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Directed diffusion-limited aggregation on the Bethe lattice: Exact results

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We study directed diffusion-limited aggregation on the Bethe lattice. An exact nonlinear recursion relation for the probability distribution of the density on finite lattices is constructed. The aggregates in a problem on the infinite lattice display marginal scaling behavior: They are compact up to a logarithmic correction. A mapping between models with perfect and partial adhesion on contact is given as a simple example of universality.

Diffusion-limited aggregation[1-3] (DLA) appears to be an effective yet conceptually simple model governing the irreversible growth of clusters in a wide range of systems: colloids, aerosols, and gels are examples. In the model, particles diffuse until they adhere to an aggregate which has grown from a seed particle and in which thermal relaxation can be neglected. As well as promising a diverse set of applications, the field has attracted considerable attention among theorists because both Monte Carlo studies[1-3] and experiments[4] have shown that the aggregates are scale invariant or fractal.

Recently, the extent to which this scale invariance is universal has been investigated.[1-3,5-8] One avenue that has been explored is the effect of a strong external field which provides a preferred direction in which the diffusion tends to take place.[7] Alternatively, motion counter to this direction is forbidden entirely.[8] Monte Carlo studies of this problem, known as directed diffusion-limited aggregation (DDLA), have suggested that the fractal dimension of these aggregates is just the Euclidean dimension. It therefore seems certain that DDLA is in a different universality class from DLA, but since it is difficult to probe for logarithmic corrections in numerical work, nontrivial marginal scaling behavior cannot be ruled out. Indeed, in the closely related problem of ballistic aggregation[9] (in which the trajectories are straight lines) such corrections may have already been observed.[10]

Although Monte Carlo work has led to a wealth of information on various models of aggregation, analytical results on this class of nonequilibrium problems have been sparse. Several mean-field treatments[11] and an approximate real space renormalization group[12] have been proposed, but few exact results have been given for nontrivial systems. In this Rapid Communication we study DDLA on the Bethe lattice. The nested structure of this lattice allows us to obtain exact results while retaining many of the essential characteristics of the problem on Euclidean lattices. We show analytically that a problem in DDLA on the Bethe lattice has a trivial fractal dimension with a logarithmic correction, and so represents a case of marginal scaling.

In the first model we shall consider, particles are introduced one at a time at the apex of a finite Bethe lattice of order l (see Fig. 1). Each executes a directed random walk (in which upward motion is forbidden) until it visits a site adjacent to a cluster and adheres, or stops at the bottom of the lattice. Once a particle has stopped it moves no further and another is added. The aggregate is complete once the apex site has been filled.[13]

Just as in models of aggregation on more connected lattices, the clusters in our problem are formed kinetically and irreversibly. In DLA and DDLA in the plane, the flux of diffusing particles received by a branch is reduced by more extended neighboring branches. This screening effect is also present in our model, but in our case it is perfect screening—branches may be cut off entirely from receiving particles by the growth of other branches. Finally, holes on all length scales can be frozen into the aggregates, presaging the existence of nontrivial scaling behavior.

The basic quantity in our work is the probability that the apex of the lattice of order l is filled by the nth particle, $P_l(n)$. The exact nonlinear recursion relation for the P_l's is simplest when written in terms of

$$G_l(n) \equiv 1-\sum_{k=0}^{n} P_l(k) = \sum_{k=n+1}^{\infty} P_l(k) \quad ,$$

which is the probability that the apex is not filled by any of the first n particles. The relation is then

$$G_{l+1}(n+1) = 2^{-n} \sum_{m=0}^{n} \binom{n}{m} G_l(m)\, G_l(n-m) \quad . \tag{1}$$

What is important to notice here is that once the number of particles entering the two subtrees beneath the apex is given, the contributions from these subtrees become statistically independent—hence, the product in Eq. (1). This decoupling does not take place in the problem of DLA on this lattice.

With the recursion relation (1) and the initial distribution $G_1(n) = \delta_{n,0}$ it becomes possible to construct the $P_l(n)$'s exactly; we have done this up to $l=15$ (see Fig. 2). Note that in this problem the exact probability distribution and all its moments can be calculated, whereas in Monte Carlo studies of aggregation not even the rms deviation from the mean has been studied because the computing time required to construct an ensemble of large clusters is prohibitively long.

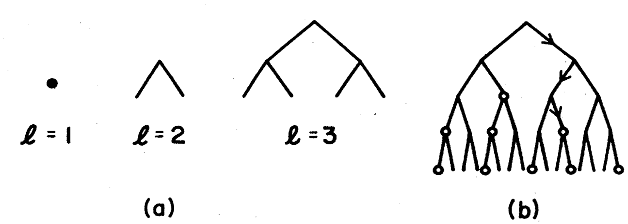

FIG. 1. (a) Finite Bethe lattices of orders $l=1,2,3$. (b) A step in the time evolution of a typical aggregate on the $l=5$ lattice.

