the integral in Eq. (A25) in powers of $1/x'$ and determine $x_0(t)$. The leading behavior in the limit of x_0 small is

$$x_0(t)\approx\frac{1}{M[1/Mx_m+(M_3/M^2)\ln x_m+O(x_m)+t_m-t]} \tag{A26}$$

which can be made to agree with the $\tau\to\tau_d$ behavior of x in region (ii) by choosing x_m appropriately. However, we must now be careful. The quantity we are eventually interested in is

$$\langle\theta\rangle_t=\frac{v}{2\pi}\int_{\text{period}}dt[x(t)-x_A(t)] \;. \tag{A27}$$

The contribution to this integral from $t\gtrsim 0$ involves

$$(v/2\pi)\int_0^{t_e}[-\Delta+x(t)]dt' \;, \tag{A28}$$

where t_e is the time at which $x(t)$ becomes of order 1 and is hence very close to $t_d=(vM)^{-1/3}\tau_d$. We thus need to know t_e to $\ln v$ accuracy in addition to the behavior of x as it becomes of order 1 in order to extract the $v\ln v$ contribution to $\langle\theta\rangle$ discussed in the text. For this purpose it is thus *not* sufficient to just match the leading behavior of x_0 from Eq. (A26) to the leading behavior of χ_0 in region (ii); we need to calculate $x(t_m)$ more accurately. We first examine the region (iii) side. The leading correction to $x(t)$ in this regime can be straightforwardly shown to be

$$x_1(t)=\int_{t_1}^{t}dt'\,t'\frac{R[x_0(t)]}{R[x_0(t')]} \;. \tag{A29}$$

Here, t_1 is a constant of integration which would need itself to be determined by matching in order to perform a full asymptotic expansion, but for our purposes we can just consider $t_1\approx t_e$ and match expansions to the desired order, for which $x_1(t)$ turns out to be unimportant. For $t\sim t_m$ [or generally $x_0(t)\ll 1$],

$$x_1(t)=O(t_e(t-t_e))+O((t-t_e)^2) \;. \tag{A30}$$

From the general form of the expansion (iii) we can conclude that it is valid if $t_e-t\ll v^{-1/3}$. Since $|t_e-t_d|$ will turn out to be $\ll v^{-1/3}$, there is an overlapping regime of validity of (ii) and (iii) when

$$v^{-1/3}\ll t_d-t\ll|\ln v| \;. \tag{A31}$$

We may thus choose any t_m satisfying this condition; it is convenient to pick

$$t_m=t_d-v^{-1/6} \tag{A32}$$

and the corresponding τ_m. At time t_m, $vx_1(t)\sim v^{1/2}$ while $v^{2/3}\chi_1((vM)^{1/3}t_m)\sim v^{1/3}\ln v$; the former can thus be ignored in determining $x(t_m)$ while the latter cannot. From region (ii) we obtain

$$x_m\approx x(t_m)=\frac{v^{1/6}}{M}-\frac{M_3}{6M^3}v^{1/3}\ln v+O(v^{1/3}) \;, \tag{A33}$$

where we have ignored $vx_1(t_m)$ in the first equality.

Before computing $\langle\theta\rangle_t$ to order $v\ln v$, we need to match expansion (iii) to expansion (i) as $x\to\Delta$. From Eq. (A25) we can see that for large times,

$$x_0(t)=\Delta-O(e^{-\Gamma(t-t_e)}) \;. \tag{A34}$$

The first correction x_1 is given by Eq. (A29) with $t_1\sim t_e$,

$$x_1(t)=\frac{t}{\Gamma}-\frac{1}{\Gamma^2}+O(e^{-\Gamma(t-t_e)}) \;, \tag{A35}$$

yielding for $t-t_e\gg 1$,

$$x(t)=\Delta+\frac{vt}{\Gamma}-\frac{v}{\Gamma^2}+O(e^{-\Gamma(t-t_e)})+O(v^2) \;. \tag{A36}$$

