Eilenberger equations for moderately dirty superconductors

V. G. Kogan
Ames Laboratory, Iowa State University, Ames, Iowa 50011
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Weak dependence of the Eilenberger Green's functions $f(\mathbf{v})$ upon the direction $\mathbf{v}$ at the Fermi surface is explored to obtain equations for the averages $\langle f(\mathbf{v})\rangle = F$ over the Fermi surface. The derivation is similar to that of Usadel for the dirty limit. The proposed equations, however, are valid not only in the extreme dirty limit, but for "moderately dirty" samples too, i.e., in the case most often encountered in experiment. The formalism also describes superconductivity in a weak field for any impurity concentration and at any temperature.

I. INTRODUCTION

A BCS superconductor with a weak coupling is described by the system of Eilenberger (E) equations[1] in any field, at any temperature, and for any concentration of impurities. The equations read

$$\mathbf{v}\cdot\mathbf{\Pi} f = \frac{2\Delta}{\hbar} g - 2\omega f + \frac{1}{\tau}(g\langle f\rangle - f\langle g\rangle) , \tag{1}$$

$$-\mathbf{v}\cdot\mathbf{\Pi}^* f^+ = \frac{2\Delta^*}{\hbar} g - 2\omega f^+ + \frac{1}{\tau}(g\langle f^+\rangle - f^+\langle g\rangle) , \tag{2}$$

$$g^2 + f f^+ = 1 , \tag{3}$$

$$\Delta \ln\frac{T_c}{T} = 2\pi T \sum_{\omega>0} \left[\frac{\Delta}{\hbar\omega} - \langle f\rangle \right] , \tag{4}$$

$$\mathbf{j} = 4\pi e N(0) T \sum_{\omega>0} \langle \mathbf{v}\,\mathrm{Im} g\rangle , \tag{5}$$

$$f^+(\mathbf{r},\omega,\mathbf{v}) = f^*(\mathbf{r},\omega,-\mathbf{v}),\quad g(\mathbf{r},\omega,\mathbf{v}) = g^*(\mathbf{r},\omega,-\mathbf{v}) . \tag{6}$$

Here $\mathbf{v}$ is the Fermi velocity, $\hbar\omega = \pi T(2n+1)$ with $n=0,1,2,\ldots$. The temperature T is measured in energy units. The gauge-invariant gradient is $\mathbf{\Pi} = \nabla - i2e\mathbf{A}/\hbar c$ and $\mathbf{A}$ is the vector potential. The angular brackets denote an average over all $\mathbf{v}$ directions on the Fermi sphere. The relaxation time for scattering by nonmagnetic impurities is $\tau = l/v$ with l being the mean free path; only S scattering is taken into account. The density of states for one spin direction at the Fermi level is $N(0)$ and $\mathbf{j}$ is the current density. The Eilenberger Green's functions f, f^+, and g are the Gor'kov functions F, F^+, and G integrated over the energy variable. The "pair potential" Δ depends on the position $\mathbf{r}$ only.

The whole phenomenon of superconductivity occurs near the Fermi surface. As a result the exact Green's functions $F(\mathbf{r})$'s and $G(\mathbf{r})$ oscillate in space with a period k_F^{-1}. On the other hand, the macroscopic observable features of superconductors change very slowly on the scale of the coherence length ξ or the penetration depth λ_H of the magnetic field, which are both much larger than k_F^{-1}.

In fact, the E equations are a semiclassical WKB approximation to the BCS theory in Gor'kov's formulation. A substantial gain in dealing with the E functions $f(\mathbf{r})$, $f^+(\mathbf{r})$, and $g(\mathbf{r})$ is that they no longer oscillate with a period k_F^{-1}. Instead, they change on distances of the order ξ, so that they are suitable for inhomogeneous problems in superconductivity. The inequality $k_F\xi \gg 1$ is actually a very weak restriction on the applicability of the E formalism. Many problems in stationary superconductivity are technically much simpler when handled with E equations in comparison with the traditional Gor'kov approach.

Still, the functions $f(\mathbf{r},\omega,\mathbf{v})$ and $g(\mathbf{r},\omega,\mathbf{v})$ are quite complicated objects primarily due to their angular ($\mathbf{v}$) dependence. In some circumstances this dependence is weak, and the E formalism can be simplified.