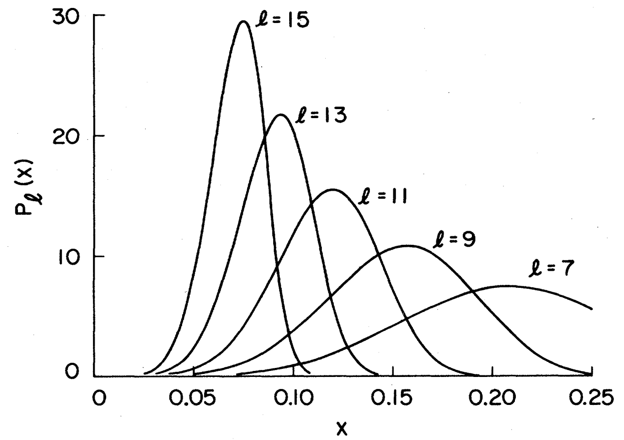

FIG. 2. Probability distribution of $x=(n+1)/2^l$ for lattices of order $l=7$, 9, 11, 13, and 15.

Figure 2 suggests that $\bar{n}_l \equiv (2^l-1)^{-1}\sum_{n=0}^{\infty} nP_l(n)$, the mean density on the lth lattice, approaches zero as l grows large. To verify this, note that when l is large $x \equiv (n+1)/2^l$ is a quasicontinuous variable which ranges between zero and one, and is equal to the density to within a correction of order 2^{-l}. We define $_l(x) \equiv G_l(n)$ and choose a fixed positive x. Equation (1) then shows that $\mathscr{G}_{l+1}(x) \cong \mathscr{G}_l^2(x)$ for sufficiently large l, since the binomial distribution is sharply peaked about its mean. This relation implies that $\mathscr{G}_l(x)$ approaches either zero or one as $l \to \infty$. We have ruled out the latter possibility by proving that $\mathscr{G}_{l+1}(x) \leq \mathscr{G}_l(x)$ and that $\mathscr{G}_l(x) < 1$ for sufficiently large l. We conclude that the probability of attaining a density greater than x [i.e., $\mathscr{G}_l(x)$] vanishes as $l \to \infty$ for any $x > 0$. In particular, $\bar{n}_l \to 0$ as $l \to \infty$, indicating that large holes appear in the aggregate with appreciable weight.

We have not been able to determine the way in which $\bar{n}_l$ tends to zero as $l \to \infty$. However, the precise scaling behavior of the density can be found in a different problem that is closely related to the one studied so far. In the new problem, the diffusing particles adhere to the aggregate as before, but now we introduce a nonzero sticking probability p that a particle simply sticks each time it arrives at a new site. Note that particles need not stick at sites adjacent to the existing aggregate. Once a particle has stuck, it remains in place and is considered to be part of the aggregate. The recursion relation for this new problem is obtained by simply inserting a prefactor q^{n+1} on the right-hand side (RHS) of Eq. (1), where $q \equiv 1-p$. In the presence of the nonzero sticking probability, the problem makes sense on the $l=\infty$ lattice, and we specialize to this limit. As $l \to \infty$ the recursion relation becomes a closed equation for $G \equiv \lim_{l\to\infty} G_l$, namely,

$$G(n+1) = \frac{q^{n+1}}{2^n} \sum_{m=0}^{n} \binom{n}{m} G(m)G(n-m) \ . \qquad (2)$$

In effect, the sharp cutoff in the lattice at the lth level in the old problem has been replaced by a smooth, probabilistic cutoff in the vicinity of the (p^{-1})th level.

Equation (2) completely specifies G and hence $P = \lim_{l\to\infty} P_l$ once the physical boundary condition $G(0) = 1$ is given. We shall show that there are solutions to (2) with the large n behavior

$$\frac{\ln G(n)}{\ln q} = \frac{n \ln n}{\ln 2} + cn + \frac{\ln n}{\ln 2} + O(1) \ . \qquad (3)$$

Here, c is a constant that must be determined by the boundary condition but which is difficult to evaluate. Although solutions to Eq. (2) with different large n behavior may exist, we have verified numerically for several values of p that (3) is the correct form when $G(0)=1$.

To see that there are, indeed, solutions to (2) with large n behavior given by (3), we introduce the new variable $k=2m-n$ and employ the Gaussian approximation to the binomial coefficient (valid for $n \gg 1$). The RHS of Eq. (2) becomes

$$\frac{2q^{n+1}}{(2\pi n)^{1/2}} \sum_k e^{-k^2/2n} G\left(\frac{n+k}{2}\right) G\left(\frac{n-k}{2}\right) \ ,$$

where $k=-n, -n+2, -n+4, \ldots, n$. Now when n is large, the exponential factor ensures that only $k \ll n$ contribute appreciably to this sum. Substituting (3) into this expression and expanding the exponents of the G's to third order in k/n, we find that (3) does indeed provide a large n solution to (2).

