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Lorenz-like truncation for convection in ^{3}He-^{4}He mixtures near the superfluid transition

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The hydrodynamical problem of the onset of convection in a ^{3}He-^{4}He mixture near the superfluid transition is reduced to a fifth-order dynamical system by a Lorenz-like truncation. The truncated system is shown to reproduce the hydrodynamical results regarding the convection thresholds and the effect of periodic modulation on the stability pattern. The reduced dynamical system is expected to facilitate the study of nonlinear effects. Preliminary results near the convection threshold are presented.

I. INTRODUCTION

The onset of convection in ^{3}He-^{4}He mixtures near the λ point T_λ is an interesting problem both theoretically and experimentally because of the unique versatility of the system. Convection in a binary mixture (a double-diffusive convection) is generally more versatile than in a single-component system because of the possibility of having both stationary and oscillatory instabilities[1-4] at the onset. Among the different binary mixtures the ^{3}He-^{4}He mixture near the superfluid transition is most striking in its properties. The mass diffusion which is small compared to the heat diffusion away from T_λ, as in any other binary fluid, increases strongly near T_λ and diverges as the λ point is reached, while the thermal conductivity at zero mass current remains finite.[5-8] The important ratio of the two diffusivities (of heat and mass) is thus inverted as T_λ is approached. Furthermore, unlike any other binary fluid the thermodiffusion coefficient becomes of order unity[9] near T_λ and thus the effect of a concentration gradient in producing a heat current can no longer be ignored. Experiments on this exceptional system were performed by Lee, Lucas, and Tyler.[10] It turns out that the qualitative features of their results can be adequately understood[11,12] from a linear-stability analysis of the hydrodynamic equations under idealized boundary conditions. It was recently pointed out[11] that a frequency modulation of the temperature difference between the plates in a Rayleigh-Benard geometry may have a particularly strong effect on the onset of convection in the nearly superfluid ^{3}He-^{4}He system. The nonlinear convective effects in this system have not been studied as yet, theoretically or experimentally.

In this work we perform a low-order truncation of the hydrodynamic equations for the above system in a manner similar to the Lorenz truncation[13,14] for the single-component system and truncation of Da Costa *et al.*[15] for the thermohaline system. The motivation behind the introduction of the truncated system is to facilitate the handling of the nonlinearities. In the Lorenz truncation for the single-component system, the nonlinearities can be treated without much difficulty, but the correspondence with the true hydrodynamical system breaks down near the onset of the time-dependent state when steady convection is destabilized. This is so because the Lorenz system does not give a stable periodic state (limit cycle) when the steady convection state is destabilized whereas the true hydrodynamic system does so. In the thermohaline system, however, the numerical solutions of the actual hydrodynamic equations by Huppert and Moore[16] bear a close resemblance to the analysis of the truncated system by Da Costa *et al.*[15] Since our system is also double diffusive, we expect the correspondence with the actual hydrodynamic system to be preserved. Here, we first carry out a linear stability analysis to establish that our truncated system yields results identical to the full hydrodynamic equations. Effects of the nonlinearity near the convection threshold are treated next. The control parameter in our system is then modulated and the effect on the convection threshold studied. Once again the results are found to be similar to those obtained from the full hydrodynamic equations establishing the faithfulness of the truncated systems in the vicinity of the convection threshold. The onset of turbulence in our truncated system is planned to be treated in a subsequent paper.

It is worth mentioning that the fifth-order dynamical system that we obtain from the truncation is, similar to the Lorenz system before it, an interesting subject of study in its own right. It is the simplest generalization of the Lorenz model which provides the possibility of a Hopf bifurcation at the onset of convective motion. It is simpler than the thermohaline system because unlike the latter this has only one control parameter. The existence of four characteristic parameters (the Prandtl number σ, the solute Prandtl number σ/s, the thermodiffusion parameter μ, and k, the ratio of concentration and thermal Rayleigh numbers) here as opposed to one (Prandtl number) in the Lorenz system gives it greater versatility. Thus, charting out the different routes to chaos for this system should be an interesting study.