In the dirty limit where the mean free path $l \ll \xi$, the $\mathbf{v}$ dependence of f and g is smeared by the strong scattering. Usadel[2] reduced the E equations for this case to a simpler set involving the averages $F(\mathbf{r},\omega) = \langle f(\mathbf{r},\omega,\mathbf{v})\rangle$ and $G(\mathbf{r},\omega) = \langle g(\mathbf{r},\omega,\mathbf{v})\rangle$:

$$-\frac{D}{2}\mathbf{\Pi}\cdot(G\mathbf{\Pi}F - F\nabla G) = \frac{\Delta}{\hbar}G - \omega F , \tag{7}$$

$$G^2 + |F|^2 = 1 , \tag{8}$$

$$\Delta \ln\frac{T_c}{T} = 2\pi T \sum \left[\frac{\Delta}{\hbar\omega} - F \right] , \tag{9}$$

$$\mathbf{j} = 4\pi e N(0) D T \sum \mathrm{Im} F^* \mathbf{\Pi} F , \tag{10}$$

where $D = vl/3$ is the diffusion coefficient. All sums hereafter are performed over $\omega > 0$, or $n = 0,1,2,\ldots$.

The Usadel equations hold in the limit $l/\xi \to 0$. More precisely, their domain of validity is given by the strong inequalities[2]

$$G \gg 2\omega\tau ,\quad F \gg 2\Delta\tau/\hbar . \tag{11}$$

The condition $l \ll \xi$ is actually met only in extreme cases of a very short mean free path, e.g., in amorphous materials. However, situations most often encountered in practice are those with l smaller than ξ, but still of the same order of magnitude. The question arises as to whether a system of equations for the averages F and G can still be formulated for "moderately dirty" materials.

Another situation where the $\mathbf{v}$ dependence of the E functions is weak arises near the critical temperature T_c. In Eq. (1) the only $\mathbf{v}$-dependent term is small:

$$\frac{1}{\omega}\mathbf{v}\cdot\mathbf{\Pi} f \sim \frac{v}{\omega\xi} f \ll f$$

due to divergence of $\xi(T)$. In this domain the E set of equations (1)—(6) yields the Ginzburg-Landau (GL) equations[3,4]

$$-\xi_{\rm GL}^2\Pi^2\Delta=\Delta\left[1-\left|\frac{\Delta}{\Delta_0}\right|^2\right], \qquad (12)$$

$$\Delta_0^2=\frac{8\pi^2T_c(T_c-T)}{7\zeta(3)},$$
$$\xi_{\rm GL}^2=\frac{7\zeta(3)\hbar^2v^2}{48\pi^2T_c(T_c-T)}\chi(\lambda). \qquad (13)$$

Here $\zeta(3)=1.202$,

$$\chi(\lambda)=\frac{8}{7\zeta(3)}\sum_{n=0}^{\infty}(2n+1)^{-2}(2n+1+\lambda)^{-1},$$

and $\lambda=\hbar/2\pi T_c\tau$ is the impurity parameter. The current density expression provides the second equation for Δ and $\mathbf{A}$; it can be found elsewhere.

II. WEAK ANISOTROPY

Let us begin with the observation that the $\mathbf{v}$ dependence of f,g is a result of inhomogeneity. Indeed, the Fermi velocity vector $\mathbf{v}$ arises in Eq. (1) only in the combination $\mathbf{v}\cdot\boldsymbol{\Pi}$ with the gauge-invariant gradient. By "inhomogeneity" we mean here the magnetic field, the persistent current, or the proximity effect in an otherwise uniform superconductor. We shall use the term "field" in a broad sense for any source of inhomogeneity.

In the uniform situation $\langle g\rangle=g$, $\langle f\rangle=f$. Equations (1)—(3) then give

$$f^{(0)}=(f^{+(0)})^*=\frac{\Delta_0}{\beta_0},\quad g^{(0)}=\frac{\hbar\omega}{\beta_0},$$
$$\beta_0^2=\hbar^2\omega^2+|\Delta_0^2(T)|, \qquad (14)$$

where Δ_0 depends only on T and satisfies the self-consistency equation (4):

$$\ln\frac{T_c}{T}=2\pi T\sum\left[\frac{1}{\hbar\omega}-\frac{1}{\beta_0}\right]. \qquad (15)$$

This yields the BCS gap $\Delta_0(T)$.

One expects the anisotropy of f and g (i.e., their $\mathbf{v}$ dependence) to be weak in a weak field or when the strong scattering in dirty samples smears this anisotropy out. For this reason we look for a solution of the E equations in the form

$$f=f_0+f_1+f_2+\cdots,$$
$$g=g_0+g_1+g_2+\cdots, \qquad (16)$$

where f_0 and g_0 are isotropic $\mathbf{v}$-independent parts; f_1 and g_1 are linear in the operator $\mathbf{v}\cdot\boldsymbol{\Pi}$; f_2 and g_2 are quadratic in components $v_i\Pi_i$, and so on. We assume the series (16) are convergent. Terms of different orders in (16) have different angular dependence on the Fermi sphere. Substituting expansions (16) in the original E system (1)—(4) one can, under assumptions made, compare terms of the same order (and of the same angular dependence) separately.

