Excitations from a filled Landau level in the two-dimensional electron gas

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We consider an interacting two-dimensional electron system, with a uniform positive background, in a strong perpendicular magnetic field at zero temperature, under conditions where an integral number of Landau levels are filled and the Coulomb energy $e^2/\epsilon l_0$ is smaller than the cyclotron energy $\hbar\omega_c$. The elementary neutral excitations may be described alternatively as *magnetoplasma modes*, or as *magnetic excitons*—i.e., a bound state of a hole in a filled Landau level and one electron in an otherwise empty level—and they are characterized by a conserved wave vector $\vec{k}$. The dispersion relations may be calculated exactly, to first order in $(e^2/\epsilon l_0)/\hbar\omega_c$, for the lowest magnetoplasmon band, which comes in to the cyclotron frequency at $k=0$. We also calculate the *spin-wave* dispersion relations for the case where one spin state of a Landau level is completely occupied, and we discuss qualitatively the exciton spectrum for a partially filled Landau level, under the conditions of the fractional quantized Hall effect.

I. INTRODUCTION

The inversion layer formed in silicon metal-oxide-semiconductor field-effect transistors (MOSFET's) or GaAs-$Al_xGa_{1-x}As$ heterojunctions is well described, at low temperatures, as a two-dimensional electron gas.[1] These are particularly interesting systems in which to observe correlation effects because the density of electrons, and hence, the relative strength of the Coulomb interaction, can be varied over a wide range by varying the gate voltage in the MOSFET's or by varying the concentration of donors in the heterojunctions. In addition, the scattering due to impurities can be very small in the best devices.

In the presence of a strong magnetic field normal to the layer, correlation effects in these systems give rise to the fractional quantum Hall effect at low densities[2,3] and are expected to cause Wigner crystallization at even lower densities.[3,4] Anomalous structure in the cyclotron resonance line shape, both in GaAs heterojunctions[5] and in Si MOSFET's,[6,7] has also been attributed to correlation effects. By Kohn's theorem,[8] electron-electron interactions cannot affect the cyclotron resonance in a system with translational symmetry. However, the presence of impurities allows coupling to magnetoplasma modes at nonzero wave vector, where correlation effects are important. Structure in the magnetoplasma modes of the two-dimensional electron gas will then lead to structure in the cyclotron resonance or optical absorption spectra.[5–7,9]

In principle one can *directly* study the finite wave-vector density response function $\chi_\rho(\vec{k},\omega)$ and obtain the magnetoplasmon dispersion curve by means of phonon absorption or reflection from the inversion layer.[10] One may also generate electromagnetic absorption at finite wave vectors by means of a grating superimposed on the sample; however, the interesting wavelengths here are on the order of 50 Å, which would be difficult to achieve in this manner.[11] In any case, it would clearly be of interest to have a theory of the excitation modes and response functions of the two-dimensional electron gas in a strong magnetic field.

In this paper we consider an ideal two-dimensional electron system, with a neutralizing positive background but no impurity scattering in the presence of a strong perpendicular magnetic field B, at a density such that an integral number of Landau levels are occupied. We consider the limit where the cyclotron energy ω_c is large compared to the Coulomb energy $e^2/\epsilon l_0$, where ϵ is the background dielectric constant, and $l_0=(eB/c)^{-1/2}$ is the magnetic length. (We set $\hbar=1$.) The elementary neutral excitations of this system can be described alternatively as "magnetoplasma modes," or as "magnetic excitons" in which one electron is excited to an unoccupied Landau level n', leaving behind a hole in a filled Landau level n. If the electron-electron interaction is ignored, then the energy of the excitation is just equal to the kinetic-energy difference $(n'-n)\omega_c$, added to the Zeeman energy change $|g\mu_B B|\delta S_z$ if a spin flip is involved. When the electron interaction is included, the energies are shifted by an amount of order $e^2/\epsilon l_0$.

Our restriction to an integral number of filled Landau levels is made because it greatly simplifies the calculation. Of course, experimentally, one is interested, at least equally, in the case of noninteger filling factors. Some discussion of the noninteger case will be given in Sec. V below.

As we shall see below, an important feature of our problem is that the neutral excitations may be classified by a conserved wave vector $\vec{k}$. Therefore, we may discuss a set of dispersion curves

$$E_m(\vec{k})=m\omega_c+|g\mu_B B|\delta S_z+\Delta E_m(\vec{k}), \tag{1.1}$$

where $m=n'-n$ is an integer ≥ 0, and $\Delta E_m(\vec{k})$ is of order $e^2/\epsilon l_0$. The function $\Delta E_m(\vec{k})$ depends on m and also on which Landau levels were initially occupied; moreover, there will be several branches to ΔE_m, in general, which we distinguish by an additional index μ when necessary.

To make our discussion concrete, let us focus our attention on the excitation spectrum at $m=1$. We consider

first the case where the initial state has both spin states equally occupied, i.e., Landau levels with index $n=0,1,\ldots,\nu_0-1$ are filled with both spin-up and spin-down electrons, and there are a total of $2\nu_0$ electrons per quantum of magnetic flux. In this situation the ground state is an eigenstate of spin angular momentum with $S=0$, and the one-electron excitations may be classified as singlet or triplet states.

