Nuclear magnetic resonance in two-dimensional superfluid ^{3}He

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We study the continuous-wave (cw) NMR spectrum of a two-dimensional p-wave superfluid ^{3}He film. Two thickness regimes are considered: thickness comparable to the interparticle separation, and thickness much larger than the interparticle separation but smaller than the superfluid coherence length. We find that the form of the dipole Hamiltonian is similar for both regimes, but numerical differences in the coefficients of various terms in some cases lead to qualitatively different results. We find remarkable differences in the NMR spectrum of the two superfluid phases of interest (Anderson-Brinkman-Morel and Balian-Werthamer types of states) and two thickness regimes. From our results it follows that cw NMR is an excellent diagnostic tool for the pairing state in films, as it is in the bulk. We emphasize the qualitative features of our results which are most likely to prove useful to experimentalists.

I. INTRODUCTION

Very thin films (one or two layers) of ^{3}He behave effectively as two-dimensional systems.[1] At temperatures $\simeq$100 mK a layer of ^{3}He adsorbed on a He II film forms a self-bound, degenerate, two-dimensional Fermi liquid.[2] With further reduction in the temperature, one can expect the transition to a paired state to take place, providing an example of a two-dimensional BCS-type superfluid. Moreover, in thicker films, containing many layers, the superfluid component will still be effectively two dimensional as long as the thickness is smaller than the superfluid coherence length ξ. This is not a severe restriction for the thickness, as ξ may be quite large. ($\xi \approx 150$–200 Å at $T=0$ for bulk ^{3}He.)

In this paper we consider the problem of how to detect the presence of the superfluid component and how to distinguish between different possible phases using the technique of continuous-wave (cw) NMR. We will concentrate our attention on two physically different situations: (i) the very-thin-film regime in which the thickness of a film is comparable to the average interparticle separation (i.e., $d \simeq k_F^{-1}$), and (ii) the thin-film regime in which the thickness is much larger than the interparticle separation, but is small compared to the coherence length (i.e., $k_F^{-1} < d < \xi$). The physical character of the superfluid state can be quite different in these two regimes since in the former the normal component behaves two dimensionally, while in the latter it is effectively three dimensional. We furthermore make the nontrivial but plausible assumption that in the very-thin-film regime the superfluid state is of the p type (i.e., $L=1, S=1$ BCS pairing).[3] No such assumption is needed for the thin films since in that case the pairing interaction is essentially the same as in the bulk.

The absolute minimum of the weak-coupling free energy in a two-dimensional p-type superfluid has twofold degeneracy.[4] Two phases, $\underline{a}$ and $\underline{b}$ [the Anderson-Brinkman-Morel (ABM) and the Balian-Werthamer (BW) types of states, respectively], appear, therefore, to be the most likely candidates for the equilibrium state in a superfluid film. The technique of cw NMR proved very powerful in the case of bulk superfluid ^{3}He,[5–7] allowing a clean distinction between the NMR responses of the normal component and the various superfluid phases. We expect cw NMR to be just as useful in the experimental study of two-dimensional p-type superfluidity, the only possible shortcoming being the weakness of the signal from the two-dimensional sample.

The relevant equilibrium properties of the $\underline{a}$ and $\underline{b}$ phases,[8] as well as Leggett's method for the calculation of the NMR response, are discussed in Sec. II. We also introduce a simple model for the superfluid component in the thin-film regime. From a general symmetry argument we give a form of the dipole Hamiltonian[5] in the two-dimensional superfluid. In Sec. III we calculate explicitly the dipole Hamiltonian for both of the regimes considered using models introduced in Ref. 8 and in Sec. II. On the basis of these explicit forms we find the cw NMR spectrum for the various cases of interest in both regimes. We find that a clear distinction in the NMR response exists between the $\underline{a}$ and $\underline{b}$ phases in very thin films; the question of which one is the stable phase could be easily resolved by experiment. In the $\underline{a}$-phase of the thin-film regime we find that the equilibrium orientation of the order-parameter vector $\vec{d}$ is perpendicular to the angular momentum vector $\vec{l}$. This unusual situation (caused by the combined action of the quasiparticle renormalization factors and different dimensionalities of the superfluid and normal component) enables one to easily distinguish the thin-film regime from, for instance, a flat slab geometry (thickness $d > \xi$, but smaller than any length associated with textures). Numerical estimates of resonance frequencies and critical fields are given and discussed, but emphasis is placed on the qualitative features independent of our models. Finally, in Sec. IV we offer some concluding remarks.

II. METHOD

In this section we consider the properties of p-type superfluid films which we need to use to calculate the cw NMR spectrum. We discuss the specific phases that we will study as well as the general method for calculating the dipole energy.

We choose the z axis to be normal to the film surface. The $L_z=0$ Cooper pairs are taken to be totally suppressed: for the case of very thin films ($d \simeq k_F^{-1}$) this suppression follows trivially from the two-dimensional character of the normal component. In the case of thin films ($k_F^{-1} < d < \xi$), pairing with $L_z=0$ is suppressed by reflection from the boundaries. If we assume that the reflection from both the free surface and the surface of the adsorbate is predominantly specular, the condensation of $L_z=\pm 1$ pairs will only be slightly affected.[9] For $d \ll \xi$, moreover, we will have a further significant reduction of the gap $\Delta(\hat{k})$ for all those directions on the Fermi sphere which are not confined to the x-y plane. Since the size of the Cooper-pair wave function is of the order of ξ, it can not fit within the transverse extension of the film, and the quasiparticles which have their momenta outside the x-y plane will be geometrically precluded from forming Cooper pairs.

For the thin-film regime, finding the exact form for the gap would involve solving the full pairing problem in that geometry. This is really not necessary for our purposes. We have performed our calculations using the following simplified model: Let us impose periodic boundary conditions in the z direction. The superfluid gap in this case is of the form $\Delta(\hat{k},n_z)$, where $\hat{k}$ is a unit vector in the x-y plane, and $k_z \equiv (2\pi/d)n_z$. We approximate $\Delta(\hat{k},n_z)$ by $\tilde{\Delta}\,\delta_{n_z,0}$, that is, we take the superfluid component as being confined to the adsorption plane. The gap parameter $\tilde{\Delta}$ (which, in the $\underline{a}$ and $\underline{b}$ phases that we will consider, is isotropic) may be thought of as being chosen so as to reproduce the overall magnitude of the gap:

$$\tilde{\Delta}=\left[\sum_{n_z=-n_{\max}}^{n_{\max}} \int_0^{2\pi} \frac{d\phi}{2\pi} \,|\, \Delta(\hat{k},n_z) \,|\, ^2\right]^{1/2} . \tag{2.1}$$

Note that $n_{\max} \simeq (k_F d)/2\pi$ may be large in the thin-film regime. In the very-thin-film limit the above assumption reduces to that of purely two-dimensional pairing in the same wave as the thin films. For the latter regime, since the normal component remains three dimensional, it is safe to assume that the nature of the pairing interaction is the same as in the bulk. Hence, the main difference between $\tilde{\Delta}$ and its bulk counterpart comes through a factor accounting for the geometrical restrictions arising from $d \ll \xi$. This will be further discussed in Sec. III B.

In practice, the limits of applicability of our method are not too narrow. In most experimentally accessible cases, the films will not be very thin: Films from 4 to about 50 layers would exhibit three-dimensional behavior for the normal component, while the coherence length $\xi = \xi_0(1-T/T_c)^{-1/2}$, which at $T=0$ is of the order of 200 Å, would still be much larger than the thickness of the film (the thickness of one layer is $\simeq 3.8$ Å). Moreover, the region of T not too far from T_c is the most accessible and probably the most attractive from the experimentalist's standpoint. Keeping this in mind, we believe that our simple model is sufficient to extract at least qualitative information concerning the NMR spectrum for thin superfluid films.

Once the film thickness becomes comparable to the interparticle spacing, so that the normal component itself is two dimensional, our model is exact (for p-wave pairing). The magnitude of the superfluid gap Δ (we remove the tilde when discussing the very-thin-film regime) is still not known due to the different character of the pairing interaction (see the Introduction). Still, the qualitative and even many quantitative features of the cw NMR spectrum do not depend on the specific mechanism which leads to the p-type BCS instability. Therefore, we expect that our results might be a useful guide in a search for the two-dimensional superfluidity.

The symmetry of the order parameter for a superfluid film is reduced from that of the bulk. For p-wave pairing the reduction is from O(3)×O(3) (spin×space) in the bulk to O(3)×O(2) for the film. Note that the lowered spatial O(2) symmetry is precisely the kind we need to observe long-range order in the two-dimensional superfluid.[10] The symmetry reduction turns out to have a nontrivial effect in the NMR spectrum.

As in the bulk case,[11] we can represent the two-dimensional superfluid order parameter by means of a complex tensor $A_{\alpha i}$ in spin and orbit space. The indices $\alpha=1,2,3$ represent spin, and $i=x,y$ are the space indices.

In the weak-coupling limit, the possible stable phases of the two-dimensional superfluid can be determined by using the free-energy functional given in Ref. 12. This was done in Ref. 4, where it was found that the weak-coupling free energy reaches an absolute minimum (there are also many relative minima) for two topologically distinct forms of the order parameter. These phases are simple generalizations of the ABM and BW (Ref. 11) types of phases in the bulk superfluid, even though in three dimensions the BW and ABM phases do not have the same weak-coupling free energy.

For the ABM-type state, which we will call the $\underline{a}$ state, we have

$$A_{\alpha i}=d_\alpha(\vec{\Delta}_1+i\vec{\Delta}_2)_i , \tag{2.2}$$

where $\vec{\Delta}_1, \vec{\Delta}_2$ are two real and orthogonal unit vectors in the x-y plane, with

$$\vec{\Delta}_1 \times \vec{\Delta}_2 = \hat{l} , \tag{2.3}$$

where $\hat{l}$ is a unit vector in the direction of the angular momentum, and $\vec{d}$ is a unit vector in spin space.

For the BW-type state, which, in two dimensions, we shall call the $\underline{b}$ state, the order parameter is specified by

$$A_{\alpha i}=\exp(i\phi)R_{\alpha i}(\hat{n},\theta) , \tag{2.4}$$

where $R_{\alpha i}(\hat{n},\theta)$ are the matrix elements (corresponding to the indicated indices) of an arbitrary[8] three-dimensional rotation around some axis $\hat{n}$, with angle θ, and ϕ is arbitrary. In what follows we will also use an alternative form of the order parameter, i.e., the order-parameter vector $d_\alpha(\hat{k})=A_{\alpha i}\hat{k}_i$,[11] where $\hat{k}$ is a unit vector in two-dimensional momentum space.

We should note the following properties of these a and b phases: Both have an isotropic gap, which is why they have the same weak-coupling energy. Both are anisotropic in spin space. Furthermore, both a and b phases correspond to equal-spin-pairing (ESP) states.[8] We consider it unlikely that strong-coupling effects could make any other phase energetically more favorable, and, for this reason, we have in this paper limited our discussion to these two states.