We introduce the vector $\mathbf{l}=\mathbf{v}\tau$ for convenience and multiply Eqs. (1) and (2) by τ to obtain

$$\mathbf{l}\cdot\boldsymbol{\Pi}f=g\tilde F-f\tilde G,\quad -\mathbf{l}\cdot\boldsymbol{\Pi}^*f^+=g\tilde F^*-f^+\tilde G \qquad (17)$$

with

$$\tilde F=F+2\Delta\tau/\hbar,\quad \tilde G=G+2\omega\tau. \qquad (18)$$

Writing Eqs. (17), we used the symmetry relations (6); e.g.,

$$F^+=\langle f^+(\mathbf{v})\rangle=\langle f^*(-\mathbf{v})\rangle=\langle f^*(\mathbf{v})\rangle=F^*.$$

Now substitute expansions (16) in Eqs. (17) and (3) and collect the isotropic terms:

$$0=g_0\tilde F-f_0\tilde G,$$
$$0=g_0\tilde F^*-f_0^*\tilde G, \qquad (19)$$
$$1=g_0^2+|f_0|^2.$$

The solutions are

$$f_0=\frac{\tilde F}{\gamma},\quad g_0=\frac{\tilde G}{\gamma},\quad \gamma^2=\tilde G^2+|\tilde F|^2. \qquad (20)$$

Note now that the term $\mathbf{l}\cdot\boldsymbol{\Pi}f_k$ is of order $k+1$ due to the assumption made about the weak anisotropy. Then for the first-order terms in Eqs. (17) and (3) we have

$$\mathbf{l}\cdot\boldsymbol{\Pi}f_0=g_1\tilde F-f_1\tilde G,$$
$$-\mathbf{l}\cdot\boldsymbol{\Pi}^*f_0^*=g_1\tilde F^*-f_1^+\tilde G, \qquad (21)$$
$$0=2g_0g_1+f_0^*f_1+f_0f_1^+.$$

Solving this system with respect to g_1, f_1, and f_1^+, we obtain the first anisotropic corrections to f_0 and g_0:

$$g_1=\frac{i}{\gamma}\mathbf{l}\cdot{\rm Im}f_0^*\boldsymbol{\Pi}f_0=i\gamma^{-3}\mathbf{l}\cdot{\rm Im}\tilde F^*\boldsymbol{\Pi}\tilde F,$$
$$f_1=-\frac{1}{2\gamma g_0}\cdot[(2g_0^2+|f_0|^2)\boldsymbol{\Pi}f_0+f_0^2\boldsymbol{\Pi}^*f_0^*], \qquad (22)$$
$$f_1^+=f_1^*(-\mathbf{l}).$$

The f_1 expression can be further simplified making use of $g_0^2+|f_0|^2=1$:

$$f_1=\gamma^{-1}\mathbf{l}\cdot(f_0\nabla g_0-g_0\boldsymbol{\Pi}f_0)$$
$$=\gamma^{-3}\mathbf{l}\cdot(\tilde F\nabla\tilde G-\tilde G\boldsymbol{\Pi}\tilde F). \qquad (23)$$

We now note that the current density $\mathbf{j}$ of Eq. (5) contains the average $\langle\mathbf{v}g\rangle=\langle\mathbf{v}g_1\rangle+O(g_3)$, because both $\mathbf{v}g_0$ and $\mathbf{v}g_2$ are odd in $\mathbf{v}$ components. Therefore, $\mathbf{j}$ can be obtained at this stage already up to the second-order terms inclusive:

$$\mathbf{j}=4\pi eN(0)DT\sum\gamma^{-3}{\rm Im}\tilde F^*\boldsymbol{\Pi}\tilde F. \qquad (24)$$

It is worth noting that this equation gives $\mathbf{j}(\mathbf{r})$ in terms of the vector potential at the same point $\mathbf{r}$, i.e., the $\mathbf{j}$-$\mathbf{A}$ connection is still local.