The first two terms are seen to be exactly the small t expansion of $X_A(vt)$ in this region, while the third is the first term of $vX_1(vt)$ [from Eq. (A6)]. Expansion (iii) will be valid provided $(t-t_e)\ll v^{-1}$ while expansion (i) in this region will be valid as long as $(t-t_e)\gg v^{-\alpha}$ (for any $\alpha>0$) so that exponential terms of the form $e^{-\Gamma(t-t_e)}$ which do *not* show up in expansion (i) are small compared to *all* terms which do. This matching of expansion (iii) to (i) is thus trivial; for convenience we pick the matching point to be $2t_e=O(v^{-1/3})$.

The contribution of the vx_1 terms in x to $\langle\theta\rangle_t$ from the time interval from t_m to $2t_e$ will be of order $v^{4/3}$ or smaller (since $vx_1^{\max}\sim v^{2/3}$) and can hence be neglected. The contribution from the region from $2t_e$ to $2\pi/v$ (i.e., 0) is of order v except for a $v\ln v$ term from $t\lesssim 0$ which can straightforwardly [from either $X_1(T)$ or $\chi_0(\tau)$] be seen to be

$$\frac{v}{2\pi}\int_{2t_e}^{2\pi/v}\theta(t)dt=\frac{v}{2\pi}\frac{1}{4M}\ln(v^{-1/3}/v^{-1})+O(v)$$
$$=\frac{v}{12\pi}\frac{\ln v}{M}+O(v) \;. \tag{A37}$$

The contribution to $\langle\theta\rangle_t$ from the rest of region (ii) is also straightforwardly obtained:

$$\frac{v}{2\pi}\int_0^{t_m}\theta(t)dt$$
$$=\frac{v}{2\pi}\left[-\Delta t_m+\frac{1}{M}\int_0^{\tau_m}d\tau[\chi_0(\tau)+v^{1/3}\chi_1(\tau)]\right]$$
$$=\frac{v}{2\pi}\left[-\Delta t_m-\frac{1}{M}\ln(\tau_d-\tau_m)+O(1)\right]$$
$$=\frac{v}{2\pi}\left[-\Delta t_m-\frac{1}{M}\frac{1}{6}\ln v+O(1)\right] \;, \tag{A38}$$

where the $\chi_1(\tau)$ term yields only a negligible $O(v^{7/6}\ln v)$.

We are thus left with only the integral from t_m to $2t_e$. This can be conveniently rewritten as an integral over x_0,

$$\int_{t_m}^{2t_e}\theta(t)dt=\int_{t_m}^{2t_e}(-\Delta+x_0)dt$$
$$=\int_{x_m}^{x_0(2t_e)}\frac{(-\Delta+x_0)}{R(x_0)}dx_0$$
$$=\int_{x_m}^{x_0(t_e)}\frac{-\Delta+x_0}{R(x_0)}dx_0+\int_{x_0(t_e)}^{x_0(2t_e)}\frac{-\Delta+x_0}{R(x_0)}dx_0 \;. \tag{A39}$$

Since both $x_0(t_e)$ and $x_0(2t_e)$ are of order 1 and $R(x_0)\approx\Gamma(\Delta-x_0)$ for $x_0\to\Delta$, the second integral is of order 1 and hence it only contributes $O(v)$ to $\langle\theta\rangle_t$. The dominant behavior of the first integral in Eq. (A39) will come from the lower limit. By expanding $R(x)$ for small x and integrating we obtain

$$\int_{x_m}^{x_0(t_e)}\frac{-\Delta+x}{R(x)}dx=-\frac{\Delta}{Mx_m}-\frac{\Delta M_3}{M^2}\ln x_m -\frac{\ln x_m}{M}+O(x_m\ln x_m)+O(1)$$
$$=-\Delta v^{-1/6}-\frac{\Delta M_3}{3M^2}\ln v -\frac{1}{6M}\ln v+O(1)\ , \qquad (A40)$$

