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Remarks on fractional statistics

D. J. Thouless

Cavendish Laboratory, Madingley Road, Cambridge CB3 OHE, England and Department of Physics FM-15, University of Washington, Seattle, Washington 98195

Yong-Shi Wu*

Department of Physics FM-15, University of Washington, Seattle, Washington 98195

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Some issues in the theory of fractional (or intermediate) statistics are discussed with an eye on possible applications to the quantum Hall effect: spins and statistics of clusters, invariant characterization of statistics, and the effect of space topology.

Recently, the subject of quantum statistics in two (spatial) dimensions has attracted more and more attention. This is because particles in two-space can obey fractional statistics which are characterized by a continuous angular parameter θ and are intermediate between the normal Bose-Einstein ($\theta=0$) and Fermi-Dirac ($\theta=\pi$) statistics. They were suggested in the flux-tube-charged-particle composite model[1,2] and $O(3)$ nonlinear σ model.[3,4] Later a general theoretical framework is established in the Feynman path-integral formalism.[5,6] The relevance of these θ statistics to the fractional quantum Hall effect has also been suggested in different contexts.[7–9] (For related discussions, see also Refs. 10–15.) In this Rapid Communication we shall discuss some issues in the theory of fractional statistics, which are of importance for possible applications to real systems and, in particular, to the quantum Hall effect.

SPIN AND STATISTICS OF A CLUSTER

Consider a two-dimensional system of indistinguishable particles which have a charge e^*, spin s^* and obey the fractional statistics with parameter θ^*. It can be described by a multivalued wave function[5,6]

$$\Psi(\vec{r}_1,\ldots,\vec{r}_N)=\Phi(\vec{r}_1,\ldots,\vec{r}_N)\exp\left(i\frac{\theta^*}{\pi}\sum_{i<j}\phi_{ij}\right)\chi_s \quad (1)$$

together with normal interactions in Hamiltonian. Here, Φ is single valued and totally symmetric for $\vec{r}_1,\ldots,\vec{r}_N$ all different; ϕ_{ij} is the azimuthal angle of $\vec{r}_i-\vec{r}_j$, and χ_s is the spin part of the wave function.

Let us first consider the statistics of a cluster of p particles. Suppose there are two such clusters, and their own sizes can be neglected compared to their separation. The exchange of the two clusters (with no change of internal states) leads, according to the wave function (1), to a phase-factor $\exp(ip^2\theta^*)$. So the θ parameter describing the statistics of the cluster is

$$\theta=p^2\theta^* \ . \quad (2)$$

To consider the spin (i.e., the total angular momentum) of the cluster, we rotate it through an angle 2π about the center of mass $\vec{r}_0$. It is easy to see that the 2π rotation leaves Φ invariant, but gives a factor $\exp(2i\theta)$ for each of the $p(p-1)/2$ pairs in the cluster. It also gives a factor $\exp(2\pi is^*)$ for each particle because of their spins. So the net result is a factor $\exp\{ip[(p-1)\theta^*+2\pi s^*]\}$; namely, the spin of the cluster is

$$S=\frac{p(p-1)\theta^*}{2\pi}+ps^*+\text{integers} \ . \quad (3)$$

Comparing this equation with Eq. (2), we reach the conclusion that the conventional spin-statistics connection is assured for the cluster [i.e., $S=\theta/2\pi \pmod 1$], if it is true for the constituent particles. Note that in this case the spin of the cluster is p^2s^* +integers; the term $p(p-1)\theta^*/2\pi$ in Eq. (3), originated from the extra angular momentum barrier between particles,[2,6] plays an essential role in recovering the spin-statistics connection for the clusters.

One can also obtain the results (2) and (3) in the effective Lagrangian formulation for θ statistics,[4,5] in which wave functions are single valued in $\vec{r}_i$'s, but a topological term

$$-\frac{\theta^*}{\pi}\frac{d}{dt}\sum_{i<j}\phi_{ij}(t) \quad (4)$$

is added to the Lagrangian. Under a 2π rotation, this term contributes a phase factor $\exp[-ip(p-1)\theta^*]$ to the path integral. So the total angular momentum of the cluster receives an extra contribution $p(p-1)\theta^*/2\pi$ in addition to the usual sum of spins ps^*, and the sum of ordinary orbital angular momenta which is always integral.