Similarly, taking the average $\langle\ \rangle$ of both sides of Eq. (1), we obtain

$$\tfrac{1}{2}\langle\mathbf{v}\cdot\boldsymbol{\Pi}f_1\rangle=\Delta G/\hbar-\omega F \qquad (25)$$

accurate up to the second-order terms inclusive. This gives, with the help of Eq. (23),

$$-\frac{\hbar D}{2}\mathbf{\Pi}\cdot\gamma^{-3}(\tilde{G}\mathbf{\Pi}\tilde{F}-\tilde{F}\nabla\tilde{G})=\Delta G-\hbar\omega F\ . \tag{26}$$

To derive another equation for F and G, we note that up to the second-order inclusive $F=f_0+\langle f_2\rangle$, $G=g_0+\langle g_2\rangle$, so that

$$G^2+|F|^2=g_0^2+|f_0|^2+2g_0\langle g_2\rangle$$
$$+f_0\langle f_2^+\rangle+f_0^*\langle f_2\rangle\ .$$

On the other hand, collecting the second-order terms in $g^2+ff^+=1$, we have

$$2g_0\langle g_2\rangle+f_0\langle f_2^+\rangle+f_0^*\langle f_2\rangle=-\langle g_1^2+f_1f_1^+\rangle\ .$$

Therefore,

$$G^2+|F|^2=1-\langle g_1^2+f_1f_1^+\rangle\ .$$

This yields, with the help of Eqs. (22) and (23),

$$G^2+|F|^2=1+\frac{l^2}{3\gamma^6}(\mathrm{Im}\tilde{F}^*\mathbf{\Pi}\tilde{F})^2+|\tilde{G}\mathbf{\Pi}\tilde{F}-\tilde{F}\nabla G|^2]\ . \tag{27}$$

Thus, the set of Eqs. (26) and (27) completed with the self-consistency Eq. (9), the current density Eq. (24), and the explicit expression

$$\gamma=\left[(G+2\omega\tau)^2+\left|F+\frac{2\tau}{\hbar}\Delta\right|^2\right]^{1/2}, \tag{28}$$

describes situations where the Eilenberger functions are only weakly anisotropic. We shall use the abbreviation EWA for this set.

In the dirty limit one sets $\tau=0$ everywhere but in $D=v^2\tau/3$. The point is that at the left of Eq. (26) we have, in fact, the ratio $D/\xi^2\propto l/\xi^2$ which should be considered as finite because $\xi^2\propto l$. Thus, in this limit $\tilde{G}=G$, $\tilde{F}=F$, $\gamma=1$, and the EWA equations reduce to the Usadel set (7)–(10). The first correction to the dirty limit would amount to keeping terms of the order $l/\xi\propto\sqrt{l}$ while neglecting those of the order $(l/\xi)^2\propto l$. We have then for the moderately dirty superconductors the set EWA where Eq. (27) is reduced to $G^2+|F|^2=1$ and

$$\gamma=1+2G\omega\tau+\frac{\tau}{\hbar}(F\Delta^*+F^*\Delta)\ . \tag{29}$$

III. GINZBURG-LANDAU DOMAIN

The derivation of the EWA equations has been based on the convergence of expansions (16). This is certainly the case near T_c, where all quantities change slowly in space [recall that the nth term in (16) consists of a combination of the nth-order derivatives]. Therefore, the validity of this system in the GL domain does not call for any restriction on the mean free path. The GL equations for *any impurity concentration* should follow from our system as $T\rightarrow T_c$.

To show this we note that as $T\rightarrow T_c$, both F and $\tilde{F}$ are small, $\gamma\simeq\tilde{G}$, the correction to unity at the right of Eq. (27) is of order $(lF/\xi\gamma^2)^2\ll F^2$ and can be neglected. Hence,

$$G=1-|F|^2/2\ . \tag{30}$$

Since the left-hand side of Eq. (26) contains already the small derivatives, one can set there $G=1$. In particular, γ of Eq. (28) reduces to $1+2\omega\tau$ so that we have

$$-\frac{\hbar D}{2(1+2\omega\tau)}\Pi^2F=\Delta(1-\tfrac{1}{2}|F|^2)-\hbar\omega F\ . \tag{31}$$

The self-consistency Eq. (9) reads

$$\frac{\Delta}{2\pi T_c}\delta t=\sum\left[\frac{\Delta}{\hbar\omega}-F\right] \tag{32}$$

with $\delta t=(T_c-T)/T_c\ll1$. Because $\Delta/T_c\ll1$ also, we have the estimate

$$F=\frac{\Delta}{\hbar\omega}+\cdots\ , \tag{33}$$

where the ellipsis indicates higher-order terms. To combine Eqs. (31) and (32), we divide Eq. (31) by $\hbar\omega$ and sum up over all $\omega>0$. Then, taking into account Eq. (33), we have

$$-\hbar D\Pi^2\Delta\sum\frac{1}{\hbar^2\omega^2(1+2\omega\tau)}=\Delta\frac{\delta t}{\pi T_c}-\Delta|\Delta|^2\sum\frac{1}{\hbar^3\omega^3}\ . \tag{34}$$

This coincides with the GL equations (12) and (13).

