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Effect of inelastic processes on localization in one dimension

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The transition of noninteracting electrons through an array of random potentials is studied. The electron wave packet contains an incoherent component, which builds up as a result of inelastic processes. The latter are represented by a random time-dependent potential which oscillates incoherently. An inelastic length is determined, beyond which a transition to Ohm's law is found. On small length scales inelastic correction to the exponential resistance is calculated. The generalization of Ohm's law and the Landauer formula to the multichannel case are discussed.

It is well known that a system of noninteracting electrons in a random one-dimensional (1D) potential at a zero temperature ($T=0$) has a conductance G, which is exponentially small with the size of the system.[1-4] A convenient way of treating this problem is to use the Landauer formula[1,2,5-7]

$$G=\frac{e^2}{2\pi\hbar}\frac{|t|^2}{|r|^2}\equiv\frac{e^2}{2\pi\hbar}\frac{T}{R} ,$$

where t and r are the transmission and reflection amplitudes,[8] respectively. To calculate these quantities, one considers a product of 2×2 transfer matrices, which connect the incoming and outgoing wave-function amplitudes at the Nth "site," to the amplitudes at the first site. Possible generalizations of this formula to the multichannel case have been discussed in the literature,[2,7] and it has been concluded that the Landauer formula does not apply in general, since G depends on the current distribution in the various incoming channels.

It is also commonly realized that inelastic processes (e.g., finite temperature effects) destroy the phase coherence and hence the effect of localization. More specifically, any inelastic process defines a certain characteristic length (or time) scale. It is believed that the system behaves according to the $T=0$ localization theory on smaller scales. G is then matched smoothly to the Ohmic behavior which is assumed to be found on larger scales. The crossover scale depends on the temperature.[9]

In this Rapid Communication we revisit the problem of localization-to-Ohmic-behavior crossover. We show explicitly (without assuming Ohm's law *a priori*) how the initially coherent electron wave function develops phase incoherent components, a process which eventually leads to an Ohmic behavior. Furthermore, we determine, on scales smaller than the inelastic length, the incoherent contributions to the $T=0$ localization behavior, and calculate the resulting leading corrections to G. When these contributions are sufficiently large, we find that the conditions for Ohm's law to be valid are satisfied. We also emphasize that the existence of inelastic processes implies nonconservation of energy. Thus, in contrast to the original Landauer picture, we have to deal here with a multichannel process in the energy space. The conductance obtained from the Landauer formalism is, therefore, a tensor in that space rather than a scalar. Different eigenmodes of this tensor (i.e., different voltage configurations in the various channels) correspond to different eigenvalues (i.e., different values of the conductance). In simple cases it is possible to obtain explicitly the (bound) spectrum of these values.

We generalize a model introduced recently by Buttiker and Landauer[10] and consider a random one-dimensional array of rectangular barriers. The jth barrier represented by the potential $V_0^{(j)}(x)$ has a height $V_0^{(j)}$ and a width $a^{(j)}$. We assume almost completely reflecting barriers, i.e., $\kappa^{(j)}a^{(j)}>>1$, where $\kappa^{(j)}\equiv(2m/\hbar)(V_0-E)^{1/2}$, and m and $E=\hbar^2k^2/2m$ are the mass and energy of the electron, respectively ($E<V_0$). We now couple to each barrier a time-dependent perturbation such that the height of the barriers oscillates as $V_0^{(j)}+V_1^{(j)}(\omega)\cos(\omega t+\phi_{(\omega)}^{(j)})$, where $\phi_{(\omega)}^{(j)}$ are random phases. $V_1^{(j)}$ represents a coupling to a classical reservoir with finite temperature. Hence, up to a constant we expect

$$\langle V_1^{(j)2}\rangle\sim\sum_\omega\frac{1}{\omega^2}\frac{\hbar\omega}{e^{\hbar\omega/k_BT}-1} . \quad (1)$$

If we further assume a Debye spectrum for the distribution of frequencies and $T<<\Theta_D$, we find $\langle V_1^{(j)^2}\rangle\sim T^2$.