where the upper limit contributes $O(1)$ and the last line is obtained by substituting for x_m from Eq. (A33). By combining the results from Eqs. (A37), (A38), and (A40) and using (A32) and (A10), we obtain

$$\langle\theta\rangle_t=-\frac{1}{2\pi}\left[\Delta M^{-1/3}\tau_d v^{2/3} + v\ln v\left[\frac{1}{6M}+\frac{\Delta M_3}{3M^2}\right]+O(v)\right]\ , \qquad (A41)$$

whence by averaging over the contributing h's (those >1) we can obtain $v(F)$ via $F_T-F=\langle\theta\rangle_{t,h}$. We conclude that

$$v=Bf^{3/2}-Cf^2\ln f+O(f^2)\ , \qquad (A42)$$

where

$$B=F_T^{3/2}\left[\frac{\tau_d}{2\pi}\int_1^{\hat h}dh\,\rho(h)\Delta_h M_h^{-1/3}\right]^{-3/2} \qquad (A43)$$

and

$$C=\tfrac{9}{4}B^2\int_1^{\hat h}dh\frac{\rho(h)}{2\pi}\left[\frac{1}{6M_h}+\frac{\Delta_h M_{3h}}{3M_h^2}\right]\ . \qquad (A44)$$

In terms of h,

$$M_{3h}=-\tfrac{1}{6} \qquad (A45)$$

independent of h,

$$M_h=\tfrac{1}{2}(h^2-1)^{1/2}\ , \qquad (A46)$$

and Δ_h is the smallest positive solution to

$$\Delta_h+h\sin[\phi_s(h)+\Delta]=h\sin\phi_s(h) \qquad (A47)$$

with

$$\frac{\pi}{2}<\phi_s(h)=\cos^{-1}\left[-\frac{1}{h}\right]<\pi\ . \qquad (A48)$$

For $h\gtrsim 1$,

$$\phi_s(h)\approx\pi-[2(h-1)]^{1/2} \qquad (A49)$$

and

$$\Delta_h\approx 3[2(h-1)]^{1/2}\ . \qquad (A50)$$

From this limit, the behavior near the multicritical point can be derived straightforwardly.

2. Weak pinning

For weak pinnning, the asymptotic evaluation of $\langle\theta\rangle_t$ is much more straightforward. For $h<1$ we again have

$$\frac{d\phi}{dt}=R_0(\phi)+vt\ , \qquad (A51)$$

however, the function $R_0(\phi)=-h\sin\phi-\phi$ now has a unique inverse. The desired $\langle\theta\rangle$ can be immediately expanded in powers of v analogously to regime (i) above. We have

$$\phi=\phi_A+\theta\ , \qquad (A52)$$

where

$$R_0(\phi_A)+vt=0 \qquad (A53)$$

and to leading order in v,

$$\theta=\frac{d\phi_A/dt}{R_0'(\phi_A)}\ , \qquad (A54)$$

and hence

$$\langle\theta\rangle_t=\frac{v}{2\pi}\int_{\text{period}}\theta(t)dt =\frac{v}{2\pi}\int_0^{2\pi}\frac{d\phi}{R_0'(\phi)}=-\frac{v}{(1-h^2)^{1/2}}\ , \qquad (A55)$$

where the change of integration variable from t to ϕ is possible in this case since the function $\phi_A(t)$ has no singular parts. From the last equality Eqs. (4.15) and (4.16) follow by averaging Eq. (A55) over h and using Eq. (4.21) with $F_T=0$.

APPENDIX B: SCALING FUNCTION FOR THE ac POLARIZABILITY NEAR THRESHOLD

In this appendix, we derive the scaling function for the low-frequency ac polarizability near threshold. By an asymptotic analysis for small ω and v which extends the results of Appendix A, it can be shown that, as claimed in the text, the most singular frequency-dependent parts of the response can be found from the change in the scaled divergence time, $\delta\tau_d$, of the solution to

$$\frac{d\chi}{d\tau}=\chi^2+\tau+\tilde a_0 e^{-i\Omega\tau}\ , \qquad (B1)$$

where we consider initially a single h so that, e.g., $\Omega=\Omega_h=C_h v^{-1/3}\omega$. For $\tilde a_0=0$ the unperturbed solution is