For future reference we now write the static susceptibility tensors for both phases as we use them in the following section. The expressions are different for the two regimes considered. For very thin films the general expression is[11] (neglecting Fermi-liquid effects)

$$\chi_{\alpha\beta}=\chi_n\left[\delta_{\alpha\beta}-\int_0^{2\pi}\frac{d\phi}{2\pi}[1-Y(\Delta(\hat{k}),T)]\frac{d_\alpha(\hat{k})d_\beta(\hat{k})}{|\vec{d}(\hat{k})|^2}\right], \tag{2.5}$$

where $Y(\Delta(\hat{k}),T)$ is the generalized Yoshida function.[11] Using the forms of the order parameter given in (2.2) and (2.4), we obtain

$$a \text{ phase: } \chi_{\alpha\beta}=\chi_n\{\delta_{\alpha\beta}-[1-Y(T)]d_\alpha d_\beta\}, \tag{2.6a}$$

$$b \text{ phase: } \chi_{\alpha\beta}=\chi_n\{\tfrac{1}{2}[1+Y(T)]\delta_{\alpha\beta}+\tfrac{1}{2}[1-Y(T)]\omega_\alpha\omega_\beta\}. \tag{2.6b}$$

In the above, χ_n is the susceptibility of two-dimensional normal-component ^{3}He for the very-thin-film regime, $Y(T)$ is the Yoshida function,[11] and $\vec{\omega}$ is a unit vector in spin space, normal to $\vec{d}(\hat{k})$ and independent of $\hat{k}$. If $\vec{\omega}$ is taken to be the axis of spin quantization, then[8] the b phase includes only the $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ members of the $S=1$ triplet.

In the thin-film regime the total susceptibility tensor can be defined as

$$\chi_{\alpha\beta}=\frac{1}{2n_{\max}}\sum_{n_z=-n_{\max}}^{n_{\max}}\chi_n\left[\delta_{\alpha\beta}-\int_0^{2\pi}\frac{d\phi}{2\pi}[1-Y(\Delta(\hat{k},n_z),T)]\frac{d_\alpha(\hat{k})d_\beta(\hat{k})}{|\vec{d}(\hat{k})|^2}\right]. \tag{2.7}$$

With the use of our model, (2.2), and (2.4), we can cast the results in the following form:

$$a \text{ phase: } \chi_{\alpha\beta}=\chi_n\{\delta_{\alpha\beta}-\pi(k_Fd)^{-1}[1-Y(T)]d_\alpha d_\beta\}, \tag{2.8a}$$

$$b \text{ phase: } \chi_{\alpha\beta}=\chi_n\{\delta_{\alpha\beta}-(\pi/2)(k_Fd)^{-1}[1-Y(T)](\delta_{\alpha\beta}-\omega_\alpha\omega_\beta). \tag{2.8b}$$

In the thin-film regime, χ_n is the normal-state susceptibility of bulk ^{3}He. The approximation for the true susceptibility tensors that our model introduces has very little effect on our results. This will become clear in Sec. III B.

In (2.6) and (2.8) we have neglected strong-coupling (Fermi-liquid) effects. In the very-thin-film regime these are unknown. For further discussion of this problem, see Sec. III A. In the thin-film regime, as will become clear in Sec. III B, the strong-coupling corrections, apart from a simple susceptibility-enhancement factor, are of little consequence for our results.

To study the NMR properties of these phases it is most convenient to use Leggett's[5] method, suitably extended to our case. The total spin $\vec{S}$ and the order-parameter vector $\vec{d}(\hat{k})$ are treated as a pair of canonically conjugate dynamical variables. The equations of the motion for $\vec{S}$ and $\vec{d}(\hat{k})$ are found to be[5] of the form

$$\dot{\vec{S}}=\gamma\vec{S}\times\vec{H}(t)+\vec{R}_{\rm dip},\quad \dot{\vec{d}}(\hat{k})=\vec{d}(\hat{k})\times\vec{H}'(t), \tag{2.9}$$

where the field $\vec{H}(t)=\vec{H}_0+\vec{H}_{\rm rf}(t)$ (uniform plus oscillating fields), and

$$\vec{H}'\equiv-\frac{\partial\mathcal{H}_{\rm tot}}{\partial\vec{S}}; \tag{2.10}$$

$\vec{R}_{\rm dip}$, the dipole torque on $\vec{S}$, is given by

$$\vec{R}_{\rm dip}\equiv-\int_0^{2\pi}\frac{d\phi}{2\pi}\left[\vec{d}(\hat{k})\times\frac{\delta\mathcal{H}_{\rm dip}}{\delta\vec{d}(\hat{k})}\right]. \tag{2.11}$$

The total Hamiltonian is defined as

$$\mathcal{H}_{\rm tot}=\tfrac{1}{2}\gamma^2\vec{S}\cdot\overleftrightarrow{\chi}^{-1}(\vec{d}(\hat{k}))\cdot\vec{S}-\gamma\vec{S}\cdot\vec{H}+\mathcal{H}_{\rm dip}(\vec{d}(\hat{k})), \tag{2.12}$$

and $\mathcal{H}_{\rm dip}$ is the dipole-interaction Hamiltonian. Obviously, knowledge of $\overleftrightarrow{\chi}^{-1}(\vec{d}(\hat{k}))$ and $\mathcal{H}_{\rm dip}(\vec{d}(\hat{k}))$ closes the set of equations (2.9), and one can study the solutions in different regimes. In order to find the resonance frequencies in cw NMR, one linearizes Eqs. (2.9) and solves for the dynamical susceptibility tensor defined by

$$\vec{S}(\omega)=\gamma^{-1}\overleftrightarrow{\chi}(\omega)\vec{H}_{\rm rf}(\omega).$$

From $\overleftrightarrow{\chi}(\omega)$ one can obtain the resonance frequencies and the weight of the different resonances. Following Leggett, we define two equilibrium tensor quantities:

$$\Omega^2_{\mu\nu}\equiv\gamma^2\chi^{-1}\frac{\partial^2\mathscr{H}_{\rm dip}}{\partial\theta_\mu\partial\theta_\nu}\ , \tag{2.13}$$

where $\partial\vec{\theta}$ stands for an infinitesimal rotation of the $\vec{d}$ vector ($\delta\vec{d}=\partial\vec{\theta}\times\vec{d}$), and

$$\alpha_{\mu\nu}\equiv\chi\,\Delta\chi^{-1}_{\mu\nu}\ , \tag{2.14}$$

where $\Delta\overleftrightarrow{\chi}^{-1}$ is the anisotropic part of the inverse susceptibility tensor and χ is the true equilibrium susceptibility.[5] These two tensors, $\overleftrightarrow{\Omega}^2$ and $\overleftrightarrow{\alpha}$, contain the required information about the equilibrium configuration of the order parameter. For all but one of the various situations that we consider in the following section (the case of the $\underline{b}$ phase in the parallel magnetic field is the exception), the principal axes of $\overleftrightarrow{\Omega}^2$ and $\overleftrightarrow{\alpha}$ coincide and are easily found.

The above method was generalized by Leggett and Takagi[7] to include relaxation effects. They employed a form of a "two-fluid" model suitably defining the superfluid and normal parts of the relevant macroscopic physical quantities (spin, susceptibility, etc.). When the NMR pulse is applied to the system, it is only the superfluid component which is involved in the "tunneling" between the up- and down-spin bands; this causes an imbalance between the superfluid and normal components in each band, which simultaneously relaxes via spin-conserving quasiparticle collisions. The characteristic relaxation time $\tau(T)$ is therefore of the order of the quasiparticle lifetime. If the frequency of the external field is not too large or too small compared to τ^{-1}, this relaxation mechanism determines the damping of the NMR.

To take care of the relaxation effects, it is convenient to introduce a new dynamical variable. The vector $\vec{\eta}$ is defined as the deviation of $\vec{S}_p$ (the superfluid component of the total spin) from its equilibrium value, i.e., $\vec{\eta}\equiv\vec{S}_p-\vec{S}^0_p$. In Ref. 7 it is shown that $\vec{S}^0_p=\overleftrightarrow{\lambda}(T)\cdot\vec{S}$ where, if one neglects Fermi-liquid corrections,

$$\overleftrightarrow{\lambda}(T)=\overleftrightarrow{\chi}_{p,0}\cdot\overleftrightarrow{\chi}_0^{-1}\ . \tag{2.15}$$

Here, $\overleftrightarrow{\chi}_{p,0}$ is the superfluid part[7] of the susceptibility tensor $\overleftrightarrow{\chi}_0$. The subscript 0 indicates that Fermi-liquid effects have been neglected. Our reasons for this have been discussed in connection with Eqs. (2.6)—(2.9). A more general expression for $\overleftrightarrow{\lambda}$ is given in Ref. 7, but it does not yield a closed-form result for the $\underline{a}$ and $\underline{b}$ phases. For very thin films Eq. (2.15) yields, in the $\underline{a}$ and $\underline{b}$ phases, the explicit results

$$\lambda_{\alpha\beta}(T)=[1-f(T)](\delta_{\alpha\beta}-d_\alpha d_\beta)\ , \tag{2.16a}$$

$$\lambda_{\alpha\beta}(T)=\frac{1-f(T)}{1+Y(T)}[\delta_{\alpha\beta}+Y(T)\omega_\alpha\omega_\beta]\ , \tag{2.16b}$$

where[7]

$$f(T)\equiv\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^\infty d\epsilon(\epsilon^2/E^2)\tfrac{1}{2}\beta\,\mathrm{sech}^2(\tfrac{1}{2}\beta E) \tag{2.17}$$

and

$$E=(\epsilon^2+\Delta^2)^{1/2}\ .$$

In the thin-film regime, using our model and retaining only terms linear in $(k_F d)^{-1}$, we obtain the following expressions for the tensor $\overleftrightarrow{\lambda}$ in the $\underline{a}$ and $\underline{b}$ phases:

$$\lambda_{\alpha\beta}(T)=\pi(k_F d)^{-1}[1-f(T)](\delta_{\alpha\beta}-d_\alpha d_\beta)\ , \tag{2.18a}$$

$$\lambda_{\alpha\beta}(T)=(\pi/2)(k_F d)^{-1}[1-f(T)](\delta_{\alpha\beta}+\omega_\alpha\omega_\beta)\ . \tag{2.18b}$$

Note that in (2.18a) and (2.18b), $f(T)$ is the same as in (2.17), but with Δ replaced by $\tilde{\Delta}$.