IV. WEAK FIELD

This is another case where the applicability of the EWA equations is not limited to moderately dirty samples. In a weak field all quantities are close to their zero-field values given in Eqs. (14) and (15):

$$F=\frac{\Delta_0}{\beta_0}+F_1\ ,\quad G=\frac{\hbar\omega}{\beta_0}+G_1\ ,\quad \Delta=\Delta_0+\Delta_1\ , \tag{35}$$

where the corrections F_1, G_1, and Δ_1 are small. Then the gradients are small too, e.g.,

$$\mathbf{\Pi}\Delta=\nabla\Delta_1-i\frac{2e}{c\hbar}\mathbf{A}\Delta_0\ . \tag{36}$$

This is not due to the slow variation as in the GL domain, but rather because of the smallness of the variable part of the function in question. Derivatives of all orders are of the same order of magnitude here, unlike the situation in the GL domain.

For this reason the coefficients of the gradients in Eqs. (24) and (26) can be set equal to their zero-field values. In particular Eq. (28) reduces to

$$\gamma=1+\frac{2\tau}{\hbar}\beta_0\ , \tag{37}$$

so that γ is $\mathbf{r}$ independent. We consider here two examples where the application of the EWA equations appears to be sufficient.

The first is the well-known but difficult problem of the penetration of a weak magnetic field into a superconducting half-space ($x>0$) bordering vacuum or insulator (see, e.g., the last section of Ref. 5). In the gauge $A_x=A_z=0$,

$A_y = A(x)$, and $A(\infty)=0$, the current density expression (24) yields readily in a weak magnetic field

$$A''(x) = \frac{32\pi^2 e^2}{c^2 \hbar} N(0) D T \Delta_0^2 \times \sum \beta_0^{-2} \left[1 + \frac{2\tau}{\hbar}\beta_0 \right]^{-1} A(x) .$$

Therefore, the penetration depth λ_H is

$$\lambda_H^{-2} = \frac{16\pi^2 e^2 v^2}{3c^2} N(0) \Delta_0^2(T) T \sum \beta_0^{-2} \left[\frac{\hbar}{2\tau} + \beta_0 \right]^{-1} \quad (38)$$

for any T and τ.

As the second example, we consider the weak proximity effect in the absence of a magnetic field. A superconductor in the half-space $x > 0$ borders a normal metal. If T is close to T_c, the order parameter in the superconductor is given by the known solution of the GL equations: $\Delta = \Delta_0 \tanh(x + x_0)/\xi_{\rm GL}\sqrt{2}$.[3,4] The constant x_0 depends on the properties of the two metals in contact as well as on their interface. At large distances from the boundary deep in the superconductor

$$\Delta = \Delta_0 - \text{const} \times \exp(-x\sqrt{2}/\xi_{\rm GL}) ,$$

so that Δ approaches its uniform value Δ_0 exponentially with the characteristic length

$$\xi_p = \xi_{\rm GL}/\sqrt{2} . \quad (39)$$

This "proximity length" is unrelated to the second material and characterizes the superconductor in question. One expects this length to be of the order of the coherence length ξ. The latter is defined, in fact, by its relation to the upper critical field $H_{c2}(T) = \phi_0/2\pi\xi^2(T)$ with ϕ_0 being the flux quantum. We are going to examine the temperature dependence of the ratio $\xi_p(T)/\xi(T)$.

Far from the interface in the superconductor, the order parameter Δ differs only slightly from Δ_0. Then one can use the "weak-field" formulas (35) and (37). The EWA equations yield

$$-\frac{\hbar D}{2\gamma_0^2} \left[G_0 \left[F_1'' + \frac{2\tau}{\hbar}\Delta_1'' \right] - F_0 G_1'' \right] = \Delta_0 G_1 + G_0 \Delta_1 - \hbar\omega F_1 , \quad (40)$$

$$\frac{\Delta_1}{2\pi T} \ln\frac{T_c}{T} = \sum \left[\frac{\Delta_1}{\hbar\omega} - F_1 \right] . \quad (41)$$

The correction to unity at the right of Eq. (27) contains $(F')^2$ and, therefore, should be neglected. Then, from $G^2 + F^2 = 1$ we obtain $G_0 G_1' = -F_0 F_1'$, so that in fact Eqs. (40) and (41) contain only two unknown functions F_1 and Δ_1. Both F_1 and Δ_1 go to zero simultaneously if the "field" vanishes. This suggests the form of a solution

$$F_1(x,\omega) = \Delta_1(x)\phi(\omega) , \quad (42)$$

where the function ϕ depends only upon ω.

