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Renormalization-group relaxational dynamics of interfaces in $4-\epsilon$ dimensions

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We present a dynamic renormalization-group calculation in $4-\epsilon$ dimensions of the capillary wave dispersion relation ω_q for the interface of an Ising-like system driven by relaxational dynamics (model A). The dispersion relation is of the form $\omega_q=-i\Gamma q^z\Omega(q\xi)$, with $\Omega(x)$ universal and $z=2+O(\epsilon^2)$, and satisfies the Goldstone theorem for the spontaneously broken Euclidean symmetry.

In recent years, the interest in the study of critical phenomena has largely shifted from bulk to surface and interfacial properties. Static critical properties in nonhomogeneous systems have been widely investigated by a variety of methods.[1] However, little progress has been achieved in the study of dynamic critical phenomena in these systems using, in particular, renormalization-group (RG) methods. A notable exception is the work by Bausch, Dohm, Janssen, and Zia[2] who employed a phenomenological drumhead model with relaxational dynamics in order to investigate, within a RG approach, the spectrum of excitations of an interface in (bulk) $d=1+\epsilon$ dimensions. These authors argue that near criticality the characteristic frequency of the interface obeys the scaling form $\omega_q\sim q^z\Omega(q\xi)$, with q the $(d-1)$-dimensional wave vector parallel to the plane of the interface and ξ the bulk correlation length, and identify the exponent z, which they calculate to second order in $\epsilon=d-1$, with the bulk system's dynamic critical exponent. The above conclusion implies (at least, in low dimensionality) that both the surface modes and the bulk critical modes are characterized by the same critical exponent z. That this is so has been formally verified only in the case of the rigid surface of a semi-infinite medium[3] described by a time-dependent Ginzburg-Landau (TDGL) model. In the case of a moving interface, there is always the possibility of a surface-wave exponent different from the bulk dynamic exponent z, if, for example, a new renormalization counterterm is required. In order to investigate this and other important issues, a more microscopic formulation of the dynamics of an interface is desirable, together with an evaluation of the unknown scaling function $\Omega(x)$. This is all the more necessary since the drumhead model has been shown[4] to correspond to the more microscopic Ginzburg-Landau-Wilson ϕ^4 model of an interface strictly in the limit of temperatures $T<<T_c$.

In this Rapid Communication, we present the first step towards this goal by calculating the dispersion relation ω_q for the interface of an Ising-like system described by the usual ϕ^4 effective Hamiltonian and model A relaxational dynamics:[5]

$$\frac{\partial\phi(x,t)}{\partial t}=-\Gamma_0\frac{\delta H}{\delta\phi(x,t)}+\eta(x,t)\ ,$$

$$H=\int d^dx\left[\tfrac{1}{2}r_0\phi^2+\tfrac{1}{2}(\nabla\phi)^2+\frac{1}{4!}\lambda_0\phi^4-h\phi\right]\ .$$

A spatial argument $x=(\rho,z)$ denotes a d-dimensional position vector. In our treatment of the TDGL model, the $h=0$ interface is introduced in a microscopic way (in analogy with equilibrium calculations[6,7]) by requiring that $\langle\phi(x,t)\rangle=M(z)$, with z as the direction perpendicular to the interface and

$$\lim_{z\to+\infty}M(z)=-\lim_{z\to-\infty}M(z)=M_B\ ,$$

the bulk equilibrium order parameter. The random noise $\eta(x,t)$ satisfies

$$\langle\eta(x,t)\eta(x',t')\rangle=2\Gamma_0\delta(x-x')\delta(t-t')\ ,$$

where Γ_0 is the kinetic coefficient. We have employed the dynamic RG technique in $4-\epsilon$ dimensions to determine the surface-wave pole $\omega=\omega_q$ of the linear response function $R(q;z,z';\omega)$. The linear response function is defined, as usual, by

$$\langle\phi(x,t)\rangle-M(z)=\int dx'dt'R(x,t;x',t')h(x',t')\ ,$$

in which case the hybrid $R(q;z,z';\omega)$ is the Fourier transform with respect to time and space (in the plane perpendicular to the z axis) in the inhomogeneous system with an interface. To first order in ϵ, the result is of the form