The extended set of dynamical variables $\vec{S}$, $\vec{d}(\hat{k})$, and $\vec{\eta}$ satisfies a system of three coupled differential equations [see Eqs. (4.20)—(4.22) and the Appendix of Ref. 7]. Since we are considering continuous-wave NMR, we need only the hydrodynamic limit ($\omega\tau\ll1$, $\omega_L\tau\ll1$) of these equations, which then can be written in the following form:

$$\begin{aligned}&\dot{\vec{S}}=\gamma\vec{S}\times\vec{H}(t)+\vec{R}_{\rm dip}\ ,\\ &\dot{\vec{d}}(\hat{k})=\vec{d}(\hat{k})\times\vec{H}'(t)\ ,\\ &\dot{\vec{\eta}}=(1-\lambda)\tau[\vec{R}_{\rm dip}+\gamma^2\vec{S}\times(\overleftrightarrow{\chi}^{-1}\cdot\vec{S})]\ ,\end{aligned} \tag{2.19}$$

where the additional term linear in $\vec{\eta}$ (Ref. 7) appears in the constitutive equation (2.10). λ is the appropriate principal value of $\overleftrightarrow{\lambda}$ in equilibrium.

Equations (2.19) can be solved with only terms of up to first order in $\omega\tau$, $\omega_L\tau$ kept in the final result. The procedure is the same as for Eqs. (2.9), the susceptibility anisotropy is treated exactly and [as in Eqs. (2.9)] the only inputs needed are the static susceptibility tensor [Eqs. (2.6) and (2.8)] and $\mathscr{H}_{\rm dip}(\vec{d}(\hat{k}))$. Given these, we can then use Eqs. (2.19) to calculate the natural widths of the cw NMR resonance peaks.

The conditions under which the two-fluid model and Eqs. (2.19) give correct results are discussed in detail by Leggett and Takagi.[7] For the cw NMR, the results for longitudinal motion are probably exact at any frequency. For the transverse motion the theory gives correct results to first order in $\omega\tau$.

In the above we have outlined the method used in our study. To actually solve the problem in a specific situation we need to know the dipolar part of the Hamiltonian. This can be found, in the thin-film and very-thin-film regimes, by explicit consideration of our specific models, as we shall see below (Sec. III and Appendixes). However, on the basis of the O(3)×O(2) symmetry of the order parameter, we know that $\mathscr{H}_{\rm dip}$ must be of the form

$$\begin{aligned}\mathscr{H}_{\rm dip}(\vec{d}(\hat{k}))=\frac{E_{\rm dip}}{2}\int_0^{2\pi}\frac{d\phi}{2\pi}[\,b\,|\,\vec{d}(\hat{k})\,|^2\ -g\,|\,\vec{d}_\perp(\hat{k})\,|^2\\ +2h\,|\,\vec{d}(\hat{k})\cdot\hat{k}\,|^2]\ .\end{aligned} \tag{2.20}$$

Here, $\vec{d}_\perp(\hat{k})$ is the projection of $\vec{d}(\hat{k})$ to the x-y plane, the integration is taken around the Fermi circle, and $E_{\rm dip}$ is some characteristic energy which measures the contribution of the dipole forces to the superfluid free energy. The factors b, g, and h are of order unity, and are different for the thin-film and very-thin-film regimes (see Appendixes A and B).

The anisotropic part of Eq. (2.20),

$$\mathscr{H}_{\rm dip}^{\rm an}(\vec{d}(\hat{k}))=\frac{E_{\rm dip}}{2}\int_0^{2\pi}\frac{d\phi}{2\pi}[-g\,|\,\vec{d}_\perp(\hat{k})\,|^2 +2h\,|\,\vec{d}(\hat{k})\cdot\hat{k}\,|^2]\,, \quad (2.21)$$

which determines the equilibrium configuration of the order parameter and the NMR response, has a form qualitatively different than that of the bulk superfluid, where it is proportional to

$$\int\frac{d\Omega}{4\pi}\,|\,\vec{d}(\hat{k})\cdot\hat{k}\,|^2\,,$$

with the integration being over the Fermi sphere. This contributes significantly to the differences that we find in the next section between the NMR results in thin films and those for the bulk.

III. RESULTS AND DISCUSSION

In this section we calculate the continuous-wave NMR frequencies for both of the thickness regimes considered. Even though the calculations are formally similar, the results are qualitatively different in many cases, and the degree for which strong quantitative statements can be made is also different, as we shall see.

A. cw NMR for very thin films

The first step is to calculate $\mathscr{H}_{\rm dip}$. For our purposes it is sufficient to use the result and simple model of Ref. 8. The coefficients of g and h and the factor $E_{\rm dip}$ which enter the expression (2.21) for the anisotropic part of the dipole energy are calculated in Appendix A. Figure 1 shows g and h as functions of k_Fd in the range which is relevant experimentally. $E_{\rm dip}$ is estimated to be $\simeq 10^{-13}(1-T/T_c)$ ergs/cm^2 (see Appendix A). $\mathscr{H}_{\rm dip}^{\rm an}$, and the magnetic part of Leggett's Hamiltonian,

$$\mathscr{H}_{\rm mag}=\frac{\gamma^2}{2}\vec{S}\cdot\overleftrightarrow{\chi}^{-1}\cdot\vec{S}-\gamma\vec{S}\cdot\vec{H}\,, \quad (3.1)$$

are responsible for the orientational effects in superfluid ^{3}He (see, for instance, Ref. 11). In Ref. 8 the equilibrium orientation of the order parameter was determined by minimization of $\mathscr{H}_{\rm dip}+\mathscr{H}_{\rm mag}$, taking into account the anisotropy of the magnetic-susceptibility tensor. Only two simple cases were considered with the external, uniform magnetic field either perpendicular or parallel to the film. Once the equilibrium configuration is known, the tensors $\overleftrightarrow{\Omega}^2$ and $\overleftrightarrow{\alpha}$ [Eqs. (2.13) and (2.14)] can be calculated, and we can find the cw NMR spectrum.

We now proceed with the discussion of the *a* and *b* phases using the corresponding forms of the order parameter. Since there are numerical uncertainties in our knowledge of quantities, such as the dipole energy or the magnetic susceptibility, our final values for the resonance frequencies are obviously to be interpreted with caution, although we believe them to be qualitatively correct. This uncertainty is caused, among other reasons, by the present unclear status of the Landau theory of the Fermi liquid in two dimensions (for a detailed discussion of the problem, see Ref. 13). The resolution of this problem is quite beyond the scope of this paper. We do not wish to engage in speculations and we have simply omitted Fermi-liquid corrections in the magnetic susceptibility and ignored (in this regime) the quasiparticle renormalization factors which should appear in $\mathscr{H}_{\rm dip}$. While this approximation might be poor when the overall magnitude of the dipole energy and magnetic susceptibility are concerned, we expect it to be more reasonable for the resonance frequencies, which are ratios of the similarly enhanced quantities.[14] We will briefly return to the quantitative effects of the Fermi-liquid corrections when we give numerical estimates of the various quantities of interest.

The order parameter for the ABM-type state is given in (2.2). We may choose, for convenience, $\vec{\Delta}_1\equiv\hat{x}$ and $\vec{\Delta}_2\equiv\hat{y}$. Then the order parameter can be written in the form

$$d_\alpha(\hat{k})=d_\alpha(\hat{k}_x+i\hat{k}_y)\,. \quad (3.2)$$

Orientational effects, from the magnetic part of the free energy, will therefore tend to orient the $\vec{d}$ vector (in spin space) in a particular direction. Since, from Fig. 1, $h(x)>g(x)$ for any x, it is easy to see that the equilibrium orientation will be that with $\vec{d}\,||\,\hat{l}$, i.e., perpendicular to the film. When a uniform, constant, external magnetic field is introduced, this orientation may change due to the orientational effects of the field. Let us now consider two characteristic cases, $\vec{H}_0$ parallel or perpendicular to $\hat{l}$.

When the applied field is parallel to $\hat{l}$, the dipole and magnetic orientational energies compete with each other. A qualitatively similar effect is found in the study of textures in superfluid ^{3}He-A near a solid wall.[15] While it would be preferable, for the dipole energy alone, to have

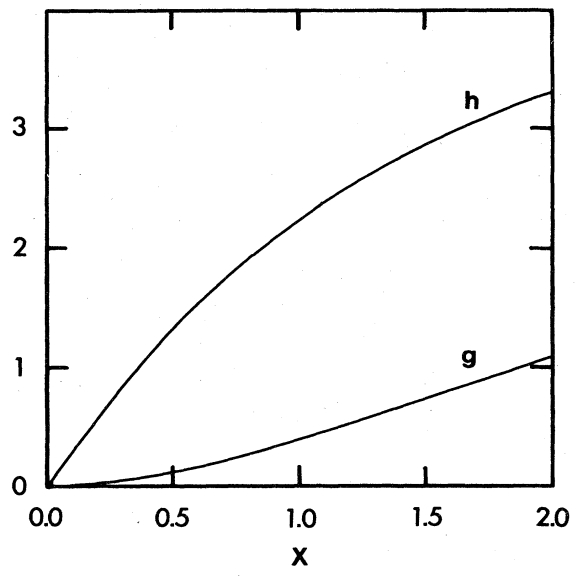

FIG. 1. Coefficients g and h which appear in Eq. (2.21) calculated in the very-thin-film regime. The parameter $x=k_Fd$ compares the spatial extent of the wave function along the z axis to the average separation of particles in the film.

$\vec{d}$ parallel to $\hat{l}$, the magnetic field pushes $\vec{d}$ into the plane of the film. There is a critical magnitude of the magnetic field $H_c^{\underline{a}}$ at which the magnetic energy prevails and the configuration of the order parameter suddenly changes from $\vec{d}\|\hat{l}$ to $\vec{d}$ in the x-y plane. By equating the anisotropic parts of the magnetic and dipole energies [Eqs. (2.21) and (3.1)], we find the critical field strength,

$$H_c^{\underline{a}}=\left[\frac{E_{\rm dip}(h-g)}{\chi_n[1-Y(T)]}\right]^{1/2}. \tag{3.3}$$

There are two separate cases, $H_0<H_c^{\underline{a}}$ and $H_0>H_c^{\underline{a}}$, which we now discuss. (The value of $H_c^{\underline{a}}$ is estimated below.) The results we obtain qualitatively agree with those obtained by Takagi[16] for a flat slab (thickness greater than the coherence length) of three-dimensional superfluid ^{3}He. For $H_0<H_c^{\underline{a}}$, $\vec{d}$ is parallel to $\hat{l}$, and it is obvious from symmetry that there is no longitudinal resonance. Leggett's equations can be solved for the dynamical susceptibilities[5] and we obtain

$$\chi_{xx}(\omega)=\chi_{yy}(\omega)=\chi_n\left\{1-D_<^{-1}(\omega)\left[(\omega^2+\alpha\omega_L^2-\Omega_{\underline{a}}^2)\left[\omega^2+\frac{\alpha}{1-\alpha}\Omega_{\underline{a}}^2\right]-\alpha(1+\alpha)\omega^2\omega_L^2\right]\right\}, \tag{3.4}$$

$$\chi_{xy}(\omega)=-\chi_{yx}(\omega)=\chi_n i\omega\omega_L\left[\omega^2-\alpha^2\omega_L^2+\frac{2\alpha}{1-\alpha}\Omega_{\underline{a}}^2\right]D_<^{-1}(\omega), \tag{3.5}$$

where

$$D_<(\omega)\equiv(\omega^2+\alpha\omega_L^2-\Omega_{\underline{a}}^2)^2-\omega^2\omega_L^2(1+\alpha)^2. \tag{3.6}$$

In the above,

$$\Omega_{\underline{a}}^2\equiv\gamma^2E_{\rm dip}(h-g)/\chi_n,$$

$\alpha\equiv1-Y(T)$, and ω_L is the Larmor frequency. Since the gap in the $\underline{a}$ phase is isotropic, the standard Yoshida function appears in the "susceptibility anisotropy" parameter α.