It has been pointed out in Ref. 10 that each harmonic component of $V_1^{(j)}$ creates an infinite discrete set of side bands. In general, an absorption (emission) of n phonons of frequency ω is associated with a factor

$$\left(\frac{V_1^{(j)}(\omega)}{\hbar\omega}\right)^n\equiv(\delta_{(\omega)}^{(j)})^n ,$$

$\delta<<1$, and a phase factor $e^{in\phi}(e^{-in\phi})$. The following expansion in δ is justified when the density of the harmonic modes vanishes fast enough in the infrared limit (as is the case for the Debye spectrum) implying that the thermal average of $V(\omega)/\hbar\omega$ is not large. To order δ^2 we have to consider the following processes: (a) static transmission and reflection without emission or absorption of a phonon with amplitudes

$$t_{00}=2\left[1+\frac{i}{2}\left(\frac{\kappa}{k}-\frac{k}{\kappa}\right)\right]^{-1}\exp(-2ika-2\kappa a) ,$$
$$r_{00}=-\frac{i}{2}\left(\frac{\kappa}{k}+\frac{k}{\kappa}\right)\left[1+\frac{i}{2}\left(\frac{\kappa}{k}-\frac{k}{\kappa}\right)\right]^{-1}e^{-2ika} , \quad (2)$$

(here and throughout we assume $\kappa a>>1$); (b) transmission and reflection with emission or absorption of one phonon

$$t_{\pm\omega}=\mp\delta\tfrac{1}{2}t_{00}\exp(\mp im\omega a/\hbar k)\times[\exp(\pm2m\omega a/\hbar\kappa)-1]e^{\pm i\phi(\omega)} ,$$
$$r_{\pm\omega}=\mp\delta\frac{m\omega}{4\hbar\kappa^2}t_{00}\exp(2\kappa a\mp im\omega a/hk)e^{\pm i\phi(\omega)} ; \quad (3)$$

(c) emission (absorption) of a phonon followed by absorption (emission) of a phonon of the same frequency. These processes renormalized the static amplitudes $t_0 = t_{00} + \Delta t$ and $r_0 = r_{00} + \Delta r$ by

$$\Delta t = \delta^2 \left(\frac{m\omega a}{\hbar\kappa} \right)^2 t_{00}, \quad \Delta r = -\delta^2 \left(\frac{m\omega}{\hbar\kappa^2} \right)^2 r_{00} \; ; \tag{4}$$

(d) two-phonon absorption or emission. These processes can be neglected.[11] Furthermore, we notice that for $\kappa a >> 1$ the inelastic reflection processes are less important than inelastic transmission processes $r_{\pm\omega}/r_0 << t_{\pm\omega}/t_0$, which simplifies the explicit expressions in the following. The last inequality means that the reflected wave is more coherent than the transmitted wave, a fact that can be verified experimentally using various interference circuits.

To the order considered the transfer matrix connecting the in and outgoing amplitudes on the left of the jth barrier at different energies $E, E+\omega, \ldots$ (l_E^i and $l_E^0, l_{E+\omega}^i, \ldots$) to those on the right is

$$\begin{pmatrix} \vdots \\ l_{E+\omega}^i \\ l_{E+\omega}^0 \\ l_E^i \\ l_E^0 \\ l_{E-\omega}^i \\ l_{E-\omega}^0 \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ \underline{\hat{Y}}_{-\omega}^{(j)} & \underline{\hat{Y}}_0^{(j)} & \underline{\hat{Y}}_\omega^{(j)} & & & \\ & \underline{\hat{Y}}_{-\omega}^{(j)} & \underline{\hat{Y}}_0^{(j)} & \underline{\hat{Y}}_\omega^{(j)} & & \\ & & \underline{\hat{Y}}_{-\omega}^{(j)} & \underline{\hat{Y}}_0^{(j)} & \underline{\hat{Y}}_\omega^{(j)} \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ r_{E+\omega}^0 \\ r_{E+\omega}^i \\ r_E^0 \\ r_E^i \\ r_{E-\omega}^0 \\ r_{E-\omega}^i \\ \vdots \end{pmatrix} , \tag{5}$$