For the fractional quantum Hall effect, it is the fractionally charged quasiparticles which supposedly obey fractional statistics.[7–9] In this case there is a uniform magnetic field $\vec{B}$ perpendicular to the plane. We assume that interactions between the electrons have axial symmetry and the one-body potential is uniform, at least locally. Then the physical system still has rotational symmetry with the gauge-modified rotation operator

$$L=\sum_j\exp\left(\frac{-ie^*\Lambda(\vec{r}_j)}{\hbar}\right)\left(-i\frac{\partial}{\partial\phi_j}\right)\exp\left(\frac{ie^*\Lambda(\vec{r}_j)}{\hbar}\right) \ ,$$

where

$$\vec{A}=(\tfrac{1}{2})\vec{B}\times\vec{r}+\text{grad}\Lambda(\vec{r}), \quad \phi_j=\tan^{-1}[(y_j-y_0)/(x_j-x_0)] \ .$$

Now it is under 2π rotation by this operator that Φ is invariant. One obtains the same conclusions as before.

Equation (2) has some importance for the discussion of the fractional quantum Hall effect. The Laughlin wave function[16] at fillings $\nu=1/p$ with p odd has been claimed to have the following two properties. (1) The θ parameter for quasiparticles is $1/p$;[8] (2) Creating p quasiparticles at the same point is equivalent to adding an electron.[17] From Eq. (2) one can easily see that for even p it is impossible to write down a wave function that satisfies both conditions, while for charged bosons p must be even, as Halperin[7] has observed. Thus, the physical reason for the nonexistence of fractional quantization with even denominator may be rather deep. An attempt is made in Ref. 9 to use this to explain the origin of the experimentally observed odd-denominator rule. However, it would seem to be easy to modify the many quasiparticle wave function in such a way that $1/p^2$ rather than $1/p$ is the value of θ^*.

INVARIANT CHARACTERIZATION OF θ STATISTICS

As emphasized before,[5] there are different ways to formulate θ statistics. One can use either wave functions having a multivalued phase together with normal interactions, or the usual totally symmetric (or antisymmetric) wave functions which are single valued in $\vec{r}_i$'s with Bohm-Aharonov-type interactions among the particles. In the former formulation, θ appears in the multivalued phase, while in the latter θ appears as a coupling constant in the Bohm-Aharonov-type terms. More generally, if one likes, one can even use a kind of mixed description[11] in which both the wave function has a multivalued phase and the Bohm-Aharonov-type interactions appear in the Hamiltonian. The physical θ parameter is then the sum of θ_{phase} in the multivalued phase and θ_{int} in the Hamiltonian. Therefore, we stress that one cannot make a judgement about quantum statistics merely from the form of permutation symmetry of the wave functions as we used to do in three-dimensional cases. Because of this situation, an invariant characterization of statistics (or the physical θ parameter) is in demand.

These various descriptions differ from each other only by "phase gauge transformations" on wave functions. So what we need is, in some sense, a "gauge-invariant" characterization of θ statistics. Recall that the configuration space of n indistinguishable particles C_n is the set of points $\{\vec{r}_1, \ldots, \vec{r}_n\}$ for all $\vec{r}_i$ different and modulo the action of the permutation group, S_n, of particle indices. The wave functions of the system are cross sections of a complex line bundle on the base space C_n with $U(1)$, the group of phase transformations as structure group.[18] We point out that the quantum of the particles can be described in terms of topological information of the principal $U(1)$ bundle associated with this complex line bundle.

Suppose that the classical Hamiltonian is $H_0(\vec{r}_i, \vec{p}_i)$. Then in the "regular gauge," namely, in the above called effective Lagrangian description, the phase of wave functions is single valued on C_n, and the Hamiltonian for particles obeying θ statistics is $H_0(\vec{r}_i, \vec{p}_i - \vec{A}_i^{\text{stat}})$, where

$$\vec{A}_i^{\text{stat}} = \frac{\theta^*}{\pi} \sum_{i<j} \frac{\vec{k}\times(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^2} \tag{5}$$

are the "statistical" potentials describing the Bohm-Aharonov-type interactions among particles. It is easy to see that these potentials are curl free. Using them we can define a curl-free connection, A^{stat}, on the principal $U(1)$ bundle mentioned above as follows. The n components of the connection on the bundle, A_i^{stat} $(i=1, \ldots, n)$, corresponding to the n coordinates $(\vec{r}_1, \ldots, \vec{r}_n)$ of the base C_n, is taken to be

$$(\vec{A}_1^{\text{stat}}, \vec{A}_2^{\text{stat}}, \ldots, \vec{A}_n^{\text{stat}}) .$$

This is to say, in this gauge the line integral of the "statistical connection" along a path $\gamma(t)$ in C_n represented by $[\vec{r}_1(t), \ldots, \vec{r}_n(t)]$, by definition, is

$$\int_\gamma A^{\text{stat}} = \sum_i \int_{\vec{r}_i(t)} \vec{A}_i^{\text{stat}} \cdot d\vec{r}_i . \tag{6}$$

Thus, the principal $U(1)$ bundle under consideration is a flat one. Though A^{stat} changes in other gauges, its holonomy is a gauge invariant concept. Recall the integrals of A^{stat} along all loops starting and ending at a fixed point in C_n, after exponentiation, form a subgroup $U(1)$, called the holonomy group of the connection. Since the statistical connection is flat, its holonomy along a loop which is continuously deformable to a point is always the identity. Therefore, the holonomy group gives us a mapping from the first homotopy group, $\pi_i(C_n)$, to $U(1)$, i.e., a character of $\pi_i(C_n)$. We know that all the characters can be parametrized by an angular parameter θ,[5] so the same is true for the all possible holonomy groups of the flat bundle of C_n, as we can directly check from Eq. (6).