The poles of the dynamical susceptibility tensor $\overleftrightarrow{\chi}(\omega)$ correspond to the resonance frequencies. Solving $D_<(\omega)=0$, one obtains two resonance frequencies,[5]

$$\omega_\pm=\frac{1}{2}\left[(1-\alpha)\left[\omega_L^2+\frac{4\Omega_{\underline{a}}^2}{(1-\alpha)^2}\right]^{1/2}\pm(1+\alpha)\omega_L\right], \tag{3.7}$$

which correspond to two circularly-polarized transverse modes. For low fields ($\omega_L\ll\Omega_{\underline{a}}$) these resonance frequencies can be approximated by $\Omega_{\underline{a}}\pm(1+\alpha)\omega_L$. On the other hand, as H_0 approaches $H_c^{\underline{a}}$, the frequency of one of the modes goes to zero, while the other is given by $(1+\alpha)\omega_L$.

Using Leggett's theory, extended to include relaxation effects (see Sec. II), one can also find the linewidths of the predicted resonance peaks. Following Ref. 7, we need only the principal values of the tensor $\overleftrightarrow{\lambda}$ (this is true also for the $\underline{b}$ phase, in a field perpendicular to the sample) and we can then solve Eqs. (2.19), taking into account the susceptibility anisotropy (as prescribed in the Appendix of Ref. 7). We obtain the following result for the linewidths of the two transverse modes:

$$\Gamma_\pm^{\underline{a}}=\frac{1-\lambda}{\lambda}\tau\Omega_{\underline{a}}^2\left[\frac{\Omega_{\underline{a}}^2}{\Delta_\pm}\left[1+\alpha^2\frac{\omega_L^2}{\omega_\pm^2}\right]+\alpha^2\omega_L^2\pm2\alpha\frac{\omega_L\Omega_{\underline{a}}^2}{\omega_\pm\Delta_\pm}\pm\alpha\omega_L\omega_\pm\right]\Big/\left[\omega_\pm^2\mp\tfrac{1}{2}\omega_L\left[\alpha^2\frac{\omega_L^2}{\omega_\pm}-\alpha\frac{\Omega_{\underline{a}}^2}{\omega_\pm}+\omega_\pm\right]\right], \tag{3.8}$$

where $\lambda(T)=1-f(T)$, $f(T)$ is defined in (2.17), and

$$\Delta_\pm=1-(\alpha^2\omega_L^2/\omega_\pm^2).$$

The relaxation time τ, introduced in Sec. II, is of order of the normal-state lifetime; for a two-dimensional Fermi liquid corresponding to a ^{3}He layer on a He II film, it is $\simeq10^{-9}$ sec at a temperature of $\simeq1$ mK. Equation (3.8) is valid in the hydrodynamic limit (see Sec. II). Furthermore, following up on our earlier discussion, the normal-state susceptibility-enhancement factor, which would ordinarily come as a prefactor in $\Gamma_\pm^{\underline{a}}$, has been taken to be 1. The rather cumbersome form of $\Gamma_\pm^{\underline{a}}$ is somewhat simplified when T is close to T_c. Then it is safe to neglect the susceptibility anisotropy, and Eq. (3.8) can be written as

$$\Gamma_\pm^{\underline{a}}\simeq\frac{1-\lambda}{\lambda}\tau\Omega_{\underline{a}}^4/(\omega_\pm^2-\tfrac{1}{2}\omega_L\omega_\pm). \tag{3.9}$$

When $H_0>H_c^{\underline{a}}$ the equilibrium direction of the vector $\vec{d}$ will be in the x-y plane. Thus the magnetic energy is minimized, while the dipole energy has its maximum value. This interesting situation was analyzed in Ref. 16, and the result for the dynamical susceptibilities can be simply transcribed to the situation under consideration here. Thus, if $\vec{d}$ is chosen along the y axis,

$$\chi_{xx}(\omega)=\chi_n\left[1-\frac{\omega^2}{D_>(\omega)}\right],$$

$$\chi_{yy}(\omega)=\chi_n\left[1-\frac{\omega^2+\Omega_{\underline{a}}^2}{D_>(\omega)}\right], \tag{3.10}$$

$$\chi_{xy}(\omega)=-\chi_{yx}(\omega)=\chi_n\frac{i\omega\omega_L}{D_>(\omega)},$$

where

$$D_{>}(\omega)=\omega^2-\omega_L^2+\Omega_{\underline{a}}^2 .$$

Again, there is no longitudinal resonance. There is a single elliptically polarized mode of frequency $(\omega_L^2-\Omega_{\underline{a}}^2)^{1/2}$. The negative shift is the consequence of the $\vec{d}$ vector oscillating about the maximum of the dipole energy. Note that $\overleftrightarrow{\chi}(\omega)$ [Eq. (3.10)] does not depend on the susceptibility anisotropy. The width $\Gamma^{\underline{a}}$ of the transverse resonance line does, however. We obtain, for $\Gamma^{\underline{a}}$ (note that the relaxation effects are not considered in Ref. 16),

$$\Gamma^{\underline{a}}=\frac{1-\lambda}{\lambda}\tau\frac{\Omega_{\underline{a}}^2(\Omega_{\underline{a}}^2+\alpha\omega_L^2)}{\omega_L^2-\Omega_{\underline{a}}^2} , \tag{3.11}$$

where

$$\alpha=[1-Y(T)]/[1+Y(T)] .$$

One should not be alarmed by the negative sign in the denominator of (3.11); for $H_0>H_c^{\underline{a}}$, ω_L is always larger than $\Omega_{\underline{a}}$ [since $1-Y(T)\leq 1$ for any T]. Equation (3.11) has been obtained under the assumption that

$$\Gamma^{\underline{a}}/(\omega_L^2-\Omega_{\underline{a}}^2)^{1/2}\ll 1 .$$

For very strong fields, i.e., $\alpha\omega_L^2\gg\Omega_{\underline{a}}^2$,

$$\Gamma^{\underline{a}}\simeq\frac{1-\lambda}{\lambda}\tau\alpha\Omega_{\underline{a}}^2 ,$$

and one finds

$$\Gamma^{\underline{a}}/(\omega_L^2-\Omega_{\underline{a}}^2)^{1/2}\simeq\frac{1-\lambda}{\lambda}\tau\alpha\Omega_{\underline{a}}^2/\omega_L .$$

This quantity is rather small, particularly close to T_c, since then both $\Omega_{\underline{a}}\tau\ll 1$ and $\alpha\ll 1$; therefore the approximations made in arriving at Eq. (3.11) are completely consistent. For intermediate field strengths, i.e., $\omega_L^2\gg\Omega_{\underline{a}}^2$, but $\alpha\omega_L^2\simeq\Omega_{\underline{a}}^2$,

$$\Gamma^{\underline{a}}\simeq\frac{1-\lambda}{\lambda}\tau\frac{\Omega_{\underline{a}}^2}{\omega_L^2}(\Omega_{\underline{a}}^2+\alpha\omega_L^2)$$

and

$$\Gamma^{\underline{a}}/(\omega_L^2-\Omega_{\underline{a}}^2)^{1/2}\simeq\frac{1-\lambda}{\lambda}\tau\frac{\Omega_{\underline{a}}^4+\alpha\omega_L^2\Omega_{\underline{a}}^2}{\omega_L^3} .$$

The upper limit on this expression is provided by either

$$2\frac{1-\lambda}{\lambda}\tau\Omega_{\underline{a}}\left[\frac{\Omega_{\underline{a}}}{\omega_L}\right]^3$$

or

$$2\frac{1-\lambda}{\lambda}\tau\alpha\omega_L\left[\frac{\Omega_{\underline{a}}}{\omega_L}\right]^2 ,$$

depending on whether $\Omega_{\underline{a}}^2>\alpha\omega_L^2$ or $\Omega_{\underline{a}}^2<\alpha\omega_L^2$. In this region the assumption works very well too. In fact, for T close to T_c, $\omega_L\gg\Omega_{\underline{a}}$ ensures that

$$\frac{\Gamma^{\underline{a}}}{(\omega_L^2-\Omega_{\underline{a}}^2)^{1/2}}\ll 1$$

within the general limits of applicability of the two-fluid model of Ref. 7.

Quantitatively, the resonance frequencies $\omega_{\pm}$, the critical field $H_c^{\underline{a}}$, and $\Gamma_{\pm}^{\underline{a}},\Gamma^{\underline{a}}$ all depend on $\Omega_{\underline{a}}$. We can get some sense of what the actual value of $\Omega_{\underline{a}}$ is by using the estimate of $E_{\rm dip}$ (see Appendix A) close to T_c. For the magnetic susceptibility we use the standard Fermi-liquid-theory expression for quasiparticles of effective mass $m^*/m=1.73$ (Ref. 2) and leave out the enhancement factor. The factor $h(x)-g(x)$ is approximately equal to 2 in the region of $x=k_Fd$ of interest here ($x\simeq 1$–2), as can be seen from Fig. 1. In this way we obtain

$$\frac{\Omega_{\underline{a}}}{2\pi}\approx(86\text{ kHz})(1-T/T_c)^{1/2} , \tag{3.12}$$

and this result is only weakly dependent on k_Fd, within the regime under consideration. The magnitude of $\Omega_{\underline{a}}$ is $\simeq 25\%$ smaller than the magnitude of its vapor-pressure counterpart in the slab geometry (without susceptibility enhancement),[11,16] but it is hard to predict to what extent the susceptibility-enhancement factor will reduce (3.12) (the reduction is about a factor of 2 in the bulk). Since ^{3}He films exhibit a very enhanced paramagnetism,[17] it is possible that this reduction is appreciable.

The critical field $H_c^{\underline{a}}$ follows from the above estimates and Eq. (3.3), and it is rather low, of the order of 20 G. Most probably, in a practical NMR experiment the external field will far exceed $H_c^{\underline{a}}$.

The case where $\vec{H}_0$ is perpendicular to $\hat{l}$ is much easier: then the equilibrium direction of $\vec{d}$ is along $\hat{l}$. This configuration minimizes both dipole and magnetic energies, and all the quantities we are interested in are identical to these for the A phase of the bulk superfluid ^{3}He, provided that we replace Leggett's frequency Ω_A by $\Omega_{\underline{a}}$.