Now Eq. (40) reduces to an equation for $\Delta_1(x)$:

$$\xi_p^2 \Delta_1'' = \Delta_1 \quad (43)$$

with a solution $\Delta_1 \propto \exp(-x/\xi_p)$ and

$$\xi_p^2 = \frac{\hbar D}{2\gamma_0^2} \frac{\phi\beta_0^2 + 2\hbar\omega^2\tau}{\phi\beta_0^3 - \hbar^2\omega^2} . \quad (44)$$

The length ξ_p must be ω independent, because Δ_1 depends only upon x. Thus, Eq. (44) determines the form of $\phi(\omega)$:

$$\phi = \left[\frac{\hbar\omega}{\beta_0} \right]^2 \frac{1 + D\tau/\gamma_0^2\xi_p^2}{\beta_0 - \hbar D/2\gamma_0^2\xi_p^2} . \quad (45)$$

This is to be substituted in Eqs. (42) and (41) to obtain an equation for ξ_p:

$$\frac{1}{2\pi T} \ln\frac{T_c}{T} = \sum \left[\frac{1}{\hbar\omega} - \phi(\omega) \right] . \quad (46)$$

Taking Eq. (15) into account we rewrite this in the more convenient form

$$\sum_{\omega > 0} \left[\frac{1}{\beta_0} - \phi(\omega) \right] = 0 . \quad (47)$$

Equations (47) and (45) can be solved numerically to find ξ_p as a function of temperature and mean free path.

One easily confirms that as $T \to T_c$, Eq. (47) generates the correct result (39) with $\xi_{\rm GL}$ given in Eq. (13) (see Appendix A). At $T=0$, the sum in Eq. (47) is replaced by an integral:

$$2\pi T \sum \to \int d(\hbar\omega) .$$

The latter can be found analytically. Two limiting cases, dirty and clean, are considered in Appendix B. Here are the results. In the dirty limit

$$\left. \frac{\xi_p}{\xi} \right|_{T\to 0,\ l\to 0} = 0.81 , \quad (48)$$

while for the clean material

$$\left. \frac{\xi_p}{\xi} \right|_{T\to 0,\ l\to \infty} = 0.61 . \quad (49)$$

Comparing these numbers with the GL value $1/\sqrt{2}$, we see that the T dependence of the ratio ξ_p/ξ is rather weak.

We close this section by indicating some of the problems that can be treated with the weak-field version of the EWA equations. These are, e. g., (a) the field dependence of the penetration depth $\lambda_H(T,H)$ which is known[3] to be weak near T_c, (b) the change in $\lambda_H(T)$ induced by proximity with a normal metal, and (c) asymptotic behavior of the order parameter at large distances from the vortex core.[6,7]

V. DISCUSSION

As we have seen the EWA equations are valid near T_c in any field as well as in a weak field at any temperature with no restriction upon the mean free path in both cases. In the dirty limit it transforms to Usadel equations and holds for any H and T. Actually, in all these cases the term $\mathbf{l}\cdot\mathbf{\Pi} f$ of Eq. (17) responsible for the anisotropy of the E function, is small. It is, however, hard to outline a region in the HT plane where the proposed equations are valid for a fixed mean free path. The general conditions under which the expansions (16) are convergent rapidly are difficult to obtain.

The moderately dirty materials are described by the theory where the terms of the order l/ξ, neglected by Usadel, are retained. We gather here the equations of this theory:

$$-\frac{\hbar D}{2}\Pi\cdot\gamma^{-3}(\tilde{G}\Pi\tilde{F}-\tilde{F}\nabla G)=\Delta G-\hbar\omega F\ ,$$

$$G^2+|F|^2=1\ ,$$

$$\frac{\Delta}{2\pi T}\ln\frac{T_c}{T}=\sum_{\omega>0}\left[\frac{\Delta}{\hbar\omega}-F\right]\ , \qquad (50)$$

$$\mathbf{j}=4\pi eN(0)DT\sum\gamma^{-3}\mathrm{Im}\tilde{F}^*\Pi\tilde{F}\ ,$$

$$\gamma=1+2\omega\tau G+\tau(F\Delta^*+F^*\Delta)/\hbar\ .$$

These equations certainly hold in the GL domain as well as in a weak field. Therefore, the most severe test one could impose on this theory is to check whether it yields correct results at $T=0$ and, in the same time, in the maximum possible field, i.e., near H_{c2}. Hence, we consider the $H_{c2}(T,l)$ problem in the frame of Eqs. (50) and compare the results with the exact $H_{c2}(0,l)$.[8]

Near H_{c2}, $G\simeq1$, while F and Δ go to 0 and $\gamma=1+2\omega\tau$. Then the first of Eqs. (50) reduces to

$$-\frac{\hbar D}{2\gamma^2}\Pi^2\left[F+\frac{2\tau}{\hbar}\Delta\right]=\Delta-\hbar\omega F\ . \qquad (51)$$