In Sec. II we give the hydrodynamic equations and introduce our truncated system of nonlinear ordinary differential equations. In Sec. III the various features of the linear stability analysis are discussed, while in Sec. IV the effects of a periodic modulation are studied. Section V provides a brief summary.

II. DYNAMICAL SYSTEM

The hydrodynamic equations governing the flow of a binary fluid have been obtained by Landau and Lifshitz[17] as

$$\frac{\partial T}{\partial t}+(\vec{v}\cdot\vec{\nabla})T=\frac{\Lambda}{C_P}\nabla^2 T+\frac{Dk_T}{\chi C_P}\nabla^2 C\ , \tag{2.1}$$

$$\frac{\partial C}{\partial t}+(\vec{v}\cdot\vec{\nabla})C=D\nabla^2 C+\frac{Dk_T}{T}\nabla^2 T\ . \tag{2.2}$$

Here, $\vec{v}$ is the velocity of flow; T, the temperature field; C, the concentration field denoting the mass fraction of ^{3}He; D, the isothermal mass-diffusion coefficient; Λ, the thermal conductivity in the absence of temperature gradient; k_T, the thermodiffusion; C_P, the specific heat at constant pressure and concentration; and χ, the isothermal susceptibility $(\partial C/\partial\mu)_T$, where μ stands for the chemical potential. The velocity follows the usual Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\frac{1}{\rho}\vec{\nabla}p+\vec{g}+\nu\nabla^2\vec{v}\ , \tag{2.3}$$

ρ being the density; p, the pressure field; $\vec{g}$, the acceleration due to gravity; and ν, the kinematic viscosity. The fluid is assumed to be incompressible,

$$\vec{\nabla}\cdot\vec{v}=0 \tag{2.4}$$

and the equation of state is

$$\rho=\rho_m[1-\alpha(T-T_m)-\beta(C-C_m)]\ , \tag{2.5}$$

where ρ_m, T_m, and C_m are the mean density, temperature, and concentration, respectively. The expansion coefficient

$$\alpha=-\frac{1}{\rho_m}\frac{\partial\rho}{\partial T}\ , \tag{2.6}$$

and

$$\beta=-\frac{1}{\rho_m}\frac{\partial\rho}{\partial C} \tag{2.7}$$

measures the change in density of the mixture with change in ^{3}He concentration. By our definition, $\beta>0$. We assume that all experiments will be carried out under the condition of vanishing mass current (steady state) and impose the restriction

$$\frac{\Delta C}{\Delta T}=-\frac{k_T}{T}\ . \tag{2.8}$$

In the Rayleigh-Benard geometry that we treat, we assume the plates to be separated in the z direction and take the axis of the convection rolls to be in the y direction. The stream function will be an ellipse in the x-z plane when convection sets in and we describe this fact in the leading order by the choice

$$\psi(x,z,t)=a(t)\sin(\pi x/a)\sin(\pi z/d) \tag{2.9}$$

of the Fourier terms. The velocity components are obtained from the stream function using

$$v_x=\frac{\partial\psi}{\partial z},\quad v_z=-\frac{\partial\psi}{\partial x}\ . \tag{2.10}$$

The temperature and concentration fields can be written as

$$\begin{aligned}T(\vec{r})&=T_1+[(T_2-T_1)/d]z\\&\quad+b(t)\cos(\pi x/a)\sin(\pi z/d)\\&\quad+c(t)\sin(2\pi z/d)\ ,\end{aligned} \tag{2.11}$$

$$\begin{aligned}C(\vec{r})&=C_1+[(C_2-C_1)/d]z\\&\quad+d(t)\cos(\pi x/a)\sin(\pi z/d)\\&\quad+e(t)\sin(2\pi z/d)\ .\end{aligned} \tag{2.12}$$