We now turn to the discussion of the cw NMR in the $\underline{b}$ phase, with an order parameter of the form (2.4). In equilibrium, with no magnetic field present, the absolute minimum of $\mathscr{H}_{\rm dip}$ is obtained when $\overleftrightarrow{R}(\hat{n},\theta)$ describes a $\pm\pi/2$ rotation about the z axis, i.e., $\hat{n}||\hat{l}$ and Leggett's "magic angle" is $\theta=\pm\pi/2$. We again must distinguish between two orientations of the static field.

When the uniform field is applied in the z direction, both $\hat{n}$ and θ remain unchanged for any magnitude of the field; both magnetic and dipole energies are minimized, as is obvious from (2.6b) and the fact that $\vec{\omega}||\hat{z}$.

In an external, longitudinal, small oscillating field $\vec{H}_{\rm rf}$, θ will be driven to follow the field oscillations, while $\hat{n}$ will remain unchanged. The dynamical susceptibility is of the standard form:

$$\chi_{zz}(\omega)=\chi_n\left[\frac{\Omega_{\underline{b}||}^2}{\Omega_{\underline{b}||}^2-\omega^2}\right] , \tag{3.13}$$

where the longitudinal resonance frequency is

$$\Omega_{\underline{b}||}^2\equiv 2\gamma^2E_{\rm dip}h(x)/\chi_n . \tag{3.14}$$

The frequency $\Omega_{\underline{b}||}$ shows a significant dependence on x, since h (as opposed to $h-g$) varies appreciably with $k_F d$. To estimate $\Omega_{\underline{b}||}$, we can use $x \simeq 1.5$;[2] then $h=2.9$. It follows that

$$\frac{\Omega_{\underline{b}||}}{2\pi} \approx (148 \text{ kHz})(1-T/T_c)^{1/2} . \tag{3.15}$$

The linewidth of the resonance peak is obtained by the same method used for the $\underline{a}$ phase. The result is

$$\Gamma_{||}^{\underline{b}} = \frac{1-\lambda}{\lambda}\tau\Omega_{\underline{b}||}^2 , \tag{3.16}$$

where $\lambda = 1-f(T)$. The existence of this longitudinal mode offers a clear distinction between the $\underline{a}$ and $\underline{b}$ phases.

The transverse components of the dynamical susceptibility tensor follow from Sec. II,

$$\begin{aligned}
&\chi_{xx}(\omega)=\chi_{yy}(\omega)=\chi_n(1-D^{-1}(\omega)\{[\omega^2-\alpha\omega_L^2-(1+\alpha)\Omega_{\underline{b}\perp}^2](\omega^2-\alpha\Omega_{\underline{b}\perp}^2)+\alpha(1-\alpha)\omega^2\omega_L^2\}) ,\\
&\chi_{xy}(\omega)=-\chi_{yx}(\omega)=\chi_n D^{-1}(\omega) i\omega\omega_L(\omega^2-\alpha^2\omega_L^2-2\alpha\Omega_{\underline{b}\perp}^2) ,
\end{aligned} \tag{3.17}$$

where

$$D(\omega)\equiv[\omega^2-\alpha\omega_L^2-(1+\alpha)\Omega_{\underline{b}\perp}^2]^2-\omega^2\omega_L^2(1-\alpha)^2 , \quad \Omega_{\underline{b}\perp}^2\equiv\gamma^2 E_{\text{dip}}(h+2g)/4\chi_n , \quad \alpha=\frac{1-Y(T)}{1+Y(T)} .$$

Thus, there are two circularly-polarized transverse modes with frequencies

$$\omega_{\pm}=\frac{1}{2}\left[(1+\alpha)\left[\omega_L^2+\frac{4\Omega_{\underline{b}\perp}^2}{1+\alpha}\right]^{1/2}\pm(1-\alpha)\omega_L\right] . \tag{3.18}$$

The expression for the linewidth $\Gamma_{\pm}^{\underline{b}}$ is as cumbersome as (3.8). To avoid writing it we note that $\Gamma_{\pm}^{\underline{b}}$ can be obtained from (3.8) if one changes α to $-\alpha$ and replaces $\Omega_{\underline{a}}^2/(1-\alpha)$ by $\Omega_{\underline{b}}^2$ [one should recall Eqs. (3.17) and (2.16b) for α and the transverse principal value of $\overleftrightarrow{\lambda}$ in the $\underline{b}$ phase].

It is very useful to examine the ratio[18]

$$\frac{\Omega_{\underline{b}\perp}^2}{\Omega_{\underline{b}||}^2}=\frac{1+2g/h}{8} . \tag{3.19}$$

Note that this ratio does not depend on the Fermi-liquid corrections to susceptibility. It is independent of the particular type of microscopic interaction which leads to the p-type instability. It is determined solely by the assumption that the normal fluid behaves two dimensionally, and our model for calculating the functions g and h.

From the values of g and h displayed in Fig. 1 the ratio $\Omega_{\underline{b}\perp}^2/\Omega_{\underline{b}||}^2$ is easily found for a given value of the single dimensionless experimental parameter $k_F d$. In the interval $1 \le k_F d \le 2$, this ratio changes monotonically from 0.17 to 0.21.

We now consider the case when the parallel external field is applied to a superfluid film in the $\underline{b}$ phase. In this case for experimentally relevant magnitudes of the external field, the principal axis of the tensors $\overleftrightarrow{\Omega}^2$ and $\overleftrightarrow{\alpha}$ are not the same, and the calculations are much more involved.

To illustrate how the above problem comes about we find the equilibrium configuration of the order parameter (2.4) in an external field $\vec{H}_0$. We take the field to be along the y axis. From the form of the susceptibility (2.6b), it is clear that $\overleftrightarrow{R}(\hat{n},\theta)$ would change from its zero-field form so as to move $\vec{\omega}$ as close to $\vec{H}_0$ as possible without an excessive cost in dipole energy. Since $\mathscr{H}_{\text{dip}}$ has axial symmetry, whatever the final form of $\overleftrightarrow{R}$ is, $\vec{\omega}$ will be in the z-y plane. If we now define

$$\cos\zeta \equiv \vec{\omega}\cdot\hat{z} , \tag{3.20}$$

we can conveniently parametrize $\overleftrightarrow{R}$ in terms of angles ζ and β, the latter describing rotations about the z axis:

$$\overleftrightarrow{R}=\begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta\cos\zeta & \cos\beta\cos\zeta & \sin\zeta \\ \sin\beta\sin\zeta & -\cos\beta\sin\zeta & \cos\zeta \end{bmatrix} . \tag{3.21}$$

Using (3.21) we can write the total anisotropic energy of the system as a function of $\cos\zeta$ and $\cos\beta$:

$$\begin{aligned}
\mathscr{H}_{\text{tot}}^{\text{an}}=\mathscr{H}_{\text{dip}}^{\text{an}}+\mathscr{H}_{\text{mag}}^{\text{an}}=\frac{E_{\text{dip}}}{2}\{&2h[\tfrac{1}{4}\cos^2\beta+\tfrac{1}{8}+\cos^2\zeta(\tfrac{1}{4}\cos^2\beta+\tfrac{1}{8})+\cos\zeta(\tfrac{1}{2}\cos^2\beta-\tfrac{1}{4})]\\
&-g(\tfrac{1}{2}+\tfrac{1}{2}\cos^2\zeta)\}+\tfrac{1}{4}\chi_n[1-Y(T)]H_0^2\cos^2\zeta .
\end{aligned} \tag{3.22}$$

To find the equilibrium form of $\overleftrightarrow{R}$, we need to minimize (3.22) with respect to $\cos\zeta$ and $\cos\beta$; the values of $\cos\zeta$ and $\cos\beta$ for which $\mathscr{H}_{\text{tot}}^{\text{an}}$ has a minimum are found, after some algebra, to be unique:

$$\cos\zeta_0=\frac{E_{\text{dip}}h}{E_{\text{dip}}(h-2g)+2\chi_n(1-Y)H_0^2} , \tag{3.23}$$

$$\cos\beta_0=0 . \tag{3.24}$$

Therefore $\overleftrightarrow{R}$ is the product of a rotation about the z axis for $\pm\pi/2$ and a rotation about the x axis for ζ_0 given by (3.23). This solution requires $|\cos\zeta_0| \le 1$; it follows that H_0 must be larger than

$$H_c^{\underline{b}} \equiv [E_{\text{dip}}g/\chi_n(1-Y)]^{1/2} .$$

Note that $H_c^b \simeq 20$ G only.

Physically what happens is the following: as the parallel magnetic field is turned on the equilibrium configuration remains the same as in the zero field as long as the magnitude of the field is below H_c^b. The vector $\vec{\omega}$ (the ESP spin-quantization axis) is still in the z direction. Once H_0 exceeds H_c^b, however, $\vec{\omega}$ leaves the z axis and moves toward the direction of the field; the angle between $\vec{\omega}$ and $\vec{H}_0$ ($\pi/2-\zeta_0$) keeps getting smaller as H_0 increases as a consequence of the increasing importance of $\mathscr{H}_{\rm mag}^{\rm an}$ relative to $\mathscr{H}_{\rm dip}^{\rm an}$. Finally, as $H_0 \to \infty$, $\vec{\omega}$ aligns parallel to $\vec{H}_0$($\zeta_0 = \pm\pi/2$). It is not difficult to reexpress $\overleftrightarrow{R}^0$ in the conventional way in terms of $\hat{n}$ and Leggett's magic angle.

The above form of $\overleftrightarrow{R}^0$ comes as compromise between the magnetic and dipole energies; neither is minimized in equilibrium. Consequently, the dipolar torque $\vec{R}_{\rm dip}$ does not vanish in equilibrium; for $H_0 > H_c^b$ it equals

$$\hat{x}\chi_n[1-Y(T)]H_0^2\cos\zeta_0\sin\zeta_0/2 \ .$$

Leggett's method, which ordinarily is applicable only when $\vec{R}_{\rm dip}=0$ in equilibrium, can be extended to this case if, in addition to the symmetric part of the tensor $\overleftrightarrow{\Phi}^0$ [defined in Ref. 5, Eq. (5.2)], one also uses the antisymmetric part. The algebra is rather complicated, but straightforward, and we do not display it here since it is not of great practical consequence. To make this clear, let us rewrite (3.23) in the form

$$\cos\zeta_0 = \frac{(H_c^b)^2(h/2g)}{(H_c^b)^2(h/2g)+[H_0^2-(H_c^b)^2]} \ . \tag{3.25}$$

For realistic values of H_0, one always has $H_0^2 \gg (H_c^b)^2$; therefore $\zeta_0 \simeq \pi/2$.