where $\underline{Y}_0, \underline{Y}_{\pm\omega}$ are given by

$$\underline{Y}_0 = \begin{pmatrix} \dfrac{1}{t_0} & -\dfrac{r_0'}{t_0} \\ \dfrac{r_0}{t_0} & t_0 \left(\dfrac{1}{1 + 2\dfrac{t_\omega t_{-\omega}}{t_0^2}} - \dfrac{r_0 r_0'}{t_0^2} \right) \end{pmatrix} \left(1 + 2\frac{t_\omega t_{-\omega}}{t_0^2} \right)$$

$$\approx \underline{Y}_{00} \left(1 + 2\frac{t_\omega t_{-\omega}}{t_0^2} \right) , \tag{6}$$

and

$$\underline{Y}_{\pm\omega} = \begin{pmatrix} \dfrac{1}{t_0} & -\dfrac{r_0'}{t_0} \\ \dfrac{r_0}{t_0} & -t_0 \left(1 + \dfrac{r_0 r_0'}{t_0^2} \right) \end{pmatrix} \left(-\frac{t_{\pm\omega}}{t_0} \right) \approx \underline{Y}_{00} \left(-\frac{t_{\pm\omega}}{t_0} \right) . \tag{7}$$

We assumed here that the band of relevant energies is narrow in comparison with the height of the barriers which allowed us to neglect the energy dependence of t_0, $t_{\pm\omega}$, and r_0. Also for the sake of simplicity, we included in Eq. (5) explicitly only one frequency component. Additional components are accounted for by further linearly independent channels. Furthermore, since $t_0^2/r_0^2 << 1$ in our model, $\underline{Y}_0$ and $\underline{Y}_{\pm\omega}$ have approximately the same matrix structure, differing only in a scalar factor. $\underline{Y}_{00}$ is the standard 2×2 static transfer matrix.[3,6] Notice that in our model no dissipative mechanism (i.e., spontaneous emission) of the electronic system exists. Thus, the electrons on the average will gain energy from the classical phonon field. However, for the questions discussed here, we expect no serious consequences.

The corresponding matrix for N barriers is $\underline{Y}_N = \prod_j \underline{Y}^{(j)}$, and to second order in δ has the same structure as $\underline{Y}^{(j)}$, with diagonal 2×2 block of the form

$$\underline{Y}_0^{(N)} = \left(\prod_i \underline{Y}_{00}^{(i)} \right) \left(1 + 2\sum_j \frac{t_{-\omega}^{(j)} t_\omega^{(j)}}{(t_0^{(j)})^2} + \sum_{i \neq j} \frac{t_\omega^{(i)} t_{-\omega}^{(j)}}{t_0^{(i)} t_0^{(j)}} \right) \tag{8}$$

and off-diagonal blocks

$$\underline{Y}_{\pm\omega}^{(N)} = \left(\prod_i \underline{Y}_{00}^{(i)} \right) \left(-\sum_j \frac{t_{\pm\omega}^{(j)}}{t_0^{(j)}} \right) . \tag{9}$$

We next go to a representation where the scattering matrix connects incoming and outgoing waves. The structure of this matrix is

$$\begin{pmatrix} \vdots \\ r_E^0 \\ l_E^0 \\ \vdots \end{pmatrix} = \begin{pmatrix} \underline{\hat{X}}_{-\omega} & \underline{\hat{X}}_0 & \underline{\hat{X}}_\omega & & \\ & \underline{\hat{X}}_{-\omega} & \underline{\hat{X}}_0 & \underline{\hat{X}}_\omega & \\ & & \underline{\hat{X}}_{-\omega} & \underline{\hat{X}}_0 & \underline{\hat{X}}_\omega \end{pmatrix} \begin{pmatrix} \vdots \\ l_E^i \\ r_E^i \\ \vdots \end{pmatrix} .$$