The terms of the form $\cos(\pi x/a)\sin(\pi z/d)$ in $T(\vec{r})$ and $C(\vec{r})$ indicate the distribution for a roll (as for the velocity field) while the $\sin(2\pi z/d)$ terms give the net heat flow and mass flow due to convection. We now introduce the above Fourier-series expansions in Eqs. (2.1)–(2.3) and equate coefficients of like terms to obtain equations for $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $e(t)$. The procedure for the single-component fluid has been discussed in detail by Mclaughlin and Martin[14] and for the thermohaline problem by Da Costa *et al.*[15] Our system differs from the thermohaline system in the existence of the two cross terms in Eqs. (2.1) and (2.2) and the imposition of the condition of Eq. (2.8). Straightforward algebra similar to Refs. 14 and 15 leads to

$$\dot{X}=\sigma(-X+Y+U)\ , \tag{2.13}$$

$$\dot{Y}=-XZ+rX-Y+sU\ , \tag{2.14}$$

$$\dot{Z}=XY-bZ+sV\ , \tag{2.15}$$

$$\dot{U}=XV-krX-sU+s\mu^2 Y\ , \tag{2.16}$$

$$\dot{V}=-XU-sbV+s\mu^2 Z\ . \tag{2.17}$$

Here, X,Y,Z,U,V are scaled versions of a,b,c,d,e, respectively, $\sigma=\nu C_P/\Lambda$ is the thermal Prandtl number, $S=DC_P/\Lambda$ is the ratio of solute to thermal diffusivities, $k=\beta k_T/\alpha T$, $\mu=k_T^2/\chi C_P T$, and

$$r=[\alpha(\Delta T)gd^3C_P/\nu\Lambda]/(27\pi^4/4)$$

is the thermal Rayleigh number, while b is a constant coming from geometrical considerations and is equal to $\frac{8}{3}$. Note that while σ, s, μ^2, and k are constants of the fluid (they depend on the mean temperature), the quantity r is the control parameter. Unlike the thermohaline problem, here we have only one control parameter because of the constraint imposed by Eq. (2.8). Equations (2.13)–(2.17) constitute our model for the onset of convection in ^{3}He-^{4}He mixtures near the superfluid transition. In the next section we provide the linear-stability analysis and study the effect of the nonlinear terms on the onset of convection.

III. THRESHOLD AND ITS VICINITY

We note that $X=Y=Z=U=V=0$ is a fixed point of Eqs. (2.13)–(2.17). This is the steady conduction state. There is no velocity field ($X=0$) and the temperature and concentration profiles are linear as is characteristic of the conduction state [see Eqs. (2.11) and (2.12)]. As long as this state is stable, the fluid is at rest; when this state is destabilized convection begins. We shall carry out a linear-stability analysis to determine at what value of r the conduction state is destabilized. To do so, we assume X, Y, Z, U, and V are very close to zero and linearize Eqs. (2.13)–(2.17) to obtain

$$\dot{X}=\sigma(-X+Y+U)\,, \tag{3.1}$$

$$\dot{Y}=rX-Y+sU\,, \tag{3.2}$$

$$\dot{U}=-krX+s\mu^2Y-sU\,. \tag{3.3}$$

We try the solutions

$$(X,Y,U)=(A_0,B_0,C_0)e^{\lambda t}\,.$$

Inserting this solution in Eqs. (3.1)–(3.3) and demanding that A_0, B_0, and C_0 be consistently determined leads to the condition

$$\det\begin{bmatrix}\lambda+\sigma & -\sigma & -\sigma\\ -r & \lambda+1 & -s\\ kr & -s\mu^2 & \lambda+s\end{bmatrix}=0\,. \tag{3.4}$$

This leads to a cubic in λ with all three roots having negative real parts for $r=0$. We now have two possibilities as r is increased.

(i) The real part of at least one root becomes zero with the imaginary part zero at the same time.

(ii) A pair of complex roots crosses the imaginary axis, so that these roots are purely imaginary at a certain value of r. If (i) holds then we have steady convection, while in case (ii) we have a Hopf bifurcation and oscillatory convection sets in.