$$\omega_q=-i\Gamma q^z\Omega(q\xi)\ , \tag{1}$$

where $z=2+O(\epsilon^2)$, and where the function $\Omega(x)$ is known to order ϵ in the form of a parametric representation in terms of multiple integrals. The limit forms of this function are

$$\Omega(x)\simeq1+\epsilon(C+O(x^2)),\quad x<<1\ ,$$

$$\Omega(x)\simeq1+\epsilon\left[\frac{C_1}{x^2}\ln x+\frac{C_2}{x^3}+O(x^{-4})\right],\quad x>>1\ , \tag{2}$$

where C, C_1, and C_2 are known constants.[8] For all values of $q\xi$, the first-order calculation is consistent with the nonperturbative result: $\lim_{q\to0}\omega_q=0$, which is the specialization of the Goldstone theorem for the spontaneously broken translational symmetry in the z direction. In the present case, the theorem can be cast in the form of a Ward identity for the response function and order parameter in the presence of a vanishingly small (translational) symmetry-breaking field $h(z)$:

$$i\Gamma_0\frac{\partial}{\partial z}M(z;h(z))-\int dz'\frac{dh(z')}{dz'}R(q=0;z,z';\omega=0;h(z))=0\ .$$

As the magnitude of $h(z)$ vanishes, the profile $M(z)$ remains; hence, the response diverges. In Eq. (1) the critical exponent for ω_q is identified with z, as scaling would suggest. A calculation to $O(\epsilon^2)$ would be a most welcome

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reinforcement in establishing this nontrivial result. Moreover, a higher-order calculation should reveal a singularity in the function $\Omega(x)$ as $x\rightarrow 0$ [for model A, however, this singularity does not show up to $O(\epsilon)$ owing to the fact that $z=2+0(\epsilon^2)$]. This interesting singularity should reflect the fact that, in general, the interfacial dispersion relation in the hydrodynamic regime ($q\xi\rightarrow 0$) has a power of q dictated by Goldstone-mode fluctuations and differing from z.[9] An analogous situation is presented by the $O(n)$ model of an isotropic ferromagnet for all $T<T_c$, in which case the hydrodynamic regime behavior of the longitudinal correlation function is $G_{\parallel}(k)\sim k^{-\epsilon}$, as prescribed by spin-wave theory.[10]

We now describe our calculation, stressing from the start that our method could be extended to more complicated and realistic models of dynamic interfaces. We employ the Martin-Siggia-Rose[11] field-theoretic dynamic RG formulation as developed by Janssen[12,13] and DeDominicis.[14,15] The random force $\eta(x,t)$ is replaced by an auxiliary field $\hat{\phi}(x,t)$ in a Lagrangian formulation with, for model A,

$$L=\Gamma_0\hat{\phi}^2-i\hat{\phi}\left[\frac{\partial}{\partial t}-\Gamma_0\nabla^2+\Gamma_0 r_0\right]\phi-i\frac{\lambda_0\Gamma_0}{3!}\hat{\phi}\phi^3 \ . \tag{3}$$

Below the critical temperature T_c, one introduces the shift $\phi(x,t)\rightarrow M(z)+\phi(x,t)$, so that Eq. (3) now becomes

$$L=\Gamma_0\hat{\phi}^2-i\hat{\phi}\left[\frac{\partial}{\partial t}-\Gamma_0\nabla^2+\Gamma_0 r_0+\frac{\lambda_0\Gamma_0}{2}M^2(z)\right]\phi -i\frac{\lambda_0\Gamma_0}{2}M(z)\hat{\phi}\phi^2-i\frac{\lambda_0\Gamma_0}{3!}\hat{\phi}\phi^3 \ . \tag{4}$$