Similar to what has been done in the preceding section, we look for $F(\mathbf{r},\omega)=\Delta(\mathbf{r})\theta(\omega)$. Then Eq. (51) reduces to the form generating Abrikosov's $\Delta(\mathbf{r})$:

$$-\xi^2\Pi^2\Delta=\Delta \qquad (52)$$

with

$$\xi^2=\frac{\hbar D(\theta+2\tau/\hbar)}{2\gamma^2(1-\hbar\omega\theta)}\ . \qquad (53)$$

The coherence length must be ω independent. This determines $\theta(\omega)$:

$$\theta=\frac{1-D\tau/\gamma^2\xi^2}{\hbar\omega+\hbar D/2\gamma^2\xi^2}\ . \qquad (54)$$

The term $D\tau/\gamma^2\xi^2=l^2/3\gamma^2\xi^2\ll1$ and should be neglected; only l/ξ terms are retained in this theory. Substituting now $F=\theta\Delta$ in the self-consistency equation [the third in the set (50)] and taking Eq. (15) into account we obtain

$$\sum_{\omega>0}\left[\frac{1}{\beta_0}-\frac{1}{\hbar\omega+\hbar D/2\xi^2(1+2\omega\tau)^2}\right]=0\ . \qquad (55)$$

This can be solved for $\xi(T,l)$. The highest field, at which Eq. (52) has a nontrivial solution $\Delta(\mathbf{r})$, is $H_{c2}=\phi_0/2\pi\xi^2$.

The roots $\xi(0,l)$ at $T=0$ of Eq. (55) have been found numerically for a number of l's. The middle curve in Fig. 1 shows the results. The mean free path l and $\xi(0,l)$ are given in units $\hbar v/2\pi T_c$. The upper curve shows $\xi_d(0,l)$ calculated with the dirty-limit formula $\xi_d^2=\hbar D/\Delta_0(0)$ or $\xi_d^2=1.187l$ in our units. The lowest curve represents the

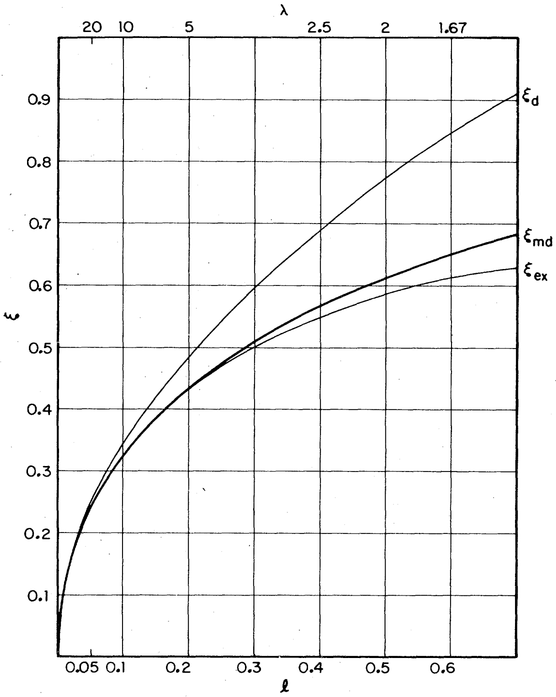

FIG. 1. Coherence length $\xi_{\rm md}(l)$ at zero temperature found by solving Eq. (55) of the theory for moderately dirty materials at $T=0$. The upper curve shows $\xi_d(l)$ for the dirty limit. The lowest curve $\xi_{\rm ex}(l)$ is the exact BCS coherence length. The ξ's and l are in units $\hbar v/2\pi T_c$. The impurity parameter $\lambda=\hbar v/2\pi T_c l$ is given on the upper scale.

exact BCS $\xi_{\rm ex}(0,l)$ obtained in Ref. 8:

$$\xi_{\rm ex}^2=\phi_0/2\pi H_{c2}\ ,\quad H_{c2}(0)=-h^*(l)\left[\frac{dH_{c2}}{dt}\right]_{t=1}\ .$$

Here $h^*(l)$ is almost constant: in the clean case $h^*(\infty)\simeq0.73$, in the dirty limit $h^*(0)\simeq0.69$. The slope

$$\left[\frac{dH_{c2}}{dt}\right]_{t=1}=\frac{\phi_0}{2\pi}\frac{d\xi_{\rm GL}^{-2}}{dt}\ ,$$

and we have $\xi_{\rm ex}^{-2}=-h^*(l)(d\xi_{\rm GL}^{-2}/dt)$. The dimensionless impurity parameter $\lambda=\hbar v/2\pi T_c l$ is shown in Fig. 1 on the upper horizontal scale. One can see that within, e.g., 1% accuracy the dirty theory applies for $\lambda\gtrsim20$, that corresponds to the ratio $l/\xi_{\rm ex}\simeq0.2$. The moderately dirty theory can be used with the same accuracy if $\lambda\gtrsim4$, for which $l/\xi_{\rm ex}\simeq0.5$.