To find the threshold value of r for steady convection, we set $\lambda=0$ in Eq. (3.4) and obtain

$$r_0=\frac{1-s\mu^2}{1+\mu^2-k[1+(1/s)]}\,. \tag{3.5}$$

For the oscillatory convection, we require a pair of roots to have the form $\pm i\omega_0$ at $r=\tilde{r}_0$ and this yields

$$\tilde{r}_0=\frac{1+(1/\sigma_1)+(\mu^2/\sigma_1)[1+(1/\sigma)-(\mu^2/\sigma_1)]^{-1}}{1-k[1+(1/\sigma)-(\mu^2/\sigma_1)]^{-1}}\,, \tag{3.6}$$

where $\sigma_1=\nu/D$ is the solute Prandtl number. The frequency of oscillations is

$$\omega_0^2=\sigma(1+s)+s(1-s\mu^2)-\sigma\tilde{r}_0(1-k)\,. \tag{3.7}$$

If $r_0<\tilde{r}_0$, then stationary convection sets in and if $r_0>\tilde{r}_0$, then the system shows oscillatory convection as the control parameter is varied. The above results are identical with those obtained from the full hydrodynamic equations.[11]

To study the nature of the bifurcation for steady convection (i.e., $r_0<\tilde{r}_0$), we set

$$r=r_0+r_2X^2 \tag{3.8}$$

and obtain r_2 by solving Eqs. (2.13)–(2.17) to $O(X^2)$ for $\dot{X}=\dot{Y}=\dot{Z}=\dot{U}=\dot{V}=0$. Straightforward algebra yields

$$r_2=\frac{Z_0(1+\mu^2)-V_0(1+s^{-1})}{1+\mu^2-k(1+s^{-1})}\,, \tag{3.9}$$

where

$$Z_0=\frac{b(r_0+s)+r_0-1}{(1+s)(b^2-\mu^2s^2)} \tag{3.10}$$

and

$$V_0=\frac{\mu^2(r_0+s)s+b(r_0-1)}{s(1+s)(b^2-\mu^2s^2)}\,. \tag{3.11}$$

Depending upon the parameters μ, r_0, s, and k, which are fixed for a given experiment but can be varied by changing the mean temperature, the bifurcation can be subcritical ($r_2<0$) or supercritical ($r_2>0$).

For the oscillatory convection, we similarly set

$$\tilde{r}=\tilde{r}_0+\tilde{r}_2X^2\,. \tag{3.12}$$

The amplitudes now have the time dependence $e^{i\omega_0 t}$ and $\tilde{r}_2$ is found to be

$$\tilde{r}_2=\mathrm{Re}\frac{\tilde{Z}_0[i\omega_0+s(1+\mu^2)]-\tilde{V}_0(i\omega_0+1+s)}{s(1-\mu^2)-k(1+s)+i\omega_0(1-k)}\,, \tag{3.13}$$

where

$$\tilde{Z}_0=\frac{\sigma(\tilde{r}_0+s)bs+(\tilde{r}_0-1)\sigma s+i\omega_0(bs^2+\tilde{r}_0\sigma-s)}{\sigma(i\omega_0+1+s)[b^2s-s^2\mu^2-\omega_0^2+i\omega_0(1+s)b]} \tag{3.14}$$

and

$$\tilde{V}_0=\frac{(\tilde{r}_0+s)\sigma\mu^2s+\sigma b(\tilde{r}_0-1)+\omega_0^2(1+\sigma+b)-\omega_0^3+i\omega_0(\sigma\tilde{r}_0+s^2\mu^2-\sigma-b-\sigma b)}{\sigma(i\omega_0+1+s)[b^2s-s^2\mu^2-\omega_0^2+i\omega_0(1+s)b]}\,. \tag{3.15}$$

For $\tilde{r}_2<0$, the Hopf bifurcation is subcritical while for $\tilde{r}_2>0$, the Hopf bifurcation is supercritical. From Eq. (3.13), one can compute the parameter values at which the bifurcation changes from supercritical to subcritical. If r is raised beyond the threshold of Hopf bifurcation, the overstable state becomes a steady state as r_0 is approached. The steady state develops for $r<r_0$ as the bifurcation at r_0 is subcritical.