As p becomes small, a large number of particles will most probably be added to the lattice before the structure is completed. Therefore, the asymptotic form (3) can be used to obtain the $p \to 0$ behavior of the mean number of particles needed to fill the apex site, $\bar{N}(p) \equiv \sum_{n=0}^{\infty} nP(n)$, and we find $\bar{N}(p) \sim \ln 2/(p|\ln p|)$ as $p \to 0$. It follows in the same way that the variance $\sigma^2(p) \equiv \sum_{n=0}^{\infty} n^2 P(n) - \bar{N}^2(p)$ diverges analogously: $\sigma(p) \sim \ln 2/(p|\ln p|)$ as $p \to 0$.

In the absence of interaction between the particles (so they simply pass through each other) but with finite sticking probability p, the mean number of particles needed to fill the top site of the lattice is just p^{-1}. Since this is an upper bound on $\bar{N}(p)$, $N \equiv p^{-1}$ may be regarded as the effective number of lattice sites available for occupation in the interacting problem. The mean density $\bar{n}(N) \equiv \bar{N}(p)/N$ then has the scaling behavior

$$\bar{n}(N) \sim \ln 2/\ln N \qquad (4)$$

as $N \to \infty$. To put this result in context, let us briefly consider the problem of DLA in a Euclidean space of dimension d. Suppose a seed particle is placed at the center of a sphere of radius R that contains N lattice sites, and the aggregation process is stopped when one of the branches touches the sphere. From Monte Carlo studies[1-3] it is known that the mean density of particles in this sphere $\bar{n}(R)$ scales like R^{D-d} or $N^{D/d-1}$, where the fractal dimension D is less than d. When interpreted in this way, our result (4) says that the fractal dimension of the Bethe lattice aggregates is just the dimension of the space they grow in (i.e., the analog of D/d is 1), so they are compact up to a logarithmic correction. Using the terminology of the theory of critical phenomena, we may state that DDLA on the $l=\infty$ Bethe lattice with sticking displays nontrivial marginal scaling behavior.

Finally, let us return to finite lattices with $p=0$ and consider a model with partial adhesion on contact. To be specific, when a diffusing particle has one filled and one vacant site immediately beneath it, it joins the aggregate with probability α but otherwise it continues its motion and moves to the vacancy. The mean number of particles needed to fill the apex of the lth lattice will be denoted $\overline{N}_l(\alpha)$. We first show that a simple correspondence exists between the problems with $\alpha=\frac{1}{2}$ and $\alpha=1$. To see this, let the analog of P_l for the $\alpha=\frac{1}{2}$ problem be P_l', and define $F_l'(n) \equiv \sum_{k=n}^{\infty} P_l'(k)$. The recursion relation for F_l' may be shown to be of precisely the same form as Eq. (1) for G_l. Because $F_1'=G_2$, we in fact have $F_l'(n)=G_{l+1}(n)$ and $P_l'(n)=P_{l+1}(n+1)$ for all n and l: the two problems are formally identical.[14] In particular,

$$\overline{N}_{l+1}(1)=\overline{N}_l(\tfrac{1}{2})+1 \quad . \tag{5}$$

To extend these considerations to α between $\frac{1}{2}$ and 1, first note that $\overline{N}_l(1) \le \overline{N}_l(\alpha) \le \overline{N}_l(\frac{1}{2})$. Combining this with Eq. (5) and the inequality $\overline{N}_{l+}(1) \le 2\overline{N}_l(1)+1$, we obtain the result $1 \le \overline{N}_l(\alpha)/\overline{N}_l(1) \le 2$. This shows that the asymptotic form of $\overline{N}_l(\alpha)$ is the same as that of $\overline{N}_l(1)$ up to a prefactor which is bounded between 1 and 2. We conclude that the problems with partial adhesion $(\frac{1}{2} \le \alpha < 1)$ have the same scaling properties as the problem with perfect adhesion $(\alpha=1)$, so we have obtained a simple but exact example of universality in aggregation. Monte Carlo studies[2,3] suggest that this result carries over to DLA on Euclidean lattices.

In conclusion, we have studied directed diffusion-limited aggregation on the Bethe lattice. In the problem on finite lattices, we constructed an exact nonlinear recursion relation for the probability distribution of the density and showed that the mean density tends to zero as the lattice size grows. The precise scaling properties of the aggregates were found for the problem on the infinite lattice with a nonzero sticking probability. The aggregates are compact up to a logarithmic correction, so this is a case of marginal scaling behavior. Finally, finite lattice problems with partial and perfect adhesion on contact were shown to have the same scaling behavior, giving a simple example of universality.

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