Of particular interest are the diagonal blocks $\underline{X}_0$

$$\underline{\hat{X}} = \begin{pmatrix} \left(\underline{Y}_{22} - \dfrac{\underline{Y}_{12}\underline{Y}_{21}}{\underline{Y}_{11}^{00}} \right) \left(1 + 2\displaystyle\sum_j \frac{t_\omega^{(j)} t_{-\omega}^{(j)}}{(t_0^{(j)})^2} + \sum_{i \neq j} \frac{t_\omega^{(i)} t_{-\omega}^{(j)}}{t_0^{(i)} t_0^{(j)}} \right) & \dfrac{\underline{Y}_{21}}{\underline{Y}_{11}} \\ -\dfrac{\underline{Y}_{12}}{\underline{Y}_{11}} & \dfrac{1}{\underline{Y}_{11}} \left(1 + \displaystyle\sum_{i \neq j} \frac{t_\omega^{(i)} t_{-\omega}^{(j)}}{t_0^{(i)} t_0^{(j)}} \right) \end{pmatrix} , \tag{10}$$

where $\underline{Y}_{ij}$ are the matrix elements of $\underline{Y}_{00}$.

One can easily obtain the relation between the incoming and outgoing *intensities*. This relation will include, in general, cross terms that mix different incoming amplitudes. For a sufficiently large number of barriers, each cross term consists of a large fluctuations part (that depends on the random phases) and a smaller, phase-independent part. Within our approximation the effect of the many frequencies will be additive in the structure of the X matrix. The condition that the ensemble average of the cross terms vanishes (we now consider again the effect of the many frequencies) is

$$\left(\left\langle \sum_\omega \sum_{ij} \frac{|t_\omega^{(i)}|^2 |t_\omega^{(j)}|^2}{|t_0^{(i)}|^2 |t_0^{(j)}|^2} \right\rangle \right)^{1/2} \gtrsim 1 , \tag{11}$$

where $\langle \; \rangle$ denotes an averaging over the random barriers. When this condition is satisfied, the magnitude of the fluctuating part becomes larger than the phase-independent part of the cross terms. The fluctuating part is then ensemble averaged to zero. Also, the phase-independent terms

$$\sum_i \frac{t_\omega^{(i)} t_{-\omega}^{(i)}}{(t_0^{(i)})^2}$$

are of order 1 and approximately cancel the 1 in the diagonal elements of X. From Eq. (11) we may define the analog of the Thouless length[9]

$$l_T \approx d \left[\sum_\omega \left\langle \frac{|t_\omega|^4}{|t_0|^4} \right\rangle \right]^{-1/2} \sim T^{-1/2} \ , \qquad (12)$$

where d is the characteristic interbarrier distance. In the last step of Eq. (12) we used Eqs. (1) and (3) and assumed a Debye spectrum. In fact, in order to achieve incoherence and a crossover to Ohmic behavior, it would be sufficient to assume a single harmonic component with random phases, though the expression for l_T in that case would be different.[12]

Notice that l_T was defined such that the interference of the cross terms averages to zero, and therefore Ohm's law is satisfied on larger scales. In the absence of multichannel inelastic processes (but with a phase averaging) one can go back to the left versus right intensities and reproduce the classical matrix of Thouless,[6] for which Ohm's law may be explicitly verified by considering the product of two such matrices. This remains true to the lowest order of our calculations.