IV. EFFECT OF CONTROL PARAMETER MODULATION

In this section we study an oscillating drive on the control parameter r. We replace r by $r[1+\epsilon\cos(\omega t)]$ and study the effect on the threshold for onset of stationary or oscillatory convection. Here, we restrict ourselves to $\epsilon\ll 1$ and apply perturbation theory. Near the onset, one can linearize Eqs. (2.13)—(2.17) in X, Y, Z, U, and V and write the modulated system equations as

$$\dot{X}=\sigma(-X+Y+U)\ , \tag{4.1}$$

$$\dot{Y}=r[1+\epsilon\cos(\omega t)]X-Y+sU\ , \tag{4.2}$$

$$\dot{U}=-kr[1+\epsilon\cos(\omega t)]X-sU+s\mu^2 Y\ . \tag{4.3}$$

For small ϵ we expand

$$\begin{aligned} X&=X_0+\epsilon X_1+\epsilon^2X_2+\cdots\ ,\\ Y&=Y_0+\epsilon Y_1+\epsilon^2Y_2+\cdots\ ,\\ U&=U_0+\epsilon U_1+\epsilon^2U_2+\cdots\ ,\\ r&=r_0+\epsilon r_1+\epsilon^2r_2+\cdots\ . \end{aligned} \tag{4.4}$$

Inserting the above expansion in Eqs. (4.1)—(4.3) and equating similar powers of ϵ, we arrive at

$$L\begin{bmatrix}X_0\\Y_0\\U_0\end{bmatrix}=0\ , \tag{4.5}$$

$$L\begin{bmatrix}X_1\\Y_1\\U_1\end{bmatrix}=\begin{bmatrix}0\\ r_0X_0\cos(\omega t)+r_1X_0\\ -kr_1X_0-kr_0X_0\cos(\omega t)\end{bmatrix}\ , \tag{4.6}$$

$$L\begin{bmatrix}X_2\\Y_2\\U_2\end{bmatrix}=\begin{bmatrix}0\\ r_1X_0\cos(\omega t)+r_0X_1\cos(\omega t)+r_1X_1+r_2X_0\\ -kr_1X_0\cos(\omega t)-kr_0X_1\cos(\omega t)-kr_1X_1-kr_2X_0\end{bmatrix}\ , \tag{4.7}$$

where L is the operator

$$L=\begin{bmatrix}\frac{\partial}{\partial t}+\sigma & -\sigma & -\sigma\\ -r_0 & \frac{\partial}{\partial t}+1 & -s\\ kr_0 & -s\mu^2 & \frac{\partial}{\partial t}+s\end{bmatrix}\ . \tag{4.8}$$

We now study separately the stationary and oscillatory instabilities.

(i) Stationary instability: To find r_c in the presence of modulation, we determine that value of

$$r=r_0+\epsilon r_1+\epsilon^2 r_2+\cdots$$

at which the state $X=Y=U=0$ is destabilized. At the leading order in ϵ, the answer is obtained from Eq. (4.5) and is the same as that obtained in Sec. III. At this point, the destabilization of $X=Y=U=0$ now implies that Eqs. (4.6) and (4.7) must be solvable for X_1, Y_1, U_1; X_2, Y_2, U_2; etc., under the condition that Eq. (4.5) is satisfied with nonzero values of X_0, Y_0, Z_0. The solvability of Eqs. (4.6), (4.7), etc., fixes the critical values of $r_1, r_2,\ldots,$ etc., and thus determines the critical value of r as an expansion in ϵ. The solvability criterion can be formulated by considering the inhomogeneous equation

$$L\,|\,g\,\rangle=|\,h\,\rangle \tag{4.9}$$

which must be solved under the constraint

$$L\,|\,f\,\rangle=0\ . \tag{4.10}$$

One can construct the vector $\langle f_1\,|$ with the property

$$\langle f_l\,|\,L=0 \tag{4.11}$$

and then it follows from Eq. (4.9),

$$\langle f_l\,|\,h\,\rangle=\langle f_l\,|\,L\,|\,g\,\rangle=0\ . \tag{4.12}$$

The orthogonality of the vectors $|\,f_l\rangle$ and $|\,h\,\rangle$ is the required solvability condition.