Correlation and response functions are averages over the fields ϕ and $\hat{\phi}$ with statistical weight $\exp(-\int dx\,dt \times L(\phi,\hat{\phi}))$. In developing a perturbation expansion in powers of λ_0, the bare propagators

$$G^{(0)}_{\alpha\beta}(1,2)=\langle\phi_\alpha(x_1,t_1)\phi_\beta(x_2,t_2)\rangle_0 \ ,$$

where $\phi_1=\phi$, $\phi_2=\hat{\phi}$, satisfy the differential equation

$$\begin{pmatrix} 0 & -i\hat{D}(x_1,-t_1) \\ -i\hat{D}(x_1,t_1) & 2\Gamma_0 \end{pmatrix}\begin{pmatrix} G^{(0)}_{11}(1,2) & G^{(0)}_{12}(1,2) \\ G^{(0)}_{21}(1,2) & G^{(0)}_{22}(1,2)\end{pmatrix}=\delta(1-2) \ ,$$

where

$$\hat{D}(x,t)=\frac{\partial}{\partial t}-\Gamma_0\nabla^2_x+\Gamma_0 r_0+\tfrac{1}{2}\lambda_0\Gamma_0 M_0^2(z) \ ,$$

with $M_0(z)$ the zero-order equilibrium profile. If we denote the eigenfunctions of $\hat{D}$ by $e^{i\omega t+iq\rho}\zeta^{(\mu)}(z)$, the $\zeta^{(\mu)}(z)$ satisfy[6,7]

$$\left[-\frac{d^2}{dz^2}+r_0+\tfrac{1}{2}\lambda_0 M_0^2(z)\right]\zeta^{(\mu)}(z)=E^{(\mu)}\zeta^{(\mu)}(z) \ .$$

Then the spectral representations of the propagators are given by

$$G^{(0)}_{11}(q;z,z';\omega)=\sum_\mu \zeta^{(\mu)}(z)\zeta^{(\mu)}(z')^*\frac{2\Gamma_0}{[\omega+i\Gamma_0(q^2+E^{(\mu)})][\omega-i\Gamma_0(q^2+E^{(\mu)})]} \ ,$$

$$G^{(0)}_{12}(q;z,z';\omega)=G^{(0)}_{21}(-q;z',z;-\omega)=\sum_\mu \zeta^{(\mu)}(z)\zeta^{(\mu)}(z')^*\frac{1}{\omega+i\Gamma(q^2+E^{(\mu)})} \ ,$$

$$G^{(0)}_{22}(q;z,z';\omega)=0 \ .$$

To order one loop, the response function $R=i\Gamma_0\langle\phi\hat{\phi}\rangle$ is given in terms of Feynman diagrams as shown in Fig. 1, where the propagators $G^{(0)}_{11}$ and $G^{(0)}_{12}$ are represented by a continuous and a continuous-wavy line, respectively, and the two different interaction vertices of Eq. (4) are represented by different circles. For $d\leq 4$ perturbation theory is divergent and a renormalization scheme must be used. We have found, at least to order one loop, that the bulk renormalization constants Z_ϕ, Z_{ϕ^2}, Z_u, and Z_Γ suffice to remove the divergences. This confirms that, at least to $O(\epsilon)$, there should be only one dynamic critical exponent, the bulk one z. (See, however, Ref. 8.) Care must be taken when renormalizing the zero-order graph in Fig. 1; the eigenfunctions $\zeta_R^{(\mu)}(z)$ and eigenvalues $E_R^{(\mu)}$ are now those appropriate to the renormalized operator

$$\hat{D}_R=\frac{\partial}{\partial t}-Z_\Gamma\Gamma\nabla^2+Z_\Gamma Z_{\phi^2}\Gamma t+\tfrac{1}{2}Z_\Gamma Z_u u\kappa_0^{-\epsilon}\Gamma M^2(z) \ ,$$

in which $\Gamma=Z_\Gamma^{-1}\Gamma_0$, $t=Z_{\phi^2}^{-1}(r_0-r_{0c})\propto T-T_c$, and $u\kappa_0^{-\epsilon}=Z_u^{-1}\lambda_0$ are the renormalized parameters, whereas $M(z)$ is the one-loop order-parameter profile.[6,7] The length scale in the renormalized theory is set by κ_0^{-1}. The complicated spectral representations for the bare propagators make the calculation of the full response function impractical even to one-loop order. Nonetheless, the surface-wave dispersion relation can be extracted from renormalized perturbation theory by making the reasonable ansatz that ω_q is the only pole of the *interface response* $\tilde{G}_R$ found by taking the full renormalized response function $R_R=i\Gamma G_{12R}$ and projecting onto the Goldstone mode eigenfunctions ($\mu=0,E^{(0)}=0$),