One should realize, however, what is meant by Ohm's law in the multichannel case. We write an infinite matrix connecting a vector whose entries are the particle densities (i.e., the voltage) in the various energy channels, both on the left and on the right to the corresponding current vector.[13] This defines the dimensionless conductance matrix G. In the one-channel case this matrix is simply

$$G = \begin{pmatrix} g & -g \\ g & -g \end{pmatrix} \ ,$$

where the scalar g is the standard conductance. However, in our case, G has the structure

$$G = \begin{pmatrix} G & -\hat{G} \\ G & -\hat{G} \end{pmatrix}$$

in the right-left space,[14] where G and $\hat{G}$ are commuting matrices in the energy space. Insensitivity of the currents to a uniform shift of the voltages and current conservation imply $\sum_j G_{ij} = 0$ and $\sum_i G_{ij} = 0$, respectively.

The matrix structure of G implies that different modes of the voltage, i.e., different densities in the various energy channels, correspond to different (eigen)values of the conductance. This is a generalization of the result obtained by Langreth and Abrahams.[7] Furthermore, the rules for combining different elements, characterized by different G_i, lead to a generalization of Ohm's law. For example, for two elements G_1 and G_2 in series we have

$$G_{\text{tot}} = G_1 - G_1(G_1+G_2)^{-1}G_1; \quad G_{\text{tot}} = G_2(G_1+G_2)^{-1}G_1 \ . \qquad (13)$$

For the case where the oscillations of the barrier are characterized by one frequency only, the linear order in T_ω/T_0 and R_ω/R_0, G and $\hat{G}$ are tridiagonal matrices given by

$$\begin{aligned} G_{i,i} &= (1+T_0^2-R_0^2)/[(1+R_0)^2-T_0^2] \ , \\ G_{i,i\pm1} &= \frac{1}{4R_0^2}[T_{\pm\omega}(1-R_0^2)-R_{\pm\omega}(1+R_0^2)] \ , \\ \hat{G}_{i,i} &= 2T_0/[(1+R_0)^2-T_0^2] \ , \\ \hat{G}_{i,i\pm1} &= \frac{1}{4R_0^2}[T_{\pm\omega}(1+R_0^2)-R_{\pm\omega}(1-R_0^2)] \ . \end{aligned} \qquad (14)$$

Thus, in the case where the voltages in all channels on the right (left) are equal the conductance is

$$\tfrac{1}{2}(G_{i,i-1}+G_{i,i}+G_{i,i+1}+\hat{G}_{i,i-1}+\hat{G}_{i,i}+\hat{G}_{i,i+1}) \ .$$

On scales l, much smaller than l_T, we can calculate corrections to the conductance, resulting from the incipient incoherent behavior. The zero-temperature localization result for the conductance is enhanced by a factor $1+l/l_T$. This is one of our main results, and it would be interesting to verify it in experiment. Though our model is oversimplified, it reflects some features characteristic of the electron-phonon interactions. Our picture is that of phonons incoherently modulating the local potential felt by the electrons. For the case where a Debye spectrum is assumed for the phonons [Eq. (1)] we find $\tau_{in} \sim (k_B T)^{-2}$. The corrections to the conductance, as well as the multichannel Ohmic behavior, are independent of the details of our model.

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[8]In fact, one has to distinguish between t,r (amplitudes from the right) and t',r' (amplitudes from the left), see, e.g., Ref. 2.

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[11]This is correct because the term $\delta e^{\pm i\phi}$ in (b) is more important. Alternatively, when the intensity is calculated, and the random phase is averaged, the contribution of (d) vanishes in the considered order.

[12]Here we do not consider higher-order processes as an absorption of a phonon at a frequency ω_1 and an emission of two phonons ω_2 and ω_3, $\omega_1=\omega_2+\omega_3$. Notice also that the relation $l_T \sim T^{-1/2}$ coincides with experimental results in the weak localization regime in one dimension.

[13]The vectors are arranged as $(\ldots, V_{E+\omega}^r, V_E^r, V_{E-\omega}^r, \ldots, V_{E+\omega}^l, V_E^l, V_{E-\omega}^l, \ldots)$.

[14]The explicit form of G given in the text reflects a left-right symmetry of the problem. General forms, which break this symmetry and with noncommuting subblocks of G may occur in general. In such cases, the value of G_{tot} for a system combined of different individual elements in general depends on their order.