Thus, we have proved that the quantum statistics can also be described in terms of the holonomy group on a flat $U(1)$ bundle on the configuration space of indistinguishable particles. When there are external electromagnetic fields, as in the case of the quantum Hall effect, we have to subtract the effects of the external potentials on the holonomy before obtaining the flat statistical connection. So far these are fancy words to describe what we have been familiar with. However, it provides us a gauge-invariant way to determine the physical θ parameter describing statistics. In fact, one needs to consider only the simplest loop in C_n, which corresponds to exchanging two of the particles along a loop with no other particles inside, and compute the holonomy along it. In this regard we mention that the Berry phase[19] in the adiabatic theorem is also a holonomy on the principal $U(1)$ bundle on the parameter space of the Hamiltonian.[20] In Ref. 8, the Berry phase is used to determine the statistics of quasiparticles in the fractional quantum Hall effect from the Laughlin wave functions.

THE EFFECT OF SPACE TOPOLOGY ON STATISTICS

In this section we shall examine a novel feature of the path-integral formalism of quantum statistics[5] that the topology of space can have significant effects on statistics for particles moving in it. The general formulation given in Ref. 5 holds good for any space M. The configuration space of n indistinguishable particles in M is instead

$$C_n \equiv (M\times M\times \cdots M - D)/S_n . \tag{7}$$

The statistics of the particles is still determined by the characters of the first homotopy group of C_n, which is again isomorphic to a braid group. But now it is the braid group on M, denoted by $B_n(M)$, which can be defined in the same way as in the R^2 case. For two-spaces of different topology, the braid group is different and so is the set of its characters. Therefore, it would be interesting to see how the topology of the two-space affects the statistics of particles moving in it. This problem is not so academic as it might look at the first glance. For two-dimensional systems with different geometry or topology could perhaps be experimentally accessible, and in the literature there have been theoretical discussions of the fractional quantum Hall effect for systems having various geometry and topology.[16,21,22]

As an example, let us consider the case[21] in which $M=S^2$ (2 sphere). Since locally one cannot distinguish between S^2 and R^2, the braid group $B_n(S^2)$ has the same generators, σ_i ($i=1,2,\ldots,n-1$), as the braid group $B_n(R^2)$ does. But, becuase of their global difference, for $B_n(S^2)$ there is one more relation among the generators in addition to those for $B_n(R^2)$. Recall that σ_i represents the exchange of the ith and the $(i+1)$th particles along a counterclockwise loop without other particles inside. It is easy to verify the relation

$$\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_{n-1}\cdots\sigma_2\sigma_1=1 \tag{8}$$

for $B_n(S^2)$. For the effect of the left side is equivalent to move only the first particle around a big loop enclosing all other particles, and on S^2 this big loop is contractible to a point. It can be proved that there is no more relation among the generators σ_i's.[23] Therefore, the characters have to satisfy only one more similar constraint with σ_i replaced by the character $\chi(\sigma_i)$. From $B_n(R^2)$ we already have $\chi(\sigma_i)=\exp(-i\theta)$. Not all characters of $B_n(R^2)$ satisfy the new constraint, so the topology of S^2 restricts the possible values of the θ parameter to be

$$\theta=\frac{m\pi}{(n-1)}\quad (m=0,1,\ldots,2n-3)\ . \tag{9}$$

Note the n dependence of the possible θ values. This n dependence could be observed, if somehow, only when the total number of particles of a particular kind on the sphere is very small. When n is very large, the set of possible values in Eq. (9) is dense everywhere in the interval $[0,2\pi)$ so that the difference between S^2 and R^2 is physically unobservable.

The above discussion can apply to any continuum space. The generators of the braid group are always the same as those of $B_n(R^2)$, but the nontrivial topology will lead to additional relations among the generators. Hence, the effect of nontrivial space topology is to restrict the possible θ values for the statistics to be a subset of the allowed values in R^2, i.e., a subset of $[0,2\pi)$. However, when the particle number is very large, this effect is expected to be physically unobservable.

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*On leave from Department of Physics, University of Utah, Salt Lake City, UT 84112; this is also the address after April 1, 1985.
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