In this limit, $\vec{R}_{\rm dip}$ nearly vanishes and no refinements are needed in Leggett's method. However, one major complication remains: the principal axes of $\overleftrightarrow{\alpha}$ and $\overleftrightarrow{\Omega}^2$ are still different. Since now $\vec{\omega} \| \vec{H}_0$, $\overleftrightarrow{\alpha}$ is diagonal. We write

$$\overleftrightarrow{\alpha} = \frac{1-Y(T)}{1+Y(T)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{3.26}$$

and, from definition (2.13), we find

$$\overleftrightarrow{\Omega}^2 = \begin{bmatrix} \Omega_x^2 & 0 & 0 \\ 0 & \Omega_y^2 & \Omega_{yz}^2 \\ 0 & \Omega_{yz}^2 & \Omega_z^2 \end{bmatrix} , \tag{3.27}$$

where, in the limit $\zeta_0 = \pm\pi/2$,

$$\begin{aligned} \Omega_x^2 &\equiv \gamma^2 E_{\rm dip}(h/2-g)/2\chi_n \ , \\ \Omega_y^2 &= \Omega_z^2 \equiv \gamma^2 E_{\rm dip} h/2\chi_n \ , \\ \Omega_{yz}^2 &\equiv \pm 3\gamma^2 E_{\rm dip} h/8\chi_n \ . \end{aligned} \tag{3.28}$$

There is no clean separation into longitudinal and transverse motion; all three resonance modes are "mixed" together and have nonzero weights for an arbitrary direction of the external oscillating field. The resonance frequencies are obtained by solving the following equation:

$$\begin{aligned} &\left[\omega^2 - \frac{(1+\alpha)\omega^2\Omega_x^2}{\omega^2-\alpha^2\omega_L^2}\right]\left[(\omega^2-\Omega_y^2)\left[\omega^2 - \frac{(1+\alpha)\omega^2\Omega_z^2}{\omega^2-\alpha^2\omega_L^2}\right] - \frac{(1+\alpha)\omega^2\Omega_{yz}^4}{\omega^2-\alpha^2\omega_L^2}\right] \\ &\quad + \left[i\omega\omega_L + \frac{\alpha(1+\alpha)\omega\omega_L\Omega_x^2}{i(\omega^2-\alpha^2\omega_L^2)}\right]\left[\frac{\alpha(1+\alpha)\omega\omega_L\Omega_{yz}^4}{i(\omega^2-\alpha^2\omega_L^2)} + (\omega^2-\Omega_y^2)\left[i\omega\omega_L + \frac{\alpha(1+\alpha)\omega\omega_L\Omega_z^2}{i(\omega^2-\alpha^2\omega_L^2)}\right]\right] = 0 \ . \end{aligned} \tag{3.29}$$

For given values of temperature and H_0, solutions of (3.29) can be obtained numerically. In order to get an idea what the NMR response is, let us consider only temperatures close to T_c (so that $\alpha \simeq 0$). Equation (3.29) then simplifies to

$$(\omega^2-\Omega_x^2)[(\omega^2-\Omega_y^2)(\omega^2-\Omega_z^2)-\Omega_{yz}^4]-\omega^2\omega_L^2(\omega^2-\Omega_y^2)=0 \ . \tag{3.30}$$

If Ω_{yz}^2 were zero, (3.30) would separate into the standard equations for the longitudinal and transverse resonance frequencies. Since this is not the case, we must solve (3.30). However, the algebra can be simplified once we realize that (3.30) holds close to T_c and for $H_0 \gg H_c^b$; then, to an excellent approximation, $\omega_L^2 \gg \Omega_x^2, \Omega_y^2, \Omega_z^2, \Omega_{yz}^2$. The frequencies of the resonance modes are given as

$$\omega_1^2 = \Omega_y^2, \quad \omega_2^2 = \omega_L^2 + \Omega_x^2 + \Omega_z^2 \ , \quad \omega_3^2 = 0 \ , \tag{3.31}$$

Thus, the finite value of Ω_{yz} does not seem to have an influence (in this limit) on the resonance frequencies. However, Ω_{yz} appears in the expressions for the components of the dynamical susceptibility tensor. It is interesting to check what is the extent of "mixing" of the three resonance modes (3.31). To this purpose, we write the diagonal components of the imaginary part of the susceptibility tensor (the results below correspond to the limit of the relaxation time going to zero. For finite τ the expressions are very cumbersome. We also set $\Omega_y = \Omega_z \equiv \Omega_1$):

$$\chi_{xx}''(\omega) = \frac{\pi}{2}\chi_n \frac{\omega[(\omega^2-\Omega_1^2)^2+\Omega_{yz}^4]}{(\omega_1^2-\omega_2^2)(\omega_2^2-\omega_3^2)(\omega_3^2-\omega_1^2)}\Delta(\omega) \ , \tag{3.32}$$

$$\chi_{yy}''(\omega) = \frac{\pi}{2}\chi_n \frac{\omega[(\omega^2-\Omega_1^2)(\omega^2-\Omega_x^2)-\omega^2\omega_L^2]}{(\omega_1^2-\omega_2^2)(\omega_2^2-\omega_3^2)(\omega_3^2-\omega_1^2)}\Delta(\omega) \ , \tag{3.33}$$

$$\chi_{zz}''(\omega) = \frac{\pi}{2}\chi_n \frac{\omega(\omega^2-\Omega_1^2)(\omega^2-\Omega_x^2)}{(\omega_1^2-\omega_2^2)(\omega_2^2-\omega_3^2)(\omega_3^2-\omega_1^2)}\Delta(\omega) \ , \tag{3.34}$$

where

$$\Delta(\omega)\equiv(\omega_2^2-\omega_3^2)\delta(\omega-\omega_1)+(\omega_3^2-\omega_1^2)\delta(\omega-\omega_2)$$
$$+(\omega_1^2-\omega_2^2)\delta(\omega-\omega_3)\ ,$$

and $\omega_1,\omega_2,\omega_3$ are roots of (3.30).

Let us imagine that the oscillating field $\vec{H}_{rf}$ is applied in the y direction. In the limit $\omega_L\to\infty$ (see above), $\omega_3\simeq 0$, and only ω_1 and ω_2 have finite weights in $\chi''_{yy}(\omega)$. From (3.33) it follows that the relative weight of the modes with frequency ω_2 and ω_1 goes as $\simeq\Omega_{yz}^4/\Omega_1\omega_L^3$. Obviously, this is a rather small number, and therefore there is a very little chance that the mixing would be detected experimentally. Similarly, the relative weight of ω_1 and ω_2 in $\chi''_{xx}(\omega)$ is $\simeq(\Omega_{yz}/\omega_L)^3$; again, a small number. Giving that ω_1 does not have any weight at all (in the limit considered) in $\chi''_{zz}(\omega)$, we conclude that for practical purposes the "longitudinal" mode of frequency ω_1 separates from the "transverse" modes with frequencies ω_2 and ω_3.

B. cw NMR for thin films

Now we turn to the case of thin superfluid films within the limits of applicability of our model as explained in Sec. II. From the experimentalist's point of view this regime is probably easier to deal with. Moreover, the nature of the pairing interaction and the relevant properties of the normal liquid are, to a good approximation, the same as in the bulk. This is of considerable help in putting the theory on a more quantitative footing. Many of the calculations needed are very similar to those performed in Sec. III A and we will avoid repetition.

As in Sec. III A, we first find the explicit forms of the dipolar part of the free energy. Using our model this is straightforward (see Appendix B). As expected, the final result has the general form required by symmetry. Explicitly,

$$\mathscr{H}_{\rm dip}^{\rm an}=\frac{\tilde{E}_{\rm dip}}{2}\int_0^{2\pi}\frac{d\phi}{2\pi}[-3(R_0+R_1)\,|\,\vec{d}_\perp(\hat{k})\,|^2$$
$$+6R_0\,|\,\vec{d}(\hat{k})\cdot\hat{k}\,|^2]\ . \qquad (3.35)$$

R_0 and R_1 are the averages over the Fermi circle of the q-dependent quasiparticle renormalization factor $R^2(q)$, as defined in Appendix B. One notices immediately that the effect of the quasiparticle renormalization factors goes beyond merely changing the overall magnitude of the dipole energy; as a consequence of the reduced symmetry, these factors qualitatively determine the equilibrium orientation. Therefore, before any further discussion we must find the values of R_0 and R_1. The quasiparticle renormalization factor $R^2(q)$ has been found explicitly by Takagi[14] within the paramagnon model, and by Fomin *et al.*[19] in the polarization-potential approach. Since, in the present case, the normal fluid has three-dimensional character, we can use the expression in Ref. 19:

$$R^2(q)=\frac{\alpha_q^a}{[1-f_q^a\chi_{\rm SC}^a(q,0)]^2}\ , \qquad (3.36)$$

where we use, for the polarization-potential parameters, the standard notation of Ref. 20. The integrals (Appendix B) required to obtain R_0 and R_1 from Eq. (3.36) can be performed numerically. We have taken α_q^a, f_q^a, and $\chi_{\rm SC}^a$ from the plots in Ref. 20. We obtain $R_0=2.0$ and $R_1=0.5$, at saturated vapor pressure (0.28 atm), as is appropriate for adsorbed films.

We now proceed with the discussion of the cw NMR properties in the *a* and *b* phases on the basis of the dipolar Hamiltonian (3.35) and Leggett's method outlined in Sec. II. In this case, because the normal component has three-dimensional character, we can say that the *a* phase is most probably stable at any temperature since the strong-coupling corrections to the free energy always favor the ABM-type state as opposed to the BW type[11] or the *b* phase. Thus, only a very brief remark concerning the properties of the *b* phase in the thin-film regime will be given at the end of this section. We now discuss the important case of the *a* phase.

The order parameter for the ABM-type state can be chosen in the form given by Eq. (3.2). In zero magnetic field, the vector $\vec{d}$ will orient itself in spin space so as to minimize $\mathscr{H}_{\rm dip}$. A quick look at (3.35) shows that, if we were dealing with particles rather than quasiparticles, our model would have no orientational effect on $\vec{d}$. In zero magnetic field, the direction of $\vec{d}$ would be unspecified. Our method would then be insufficient for the discussion of orientational effects and we would have to include terms of higher order in $(k_Fd)^{-1}$ and d/ξ in our expression for $\mathscr{H}_{\rm dip}$ (see Appendix B). Fortunately, quasiparticle effects are present in $\mathscr{H}_{\rm dip}$; in fact, it is the quasiparticle renormalization factor which determines the orientation of $\vec{d}$ in the lowest order.

Using the values for R_0 and R_1 listed above and Eq. (3.2), we can find $\mathscr{H}_{\rm dip}^{\rm an}$ in terms of the relative orientation of $\vec{d}$ and $\hat{l}(\hat{z})$. After some algebra and integration around the Fermi circle, we obtain

$$\mathscr{H}_{\rm dip}^{\rm an}=\frac{\tilde{E}_{\rm dip}}{2}3R_1(\vec{d}\cdot\hat{l})^2=\tfrac{3}{4}\tilde{E}_{\rm dip}(\vec{d}\cdot\hat{l})^2\ . \qquad (3.37)$$

Obviously, $\mathscr{H}_{\rm dip}^{\rm an}$ has a minimum for $\vec{d}$ perpendicular to $\hat{l}$. Since $\hat{l}$ is perpendicular to the surface of the film, $\vec{d}$ is confined to the x-y plane. As far as the dipole energy is concerned, all directions in this phase are degenerate. This is a rather surprising result, as it is just the opposite of what happens in the very-thin-film regime and in the bulk. As we have already emphasized, the quasiparticle renormalization effects and the reduced symmetry of the order parameter are responsible for this equilibrium orientation of $\vec{d}$.