In fact, the curve $\xi(0,l)$ obtained from Eq. (55) follows pretty closely the exact $\xi_{\rm ex}(0,l)$ up to $l/\xi_{\rm ex}\simeq1$. This, however, should not mislead one to exaggerate the accuracy of the theory proposed. Equation (55), which yields $\xi(0,l)$, is correct only as long as $l^2/\xi^2\ll1$. On the other hand, the accuracy of Eqs. (50) should improve out of the domain $T\sim0$ K and $H\sim H_{c2}(0)$.

To conclude, we propose a scheme [Eqs. (9), (24), (26), and (27)] to describe a BCS superconductor in terms of the Eilenberger functions averaged over the Fermi surface: $\langle f(\mathbf{r},\omega,\mathbf{v})\rangle$. The theory holds when the $\mathbf{v}$ dependence of f is weak (GL domain, weak magnetic field, small persistent current, weak proximity effect). In particular, it applies to moderately dirty materials [Eqs. (50)]. In fact, it improves the Usadel theory retaining terms on the order l/ξ neglected in the dirty limit. The current-field relation remains local (as in the dirty limit) as long as the condition $l^2/\xi^2 \ll 1$ is satisfied.

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APPENDIX A

Equation (47) reads

$$0=\sum\left[\frac{1}{\beta_0}-\frac{\hbar^2\omega^2(1+D\tau/\gamma_0^2\xi_p^2)}{\beta_0^2(\beta_0-\hbar D/2\gamma_0^2\xi_p^2)}\right]$$

$$=\sum\frac{\Delta_0^2-\hbar^2\omega D/2\gamma_0\xi_p^2}{\beta_0^2(\beta_0-\hbar D/2\gamma_0^2\xi_p^2)}\ .$$

In the GL domain the numerator is small. Therefore, in the denominator one can set Δ_0 and ξ^{-2} as zero. Then one obtains

$$\Delta_0^2\sum(\hbar\omega)^{-3}=\frac{\hbar D}{2\xi_p^2}\sum(\hbar\omega)^{-2}(1+2\omega\tau)^{-1}\ .$$

This yields $\xi_p^2=\xi_{GL}^2/2$ with the help of Eq. (13).

APPENDIX B

In the dirty limit $\gamma=1$. At $T=0$ we have instead of the sum (47)

$$\int_0^\infty dx\left[\frac{1}{(x^2+\Delta_0^2)^{1/2}}-\frac{x^2}{(x^2+\Delta_0^2)[(x^2+\Delta_0^2)^{1/2}-x_0]}\right]=0\ ,$$

where $x_0=\hbar D/2\xi_p^2$. With $x=\Delta_0\tan\phi$ this becomes

$$\int_0^{\pi/2}d\phi\frac{\Delta_0\cos\phi-x_0}{\Delta_0-x_0\cos\phi}=\frac{\Delta_0}{x_0}\left[2(1-x_0^2/\Delta_0^2)^{1/2}\tan^{-1}\left[\frac{1+x_0/\Delta_0}{1-x_0/\Delta_0}\right]^{1/2}\right]=0\ .$$

The last equation is further reduced to $2(\pi-\theta)\sin\theta=\pi$ by denoting $x_0/\Delta_0=\cos\theta$. The root is $\theta=0.8725$. This yields $\xi_p^2=\hbar D/1.532\Delta_0$. The dirty-limit coherence length at $T=0$ is $\xi^2=\hbar D/\Delta_0$.

In the clean case $\gamma=2\tau\beta_0/\hbar$. Then the function $\phi(\omega)$ of Eq. (45) assumes the form

$$\phi=\frac{\hbar^2\omega^2}{\beta_0^3}\left[1+\frac{\hbar^2v^2}{12\beta_0^2\xi_p^2}\right]\ .$$

Being substituted in Eq. (47) this yields

$$\Delta_0^2\sum\frac{1}{\beta_0^3}=\frac{\hbar^2v^2}{12\xi_p^2}\sum\frac{\hbar^2\omega^2}{\beta_0^5}\ .$$

The sums are easily found at $T=0$. As a result we have $\xi_p=\hbar v/6\Delta_0$. The clean-limit coherence length at $T=0$ is $\xi=0.27\hbar v/\Delta_0$. The latter is taken from Ref. 8 (see Discussion).

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