To implement the above, we first apply the solvability criterion on Eq. (4.6). In this case X_0, Y_0, U_0 are time independent and hence $\langle f_l\,|=(a_0b_0c_0)$, where a_0, b_0, and c_0 are constants. The vanishing of the scalar product (which is a time average over one period of the modulation) of Eq. (4.12) now leads to

$$r_1=0\ . \tag{4.13}$$

We can now solve Eq. (4.6), X_1, Y_1, and U_1 and obtain

$$X_1=r_0X_0\mathrm{Re}\frac{i\omega G_1+G_2}{L_1+iL_2}e^{i\omega t}\ , \tag{4.14}$$

where

$$L_1=s\sigma(1-s\mu^2)-\sigma r_0 s\left[1+\mu^2-k\left[1+\frac{1}{s}\right]\right]-\omega^2(1+\sigma+s)\ , \tag{4.15}$$

$$L_2=-\omega^3+\omega[(1+s)\sigma+s(1-s\mu^2)+\sigma r_0(k-1)]\ , \tag{4.16}$$

$$G_1=\sigma(1-k)\ , \tag{4.17}$$

and

$$G_2=\sigma s(1+\mu^2)-k\left[1+\frac{1}{s}\right]\ . \tag{4.18}$$

Using Eqs. (4.13) and (4.14) in the right-hand side of Eq. (4.7) and applying the solvability criterion, we now obtain

$$r_2=\frac{r_0^2}{2}\frac{(G_2L_1-\omega G_1L_2)}{L_1^2+L_2^2}\ . \tag{4.19}$$

The correction can be of either sign depending on the frequency and the parameters s,k,σ,μ. By a proper choice of the parameters we can obtain a stabilization or a destabilization and also control the magnitude of the effect.[11]

(ii) Oscillatory instability: In this case the system has a natural frequency ω_0 and the response of the system depends very strongly on the frequency of the modulation. As is usual in such cases, parametric resonance occurs at frequencies $2\omega_0/n$, where $n=1,2,\ldots,$ etc. The response is strongest for $n=1$. This is manifested by the fact that there is a correction to $\tilde{r}_0$ at $0(\epsilon)$ when the modulating frequency is $\omega=2\omega_0$. The correction can be worked out by applying the solvability criterion. The form of the vector $\langle f_l \mid$ is $(\tilde{a}\tilde{b}\tilde{c})e^{i\omega_0 t}$ and we obtain

$$\frac{1}{T}\int_0^T dt[\tilde{r}_0\tilde{X}_0\tilde{b}\cos^2\omega_0 t\cos(2\omega_0 t)+\tilde{r}_1\tilde{X}_0 b\cos^2(\omega_0 t) - k\tilde{r}_0\tilde{X}_0\tilde{c}\cos^2(\omega_0 t)\cos(2\omega_0 t) - k\tilde{r}_1\tilde{X}_0\tilde{c}\cos^2(\omega_0 t)]=0\ , \tag{4.20}$$

leading to

$$\tilde{r}_1=-\frac{\tilde{r}_0}{2}\ . \tag{4.21}$$

For frequencies $\omega\neq 2\omega_0$ or more precisely outside a band of $O(\epsilon\omega_0)$ about $2\omega_0$, the correction r_1 vanishes. In general, for a frequency $\omega=2\omega_0/n$, the first correction occurs at $O(\epsilon^n)$. Note that at $\omega=2\omega_0$ and for $\epsilon\ll 1$, the effect is always one of destabilization, i.e., the Hopf bifurcation should set in before the threshold for the unmodulated case is reached. Hopf bifurcation in the Lorenz system under a modulation shows a similar behavior for small amplitudes ϵ.[18]

V. SUMMARY

To study the onset of convection and turbulence in a ^{3}He-^{4}He mixture near the superfluid transition, we have reduced the nonlinear partial differential equations of hydrodynamics to a fifth-order dynamical system of coupled nonlinear ordinary differential equations (2.13)–(2.17). Linear stability analysis for convection thresholds of this dynamical system and the effects of a periodic modulation of the control parameter yield results identical with those for the true hydrodynamic system. The truncated set allows for the exploration of nonlinear effects. In Sec. III we have studied the effect near the convection threshold and in a subsequent publication we plan to report on the effect near the onset of turbulence.

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