$$\chi_0=\frac{\mathrm{Ai}'(-\tau)}{\mathrm{Ai}(-\tau)} \qquad (B2)$$

which diverges at scaled time $\tau_d\simeq 2.338$. The linearized deviation $\delta\chi=\chi-\chi_0$ from this solution obeys

$$\frac{d\delta\mathcal{X}}{d\tau}=2\mathcal{X}_0\delta\mathcal{X}+\tilde{a}_0 e^{-i\Omega\tau} \tag{B3}$$

which can be solved, with the boundary condition (at this order in v) $\delta\mathcal{X}(\tau=-\infty)=0$, yielding

$$\delta\mathcal{X}(\tau)=\frac{\tilde{a}_0}{[\mathrm{Ai}(-\tau)]^2}\int_{-\infty}^{\tau} d\tau' e^{-i\Omega\tau'}[\mathrm{Ai}(-\tau')]^2 . \tag{B4}$$

Since $\mathcal{X}_0(\tau)$ diverges near τ_d as $(\tau_d-\tau)^{-1}$, $\delta\mathcal{X}$ will diverge as

$$\delta\mathcal{X}(\tau)\approx\frac{\tilde{a}_0 e^{-i\Omega\tau_d}}{(\tau-\tau_d)^2}\frac{1}{[\mathrm{Ai}'(-\tau_d)]^2}\int_{-\infty}^{\tau_d} d\tau' e^{-i\Omega(\tau-\tau_d)}[\mathrm{Ai}(-\tau')]^2\equiv\frac{\tilde{a}_0 e^{-i\Omega\tau_d}}{(\tau-\tau_d)^2}Y(\Omega) , \tag{B5}$$

where we have defined the scaled integral $Y(\Omega)$. For small $\tilde{a}_0$ the solution to Eq. (B1) will thus diverge at a scaled time $\tau_d-\delta\tau_d$, where

$$\delta\tau_d=\tilde{a}_0 e^{-i\Omega\tau_d}Y(\Omega)+O(\tilde{a}_0^2) \tag{B6}$$

corresponding, for a given h, to a change in time of

$$\delta t_d=C_h v^{-1/3}\delta\tau_d . \tag{B7}$$

A reduction of the divergence time by δt_d implies that for an extra time δt_d the perturbed phase will be near $\phi_r(h)$ rather than $\phi_s(h)$ hence the singular contribution to $\langle\eta(t,h)e^{i\omega t}\rangle_t=\eta_0(h)$ [from Eq. (5.10)] will be

$$[\eta_0(h)]_{\text{singular}}=\frac{v}{2\pi}\Delta_h e^{i\omega t_d(h)}\delta t_d(h) \tag{B8}$$

which after substituting for the scaled $\tilde{a}_0$ from Eq. (5.24) and δt_d from Eqs. (B6) and (B7) yields with $\Delta_h\equiv\phi_r(h)-\phi_s(h)$ the result in Eq. (5.26) in the text:

$$[\eta_0(h)]_{\text{singular}}=\frac{1}{2\pi}\frac{\Delta_h}{C_h}\tilde{A}_0 Y(C_h\omega/v^{1/3}) . \tag{B9}$$

Since the local response $K_h(\omega)$ has a nonsingular part which makes $K_h(\omega=0)=1$, the leading behavior for the polarizability at small ω and v is given by

$$\mathcal{X}^{\text{singular}}(\omega,v)=\frac{1-\int\rho(h)(\Delta_h/2\pi)[Y(0)-Y(\Omega_h)]dh}{\int\rho(h)(\Delta_h/2\pi)[Y(0)-Y(\Omega_h)]dh} \tag{B10}$$

from which Ξ can be obtained by changing variables via $v=Bf^{3/2}$.