In a cw NMR experiment, a constant, uniform magnetic field $\vec{H}_0$ is applied to the system. We will again consider only two simple possibilities for the direction of $\vec{H}_0$: $\vec{H}_0$ either parallel or perpendicular to $\hat{l}$. Since the order parameter in a magnetic field takes a form which maximizes the susceptibility, it is clear from (2.8) that $\vec{d}$ will tend to align perpendicularly to $\vec{H}_0$.

When $\vec{H}_0$ is parallel to $\hat{l}$, both magnetic and dipole energies are minimized when $\vec{d}$ lies in the x-y plane. The two vectors $\vec{d}$ and $\vec{S}$ remain perpendicular to each other

throughout the motion (assuming that the applied field is weak, compared to H_0). Equations (2.9) can then be linearized and solved for the dynamical susceptibility tensor. Since rotation of $\vec{d}$ about the z axis ($\hat{l}$) does not change $\mathscr{H}_{\rm dip}$, there is clearly no longitudinal resonance. The components of $\vec{\chi}(\omega)$ are given by Eqs. (3.10), in which we change $\Omega_{\underline{a}}^2$ to $-\tilde{\Omega}_{\underline{a}}^2$ where

$$\tilde{\Omega}_{\underline{a}}^2 \equiv \gamma^2 \tilde{E}_{\rm dip}(3R_1)/\chi_n = \tfrac{3}{2}\gamma^2 \tilde{E}_{\rm dip}/\chi_n \ . \tag{3.38}$$

Thus, we find a single, elliptically polarized transverse mode of frequency

$$\omega = (\omega_L^2 + \tilde{\Omega}_{\underline{a}}^2)^{1/2} \ . \tag{3.39}$$

In contrast to Takagi's[16] result for a superfluid slab, where in the strong perpendicular fields there is a negative shift in the transverse mode [$\omega = (\omega_L^2 - \Omega_A^2)^{1/2}$, where Ω_A is the Leggett frequency; this result has been confirmed experimentally[21]], we predict that once the thickness of the film becomes significantly smaller than the coherence length, the NMR signal would exhibit a positive shift in the transverse resonance. Therefore, at least within the limits of applicability of our method, there is an easy way to distinguish between a slab (with a three-dimensional superfluid component) and a thin film (with a two-dimensional superfluid component).

The susceptibility anisotropy has no effect on (3.39). Therefore, the only uncertainty that our model introduces in the value of the resonance frequency is that which is contained in (3.37). The frequency $\tilde{\Omega}_{\underline{a}}$ which determines the shift can be conveniently written in the following form:

$$\tilde{\Omega}_{\underline{a}}^2 = r\Omega_A^2 \ , \tag{3.40}$$

where $r \cong \tilde{\psi}_{\underline{a}}^2/\psi_A^2$ or, at least close to T_c, $r \cong \tilde{n}_{\underline{a}}/n_A$, where $\tilde{n}_{\underline{a}}$ is the superfluid density in the $\underline{a}$ phase of the thin film and n_A is the corresponding quantity for the bulk A phase. The ratio r is less than unity, and is temperature dependent, but its other properties are difficult to elucidate. In the regime we consider in this subsection it is possible that r is significantly less than 1 and, consequently, any search for the transverse NMR shift of type (3.39) would have to be conducted in a fairly low field.

As in Sec. III A, we can find the linewidth corresponding to the resonance (3.39). The effect of the susceptibility anisotropy can be neglected here, since it is clear from (2.8a) that $\alpha \simeq O((k_F d)^{-1})$. The result is

$$\tilde{\Gamma}_{\perp} = \frac{1-\lambda}{\lambda}\tau \frac{\tilde{\Omega}_{\underline{a}}^4}{\omega_L^2 + \tilde{\Omega}_{\underline{a}}^2}\frac{\chi_n}{\chi_{n0}} \ , \tag{3.41}$$

where λ is given in (2.16a).

If the applied field is in the plane of the film the equilibrium configuration of $\vec{d}$ is along ($\hat{l} \times \vec{H}_0$). Both magnetic and dipole energy are minimized. The equations of motion for $\vec{d}$ and $\vec{S}$ are easily solved and we find that the signal for the transverse NMR is the same as in the normal system; the resonance occurs at ω_L and there is no shift caused by the dipole forces. There is a single longitudinal resonance mode at

$$\omega^2 = \tilde{\Omega}_{\underline{a}}^2 \ . \tag{3.42}$$

The linewidth of this mode is

$$\tilde{\Gamma}_{\parallel} = \frac{1-\lambda}{\lambda}\tau \tilde{\Omega}_{\underline{a}}^2 \frac{\chi_n}{\chi_{n0}} \ . \tag{3.43}$$

Therefore, the peculiar arrangement of vectors $\hat{l}$, $\vec{d}$, and $\vec{H}_0$ in the orthogonal triad produces the same qualitative effect in the cw NMR as the isotropic bulk B phase.

As we have mentioned above, it is unlikely that the results for the $\underline{b}$ phase in this regime are of experimental interest and, for the sake of brevity, we will omit them. The calculations are quite similar to those performed in Sec. III A for the case of very thin films and the results are qualitatively similar.

IV. CONCLUSION

We have studied the cw NMR response of a two-dimensional superfluid, ^{3}He. Two thickness regimes were considered, the very-thin-film ($d \simeq k_F^{-1}$) and the thin-film case ($k_F^{-1} < d < \xi$) with two- and three-dimensional normal components, respectively. There is, as expected, a significant qualitative difference in the NMR spectrum between the $\underline{a}$ and the $\underline{b}$ phases. This difference forms a basis for an experimental identification of the stable phase in a superfluid film; we believe that our results can serve as a useful reference in this sense. Caution is needed when our numerical estimates for resonance frequencies are used; as discussed in Secs. II and III, there are various sources of possible errors which could, at least in principle, influence our numerical estimates. However, it is important to emphasize that there are very many features in the NMR response which are quite insensitive to the approximations we employed. For example, the ratio of the longitudinal and transverse resonance frequencies in the $\underline{b}$ phase of the very-thin-film regime depends only on our model for the two-dimensional Fermi liquid. We do not expect that a more sophisticated model (with a more realistic wave function and included renormalization factors) would produce any major changes in the value of the ratio $\Omega_{\underline{b}\perp}^2/\Omega_{\underline{b}\parallel}^2$ in Sec. III A. In fact, a cw NMR experiment would be quite conclusive in this case; observing the value of the above ratio in the neighborhood of 0.2 would positively identify the $\underline{b}$ phase. It could be even possible to use the deviation from the bulk value to test the model used for the two-dimensional Fermi liquid. Similarly, the $\underline{a}$ phase in the very-thin-film regime could be easily identified by the negative shift in the transverse resonance. The $\underline{a}$ phase in the thin-film regime also has a very distinctive NMR response.

Therefore, we expect that a cw NMR experiment in the two regimes that have been discussed would yield useful information about the two-dimensional superfluidity and the character of the superfluid and normal components in ^{3}He films. From an experimentalist's viewpoint, the major obstacle in designing the experiment would probably be the weakness of the signal from a two-dimensional sample. This, however, is the problem which has appeared frequently in two-dimensional physics and has

been repeatedly overcome. We think that such an experiment is within the reach of today's experimental capabilities and we hope that it will be done in the near future.

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APPENDIX A

The contribution of the dipole interaction to the superfluid free energy can be written in the following form:[11]

$$\mathscr{H}_{\rm dip}=\tfrac{1}{8}\int d^3R\int d^3r\, g_{\alpha\beta}(\vec{r})(\sigma_\alpha)_{\gamma\delta}(\sigma_\beta)_{\mu\nu}\langle\psi^\dagger_\gamma(\vec{R}+\vec{r}/2)\psi^\dagger_\mu(\vec{R}-\vec{r}/2)\rangle\langle\psi_\nu(\vec{R}-\vec{r}/2)\psi_\delta(\vec{R}+\vec{r}/2)\rangle\ , \tag{A1}$$

where

$$g_{\alpha\beta}(\vec{r})=\frac{\gamma^2\hbar^2}{r^3}(\delta_{\alpha\beta}-3\hat{r}_\alpha\hat{r}_\beta)\quad(\hat{r}\equiv\vec{r}/r)\ . \tag{A2}$$

σ_α ($\alpha=1,2,3$) are the Pauli matrices, $\langle\ \rangle$ denotes an anomalous average, and γ is the gyromagnetic ratio. We introduce the spin tensor $F_{\mu\nu}$, defined as

$$F_{\mu\nu}(\vec{r},\vec{R})\equiv\langle\psi_\mu(\vec{R}-\vec{r}/2)\psi_\nu(\vec{R}+\vec{r}/2)\rangle\ . \tag{A3}$$

If a system has full translational symmetry, $F_{\mu\nu}$ will be independent of $\vec{R}$. However, if this symmetry is broken, as in the thin-film geometry, $F_{\mu\nu}$ will generally depend on $\vec{R}$.

In the very-thin-film regime we, in fact, consider a monolayer of ^{3}He on a substrate, and assume that the system behaves like a two-dimensional Fermi liquid. Then the field operator can be expanded as

$$\psi_\alpha(\vec{r})=\sum_{\vec{k}}\frac{1}{\sqrt{A}}a_\alpha(\vec{k})e^{i\vec{k}\cdot\vec{\rho}}\Phi(z)\ , \tag{A4}$$

where A is the area of the film, $\vec{k}$ is the wave vector in the x-y plane, and $\vec{\rho}$ is the projection of $\vec{r}$ on the x-y plane. $\Phi(z)$ is the ground-state wave function for an ^{3}He atom bound on the host surface.