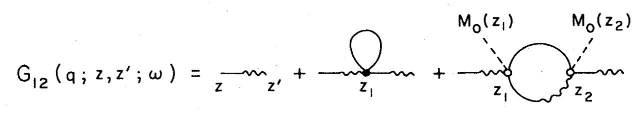

FIG. 1. Diagrammatic representation of the response function to order one loop for $T<T_c$. The circles denote the third- and fourth-order interaction couplings in Eq. (4).

which generate localized interface distortions, i.e.,

$$\tilde{G}_R(q;\omega)=\int dz\, dz'\zeta_R^{(0)}(z)\,G_{12R}(q;z,z';\omega)\,\zeta_R^{(0)}(z')^* \;.$$

After a computation far more laborious than for equilibrium profile calculations, the result for $\tilde{G}_R(q;\omega)$ may be written

$$\tilde{G}_R(q;\omega)=\frac{1}{\omega+i\Gamma q^2}-u\frac{i\Gamma\kappa^2}{(\omega+i\Gamma q^2)^2}\Phi\left[\frac{q^2}{\kappa^2},i\frac{\omega}{\Gamma\kappa^2}\right] \;, \qquad (5)$$

where $\kappa=\xi^{-1}$, $u\rightarrow u^*=2\epsilon/3$ is the dimensionless renormalized coupling (u^* being the fixed point), and where $\Phi(x,y)$ is a universal function given in terms of lengthy multiple integrals. The dispersion relation is the solution of $\tilde{G}_R(q;\omega_q)^{-1}=0$, and from Eq. (5) one finds

$$\omega_q=-i\Gamma\left[q^2+\tfrac{2}{3}\epsilon\kappa^2\Phi\left[\frac{q^2}{\kappa^2},\frac{q^2}{\kappa^2}\right]\right]=-i\Gamma q^2\Omega(q\xi) \;.$$

The limiting forms of $\Omega(x)$ have been given in Eq. (2). Details of the calculation, as well as the form of the function $\Omega(x)$, will be given in a forthcoming publication.

We have reported for the first time a calculation of dynamic interfacial properties of an Ising-like system described by a time-dependent Ginzburg-Landau model in $4-\epsilon$ dimensions. Purely relaxing interfaces of this kind characterize domains in Ising antiferromagnets or in order-disorder systems. The calculation is the first to make use of the static and dynamic equations of the bulk system as the starting point and the approach could be generalized to the study of interfaces in binary alloys (model B) or liquid-gas–liquid mixture systems (model H).[5] Calculations on such systems would allow a close look at the nature of the hydrodynamic limit.

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[1]See, e.g., the reviews (and references cited): K. Binder, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. Lebowitz (Academic, New York, 1984), Vol. 8; D. Jasnow, Rep. Prog. Phys. **47**, 1059 (1984).

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[8]Note, however, that the *amplitude* of the surface singularity in $R(q;z,z';\omega)$ may vanish as the critical point is approached and the interface disappears. This can be seen explicitly in the zero-order calculation. What survives is a branch cut at the characteristic *bulk* frequency $\omega_B(q)\sim q^z$.

[9]Unfortunately, in model A the mean-field value of $z(=2)$ happens to coincide with the power of q in ω_q as $q\xi\rightarrow 0$. However, in the case of model B for a conserved order parameter, the interface relaxes as $\omega_q\sim q^3$, for $q\xi\rightarrow 0$, while the bulk modes relax according to $\omega_B(k)\sim k^4$, suggesting a singularity $\Omega(x)\sim x^{-1}$ as $x\rightarrow 0$.

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