The limiting behavior of the singular part of $\mathcal{X}$ can be straightforwardly found from the Ω dependence of $Y(\Omega)$. We can expand $Y(\Omega)$ about $\Omega=0$ as

$$Y(\Omega)=\sum_{n=0}^{\infty}Y_n(i\Omega)^n . \tag{B11}$$

After defining

$$f_n=\frac{1}{[\mathrm{Ai}'(-\tau_d)]^2}\int_{-\infty}^{\tau_d}(-\tau)^n[\mathrm{Ai}(-\tau')]^2 d\tau' \tag{B12}$$

one can derive the recursive formula

$$f_n(2n+1)=(-\tau_d)^n+\frac{f_{n-3}}{2}(n-2)(n-1)n , \tag{B13}$$

whence

$$\begin{aligned}
&f_0=1,\quad Y_0=1 ,\\
&f_1=-\tfrac{1}{3}\tau_d,\quad Y_1=\tfrac{2}{3}\tau_d ,\\
&f_2=\tfrac{1}{5}\tau_d^2,\quad Y_2=\tfrac{4}{15}\tau_d^2 ,\\
&f_3=\tfrac{3}{7}-\tfrac{1}{7}\tau_d^3,\quad Y_3=\tfrac{1}{14}+\tfrac{8}{105}\tau_d^3 ,
\end{aligned} \tag{B14}$$

etc.

The small Ω form of Y yields the differential ac conductivity as low frequencies $\sigma(\omega)=-i\omega\mathcal{X}(\omega)$. This is given by

$$\sigma(\omega)\approx\frac{1}{\langle(\Delta_h/2\pi)Y_1C_h v^{-1/3}\rangle_h}+O(\omega^2) \tag{B15}$$

which from Eqs. (4.43) and (4.35) is equal to the dc differential conductivity dv/dF as expected. The high-frequency (relative to the scale $v^{1/3}$) limit of the scaling form of $\mathcal{X}^{\text{singular}}$ is obtained simply by noting that for $\Omega\gg 1$, the integral Eq. (B5) determining $Y(\Omega)$ is dominated by the upper cutoff yielding

$$Y(\Omega)=\frac{2}{(-i\Omega)^3}+O(\Omega^{-4}) . \tag{B16}$$

Thus, the real part of $\bar{K}$ will be dominated by the constant nonsingular part at high Ω and hence for $v^{1/3}\ll\omega\ll v$,

$$\mathcal{X}^{\text{singular}}(\omega,v)\approx\frac{1-\langle\Delta_h/2\pi\rangle_h}{\langle\Delta_h/2\pi\rangle_h}+\frac{v}{(-i\omega)^3}\frac{\langle(\Delta_h/2\pi)(2/C_h^3)\rangle_h}{\langle\Delta_h/2\pi\rangle_h^2} . \tag{B17}$$

Note that for $1\gg\omega\gg\Omega_h$ this implies that $\omega\mathrm{Im}\mathcal{X}(\omega)<0$. For an equilibrium thermodynamic system, this would violate the usual stability criterion as it would imply that the system does work on the perturbing source. However, for a moving state which is far from equilibrium and already dissipating energy, there is no reason why the perturbed system should dissipate more energy than the unperturbed system and hence $\omega\mathrm{Im}\mathcal{X}(\omega)<0$ is *not* inconsistent with stability. This does suggest, however, that "maximum entropy production" principles cannot be valid for this system since presumably they would predict that the system would try to maximize the energy dissipation rate.

[1]P. Monceau, N. P. Ong, A. M. Portis, A. Meerschaut, and J. Rouxel, Phys. Rev. Lett. **37**, 602 (1976); N. P. Ong and P. Monceau, Phys. Rev. B **16**, 3443 (1977). For a recent review see G. Grüner, Physica **8D**, 1 (1983).

[2]A. H. Thompson, A. Zettl, and G. Grüner, Phys. Rev. Lett. **47**, 64 (1981); Z. Z. Wang *et al.*, Solid State Commun. **46**, 325

(1983); **47**, 439 (1983); J. Dumas, C. Schlenker, J. Marcus, and R. Buder, Phys. Rev. Lett. **50**, 757 (1983).