Combining (A4) with (A3), we obtain

$$F_{\alpha\beta}(\vec{\rho},z;Z)=\sum_{\vec{k}}e^{i\vec{k}\cdot\vec{\rho}}\Phi(Z-z/2)\Phi(Z+z/2)F_{\alpha\beta}(\vec{k})\ , \tag{A5}$$

where $Z\equiv\vec{R}\cdot\hat{z}$ and

$$F_{\alpha\beta}(\vec{k})\equiv\langle a_\alpha(-\vec{k})a_\beta(\vec{k})\rangle\ .$$

It is also useful to define the spin vector $\vec{F}$:

$$\vec{F}(\vec{k})\equiv-\frac{i}{2}(\sigma_2\vec{\sigma})_{\alpha\beta}F_{\alpha\beta}(\vec{k})\ . \tag{A6}$$

With the help of (A5) and (A6), after somewhat lengthy algebra, $\mathscr{H}_{\rm dip}$ can be written as

$$\mathscr{H}_{\rm dip}=-\tfrac{1}{2}\gamma^2\hbar^2\sum_{\vec{k}}\sum_{\vec{k}'}F^*_\alpha(\vec{k}')F_\beta(\vec{k})K_{\alpha\beta}(\vec{k}-\vec{k}')\ , \tag{A7}$$

with

$$K_{\alpha\beta}(\vec{q})\equiv\int d^2\rho\, e^{-i\vec{q}\cdot\vec{\rho}}\int dz\, g_{\alpha\beta}(\vec{r})\int dZ\,\Phi^2(Z-z/2)\Phi^2(Z+z/2)\quad(\vec{q}\equiv\vec{k}-\vec{k}')\ . \tag{A8}$$

In order to find $K_{\alpha\beta}(\vec{q})$ we need to know $\Phi(z)$. We assume that we can approximate $\Phi(z)$ by

$$\Phi(z)=\begin{cases}0 & \text{for } |z|>d/2,\\ 1/\sqrt{d} & \text{for } |z|\le d/2\ .\end{cases} \tag{A9}$$

Although this approximation is rather crude, we do not believe any significant error is introduced in our results by making it; it clearly contains the relevant physics. Equation (A9) can be inserted in (A8) with the result

$$K_{\alpha\beta}(\vec{q})=\frac{2\pi}{d}\left\{-\delta_{\alpha\beta}+2\left[1-\exp\left[-\frac{qd}{2}\right]\right]\hat{q}_\alpha\hat{q}_\beta+\left[1+2\exp\left[-\frac{qd}{2}\right]\right]\delta_{\alpha z}\delta_{\beta z}\right\}\quad(\hat{q}\equiv\vec{q}/q)\ . \tag{A10}$$

Finally, we can perform the sum over $|\vec{k}|$ and $|\vec{k}'|$ in (A7) and, using the definition,[11]

$$\vec{d}(\hat{k})\equiv\psi^{-1}\sum_{|\vec{k}|}\vec{F}(\vec{k})\ , \tag{A11}$$

where ψ is the overall magnitude of the order parameter, we obtain, from the explicit form (A10),

$$\mathscr{H}_{\rm dip}=\pi\frac{\gamma^2\hbar^2\psi^2}{d}\int\frac{d\phi}{2\pi}\int\frac{d\phi'}{2\pi}\left\{\vec{d}^{\,*}(\hat{k}')\cdot\vec{d}(\hat{k})\right.$$
$$\left.-\left[1+2\exp\left[-\frac{qd}{2}\right]\right]d_z^*(\hat{k}')d_z(\hat{k})-2\left[1-\exp\left[-\frac{qd}{2}\right]\right]\hat{q}\cdot\vec{d}^{\,*}(\hat{k}')\hat{q}\cdot\vec{d}(\hat{k})\right\}. \tag{A12}$$

To put (A12) into the desired form (2.20), we now use $d_\alpha(\hat{k})=A_{\alpha i}\hat{k}_i$ and insert this in (A12). We obtain

$$\mathscr{H}_{\rm dip}=\frac{\pi\gamma^2\hbar^2\psi^2}{d}\left\{-2A_{zi}^*A_{zj}\int\frac{d\phi}{2\pi}\int\frac{d\phi'}{2\pi}\hat{k}'_i\hat{k}_j\exp\left[-\frac{qd}{2}\right]-2A_{ij}^*A_{ml}\int\frac{d\phi}{2\pi}\int\frac{d\phi'}{2\pi}\hat{q}_i\hat{q}_m\hat{k}'_j\hat{k}_l\left[1-\exp\left[-\frac{qd}{2}\right]\right]\right\}. \tag{A13}$$

There are two types of tensors appearing in (A13):

$$P_{ij}\equiv\int\frac{d\phi}{2\pi}\int\frac{d\phi'}{2\pi}Q(\hat{k}\cdot\hat{k}')\hat{k}'_i\hat{k}_j\ , \tag{A14}$$

$$S_{imjl}\equiv\int\frac{d\phi}{2\pi}\int\frac{d\phi'}{2\pi}Q(\hat{k}\cdot\hat{k}')\hat{q}_i\hat{q}_m\hat{k}'_j\hat{k}_l\ . \tag{A15}$$

From symmetry considerations, it follows that

$$P_{ij}=\tfrac{1}{2}Q_1\delta_{ij}\ , \tag{A16}$$

$$S_{imjl}=\tfrac{1}{4}(Q_1+\tfrac{1}{2}Q_0)\delta_{im}\delta_{jl}-\tfrac{1}{8}Q_0(\delta_{ij}\delta_{ml}+\delta_{il}\delta_{mj})\ , \tag{A17}$$

where

$$Q_0\equiv\int\frac{d\phi}{2\pi}\int\frac{d\phi'}{2\pi}Q(\hat{k}\cdot\hat{k}'),\quad Q_1\equiv\int\frac{d\phi}{2\pi}\int\frac{d\phi'}{2\pi}Q(\hat{k}\cdot\hat{k}')\hat{k}\cdot\hat{k}'\ . \tag{A18}$$

From (A16)–(A18),

$$\mathscr{H}_{\rm dip}=\frac{2\pi\gamma^2\hbar^2\psi^2}{d}[-\tfrac{1}{2}Q_1A_{zi}^*A_{zi}+(-\tfrac{1}{8}+\tfrac{1}{4}Q_1+\tfrac{1}{8}Q_0)A_{ij}^*A_{ij}+(\tfrac{1}{8}-\tfrac{1}{8}Q_0)(A_{ij}^*A_{ji}+A_{ii}^*A_{jj})]\ , \tag{A19}$$

where

$$Q\equiv\exp\left[-\frac{k_Fd}{\sqrt{2}}(1-\hat{k}\cdot\hat{k}')^{1/2}\right].$$

It is a matter of straightforward algebra to show that (A19) can be obtained from

$$\mathscr{H}_{\rm dip}=\frac{2\pi\gamma^2\hbar^2\psi^2}{d}\int\frac{d\phi}{2\pi}[-Q_1\,|\,\vec{d}(\hat{k})\,|^2+(\tfrac{3}{2}Q_1+\tfrac{1}{2}Q_0-\tfrac{1}{2})\,|\,\vec{d}_\perp(\hat{k})\,|^2+2(\tfrac{1}{2}-Q_0/2)\,|\,\vec{d}(k)\cdot\hat{k}\,|^2]\ , \tag{A20}$$

again by using $d_\alpha(\hat{k})=A_{\alpha i}\hat{k}_i$.

Therefore we have finally arrived at the expected form, (2.20), for $\mathscr{H}_{\rm dip}$. We now identify

$$E_{\rm dip}\equiv0.4\frac{\pi\gamma^2\hbar^2\psi^2(T)}{d},\quad b\equiv-10Q_1\ ,\quad g\equiv10(-\tfrac{3}{2}Q_1-\tfrac{1}{2}Q_0+\tfrac{1}{2}),\quad h\equiv10(-Q_0/2+\tfrac{1}{2})\ . \tag{A21}$$

The factor of 10 was introduced to make g and h numbers of order unity. g and h as functions of $x=k_Fd$ are plotted in Fig. 1.

In Ref. 8, $E_{\rm dip}$ was erroneously estimated to be $\simeq10^{-11}(1-T/T_c)$ ergs/cm^2, close to T_c (for ^{3}He layer on He II film[2]). The error was caused by a misplaced factor of $(2\pi)^2$. The correct estimate is

$$E_{\rm dip}\simeq10^{-13}(1-T/T_c)\ \text{ergs/cm}^2\ . \tag{A22}$$

APPENDIX B

We start from the general expression (A7):

$$\mathscr{H}_{\rm dip}=-\tfrac{1}{2}\gamma^2\hbar^2\sum_{\vec{k}}\sum_{\vec{k}'}F_\alpha^*(\vec{k}')F_\beta(\vec{k})R^2(\,|\,\vec{k}-\vec{k}'\,|\,)K_{\alpha\beta}(\vec{k}-\vec{k}')\ , \tag{B1}$$

where now, since the normal component is effectively three dimensional, $K_{\alpha\beta}$ is given by its bulk form,[11]

$$K_{\alpha\beta}(\vec{k}-\vec{k}')=-\frac{4\pi}{3}(\delta_{\alpha\beta}-3\hat{q}_\alpha\hat{q}_\beta)\ , \tag{B2}$$

and $R^2(q)$ is the quasiparticle renormalization factor in the three-dimensional Fermi liquid. If we use the model introduced in Sec. II, we have that $F_\alpha(\vec{k}) \propto \delta_{k_z,0}$; from this and from definition (A11) we obtain

$$\mathscr{H}_{\rm dip} = -\tfrac{1}{2}\gamma^2\hbar^2\tilde{\psi}^2 \int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^{2\pi}\frac{d\phi'}{2\pi}R^2(q)K_{\alpha\beta}(\hat{q})d^*_\alpha(\hat{k}')d_\beta(\hat{k})$$
$$= \frac{2\pi}{3}\gamma^2\hbar^2\tilde{\psi}^2\int\frac{d\phi}{2\pi}\int\frac{d\phi'}{2\pi}R^2(q)[d^*_\alpha(\hat{k}')d_\alpha(\hat{k})-3\hat{q}\cdot\vec{d}^*(\hat{k}')\hat{q}\cdot\vec{d}(\hat{k})]\,, \tag{B3}$$

where $\tilde{\psi}$ is the overall magnitude of the superfluid order parameter in the thin-film regime. Equation (B3) has the same general form as (A12). Using Eqs. (A14)—(A19), we can transform it in the following final form:

$$\mathscr{H}_{\rm dip} = \frac{\tilde{E}_{\rm dip}}{2}\int\frac{d\phi}{2\pi}[2R_1|\vec{d}(\hat{k})|^2-3(R_0+R_1)|\vec{d}_\perp(\hat{k})|^2+6R_0|\vec{d}(\hat{k})\cdot\hat{k}|^2]\,, \tag{B4}$$

where

$$R_0 \equiv \int\frac{d\phi}{2\pi}\int\frac{d\phi'}{2\pi}R^2(q),\quad R_1 \equiv \int\frac{d\phi}{2\pi}\int\frac{d\phi'}{2\pi}R^2(q)\hat{k}\cdot\hat{k}'\,, \tag{B5}$$

and

$$\tilde{E}_{\rm dip} \equiv \frac{2\pi}{3}\gamma^2\hbar^2\tilde{\psi}^2(T)\,. \tag{B6}$$

Note that (B4) is exact in the limit $d/\xi \to 0$, $(k_F d)^{-1} \to 0$. For realistic situations, therefore, our model amounts to retaining only the zeroth-order term in the expansion of $\mathscr{H}_{\rm dip}$ in terms of small quantities d/ξ and $(k_F d)^{-1}$; this is sufficient for the level of accuracy we require in our discussion.

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