[3]R. M. Fleming, D. E. Moncton, and D. B. McWhan, Phys. Rev. B **18**, 5560 (1978); R. M. Fleming, D. E. Moncton, J. D. Axe, and G. S. Brown, Phys. Rev. B **30**, 1877 (1984).

[4]R. M. Fleming and C. C. Grimes, Phys. Rev. Lett. **42**, 1423 (1979).

[5]J. W. Brill, N. P. Ong, J. C. Eckert, J. W. Savage, S. K. Khanna, and R. B. Somoano, Phys. Rev. B **23**, 1517 (1981).

[6]P. Monceau, J. Richard, and M. Renard, Phys. Rev. Lett. **45**, 43 (1980).

[7]J. C. Gill, Solid State Commun. **39**, 1203 (1981); A. Zettl and G. Grüner, Phys. Rev. B **26**, 2298 (1982).

[8]R. M. Fleming, Solid State Commun. **43**, 167 (1982).

[9]R. M. Fleming and L. F. Schneemeyer, Phys. Rev. B **28**, 6996 (1983).

[10]J. Richard, P. Monceau, and M. Renard, Phys. Rev. B **25**, 948 (1982).

[11]G. Grüner, A. Zawadowski, and P. M. Chaikin, Phys. Rev. Lett. **46**, 511 (1981).

[12]H. Fukuyama and P. A. Lee, Phys. Rev. B **17**, 535 (1977).

[13]P. A. Lee and T. M. Rice, Phys. Rev. B **19**, 3970 (1979).

[14]L. Sneddon, M. C. Cross, and D. S. Fisher, Phys. Rev. Lett. **49**, 292 (1982). In this paper a more realistic model was analyzed in which the equations of motion are expressed in the moving frame of the CDW. This is necessary in order to correctly describe the details of the ac-dc interference effects and also the phase correlations in the moving CDW. Near threshold, however, the differences between the models are formally irrelevant. The model in this reference must be considered, however, to properly analyze neutron scattering off the sliding CDW.

[15]L. Sneddon, Phys. Rev. B **29**, 719 (1984); **29**, 725 (1984).

[16]R. A. Klemm and J. R. Schrieffer, Phys. Rev. Lett. **51**, 47 (1983).

[17]D. S. Fisher, S. N. Coppersmith, and M. C. Cross, Phys. Rev. Lett. **52**, 481 (1984).

[18]M. Weger and B. Horovitz, Solid State Commun. **43**, 583 (1982).

[19]S. E. Burkov and V. L. Pokrovsky, Solid State Commun. **46**, 609 (1983).

[20]P. Bak, Phys. Rev. Lett. **48**, 692 (1982).

[21]J. Bardeen, Phys. Rev. Lett. **42**, 1498 (1979); **45**, 1978 (1980); J. R. Tucker, IEEE J. Quantum Electron. **QE-15**, 1234 (1979).

[22]G. Grüner, A. Zettl, W. G. Clark, and J. Bardeen, Phys. Rev. B **24**, 7247 (1981).

[23]S. E. Barnes and A. Zawadowski, Phys. Rev. Lett. **51**, 1003 (1983); **51**, 1499(E) (1983).

[24]J. B. Sokoloff, Phys. Rev. B **23**, 1992 (1981); L. Pietronero and S. Strässler, *ibid.* **28**, 5863 (1983); P. B. Littlewood, in *Proceedings of the International Conference on Charge Density Waves in Solids,* edited by G. Hutiray (Springer, Berlin, in press).

[25]S. N. Coppersmith, Phys. Rev. B **30**, 410 (1984); S. N. Coppersmith and D. S. Fisher (unpublished); L. Sneddon, Phys. Lett. **52**, 65 (1984).

[26]D. S. Fisher, Phys. Rev. Lett. **50**, 1486 (1983).

[27]See, for example, J. B. Sokoloff, Ref. 24.

[28]D. S. Fisher (unpublished).

[29]For large $h, \Upsilon(\alpha,h) \simeq -h$ so that the threshold force will be *finite* as long as $\int_0^\infty dh h\rho(h) < \infty$.

[30]It is not *a priori* obvious that there cannot exist steady-state moving solutions with F less than the threshold field F_T at which the last static solution disappears. From Eq. (4.21) their existence would imply that $\langle \theta \rangle < 0$. A simple argument shows that, in fact, $\theta(h,t)$ is positive in steady state for all h and t, from which the absence of coexisting uniformly moving and static solutions follows: The curve $\phi_A(t)$ in the ϕ,t plane (for each β,h) is part of the single continuous curve which is the locus of zeros of the right-hand side of the equation of motion for ϕ. Above this curve $d\phi/dt$ will be strictly negative and below it strictly positive. Thus, any solution which starts above the curve will eventually cross it in finite time and henceforth remain below. Once it crosses this curve, then at most a finite time later, θ will be positive and then remain so.

[31]The definition of Υ for the δ-function distribution (case 1) is somewhat arbitrary. Since χ_0 tends to a constant, which does not appear singular, one could define γ to be $-\frac{1}{2}$ from the obviously singular part of χ_0 which goes as $|f|^{1/2}$. However, this square-root singularity arises from the *cusp* in $\Upsilon(\alpha)$ for $\alpha \to \alpha_s^-$, while the finiteness of $\chi_0(F_T)$ and the form of the divergence of χ_0 for other distributions *both* are due to the jump in $\Upsilon(\alpha)$. Thus, it is more consistent to identify $\chi_0 \to \text{const} \sim |f|^0$ as the dominant singularity and define $\gamma = 0$ for case 1.

[32]K. B. Efetov and A. I. Larkin, Zh. Eksp. Teor. Fiz. **72**, 2350 (1977) [Sov. Phys.—JETP **45**, 1236 (1977)].

[33]For $d \leq 2$ there may be additional divergences; these should not invalidate the following scaling arguments.

[34]A. I. Larkin and Yu. N. Ovchinikov, Zh. Eksp. Teor. Fiz. **65**, 1764 (1973) [Sov. Phys.—JETP **38**, 854 (1974)]; **41**, 960 (1975) [**68**, 1915 (1975)].

[35]R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).

[36]N. P. Ong, G. Verma, and K. Maki, Phys. Rev. Lett. **52**, 663 (1984).

[37]P. Monceau, H. Salva, and Z. Z. Wang, J. Phys. Suppl. **44**, C3-1639 (1983).

[38]A useful conversation with Frank DiSalvo on this point is gratefully acknowledged.

[39]G. Grüner, A. Zettl, W. G. Clark, and A. H. Thompson, Phys. Rev. B **23**, 6813 (1981).

[40]G. Verma and N. P. Ong, Phys. Rev. B **30**, 2928 (1984).

[41]C. C. Grimes (private communication).

[42]R. J. Cava, R. M. Fleming, P. B. Littlewood, E. A. Rietman, L. F. Schneemeyer, and R. G. Dunn, Phys. Rev. B (to be published); W. Wu, L. Mihály, G. Mozurkewich, and G. Grüner, Phys. Rev. Lett. **52**, 2382 (1984).

[43]R. M. Fleming and L. F. Schneemeyer (private communication).

[44]G. Mozurkewich and G. Grüner, Phys. Rev. Lett. **51**, 2206 (1983).

[45]P. B. Littlewood and C. M. Varma (unpublished); see P. B. Littlewood, Ref. 24.

[46]A. T. Fiory, A. F. Hebard, and W. I. Glaberson, Phys. Rev. B **28**, 5075 (1983).

[47]See D. S. Fisher, Phys. Rev. B **22**, 1190 (1980), and references therein.

[48]A related, although simpler, asymptotic expansion exists in the literature. The Dorodnicyn expansion for the period of the strongly damped Rayleigh oscillator discussed in C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw Hill, New York, 1978), pp. 468–483.