The $m=1$ singlet exciton branch $E_1^s(\vec{k})$ is the familiar magnetoplasma mode, which couples to the charge-density operator $\rho(\vec{k})$, and which gives rise to a pole at $\omega=E_1^s(\vec{k})$ in the density response function $\chi_\rho(\vec{k},\omega)$. The triplet exciton states do not affect the response to a density perturbation $\rho(\vec{k})$, but they do appear in the response functions associated with the spin-density operators $\vec{\sigma}(\vec{k})$. If we denote by $E_1^t(\vec{k})$ the energy of the triplet exciton with $\delta S_z=0$, then the spin-density response function $\chi_{\sigma_z}(\vec{k},\omega)$ has a pole at $\omega=E_1^t(\vec{k})$. The triplet excitons with $\delta S_z=\pm 1$ have energies shifted from $E_1^t(\vec{k})$ by the Zeeman energy $|g\mu_B B|\,\delta S_z$, and these states therefore give rise to poles at $\omega=E_1^t(\vec{k})\pm g\mu_B B$ in the transverse spin-density response functions $\chi_{\sigma_\pm}(\vec{k},\omega)$.[12]

When the ground state of the system has *different* occupations for the two spins, the classification of excitons into singlet and triplet is no longer valid. The case of greatest interest has $\nu_\downarrow=\nu_\uparrow+1$ where $\nu_\uparrow$ and $\nu_\downarrow$ are the number of occupied Landau levels for spin up and spin down, respectively, and the total number of electrons per flux quantum is $(2\nu_\uparrow+1)$. If $\nu_\uparrow>0$, so that there are electrons present with both spins, then we find *two* branches $E_1^1(\vec{k})$ and $E_1^2(\vec{k})$ of the exciton spectrum at $m=1$, *both* of which appear as poles in the density response function $\chi_\rho(\vec{k},\omega)$. The spin quantum numbers of both the ground state and the two branches of excited states are $S=-S_z=N_\phi/2$, where N_ϕ is the number of flux quanta in the system. There are also corresponding excited states with $S=N_\phi/2$ and $S_z=-S+1$; these give rise to poles at $E_1^1(\vec{k})+|g\mu_B B|$ and $E_1^2(\vec{k})+|g\mu_B B|$ in the spin-density response $\chi_{\sigma_+}(\vec{k},\omega)$.

When $\nu_\uparrow=0$, so that only one spin state of the lowest Landau level is occupied, then there is only one exciton branch $E_1^\rho(\vec{k})$ appearing in $\chi_\rho(\vec{k},\omega)$, and one branch with energy $E_1^{\sigma+}(\vec{k})$ appearing in the spin-density response function $\chi_{\sigma_+}(\vec{k},\omega)$, at $m=1$. The branch $E_1^\rho(\vec{k})$ also appears in χ_{σ_z}.

We shall find that when the Zeeman energy is omitted, the various branches of the $m=1$ exciton spectrum have $E_1^\mu(\vec{k})\to\omega_c$ for $k\to 0$ in all cases. Thus Kohn's theorem is clearly satisfied for the $m=1$ modes appearing in the density response at $k\to 0$. [Excitations with $m\neq 1$ give vanishing contribution to $\chi_\rho(\vec{k},\omega)$ in the limit $k\to 0$.]

In the absence of impurities, and at $T=0$, the $m=1$ excitons have infinite lifetimes, at least if the parameter $(e^2/l_0\epsilon\omega_c)$ is not too large, because there are no other states possible with the same energy, wave vector, and spin quantum numbers.

Excitation modes with $m=0$ do not exist if the initial state has equal (integer) occupation numbers of the Landau levels of both spins. The $m=0$ mode is meaningful, however, if we have unequal occupation numbers, $\nu_\downarrow=\nu_\uparrow+1$. Excitations with an energy $E_0(\vec{k})=\Delta E_0(\vec{k})+|g\mu_B B|$ appear as poles in the spin-density response $\chi_{\sigma_+}(\vec{k},\omega)$, and the corresponding excited states have spin quantum numbers $S=-S_z=\frac{1}{2}N_\phi-1$. These $m=0$ excitons are just *spin waves* in the ferromagnetic ground state.

For values of $m\geq 2$, however, an excitation described by Eq. (1.1) can generally decay into two excitations with lower indices m' and $m''=m-m'$, in such a way that the spin quantum numbers, the total energy, and the total wave vector are conserved. Since we estimate the decay rate for these processes to be of order $e^2/\epsilon l_0$, the excitations with $m\geq 2$ may have a width in energy comparable to the energy shift $\Delta E_m(\vec{k})$.

In this paper we shall not calculate any decay rates, but we shall derive formulas for the energy shifts $\Delta E_m(\vec{k})$, which are correct to first order in the parameter $e^2/\epsilon l_0\omega_c$, for the cases $m=0$ and $m=1$. There are in fact three contributions to the energy shift $\Delta E_m(\vec{k})$ which we must take into account.

(a) A constant, independent of wave vector $\vec{k}$ and frequency ω, which represents the difference of the exchange self-energy of an electron in the excited Landau level and the self-energy in the level from which the electron is removed.

(b) The direct Coulomb interaction of the excited electron and hole.

(c) An "exchange energy," arising from terms in the Hamiltonian where the electron and hole annihilate each other at one point in space, and an electron-hole pair are created simultaneously at another point.

The third term (c) is the *only* term taken into account in the commonly employed random-phase approximation (RPA) for the density response function $\chi_\rho(\vec{k},\omega)$. The second term (b) is represented in diagrammatic perturbation theory by the "ladder diagrams" omitted from the RPA. (Note that direct interactions between a particle and hole are generally considered "exchange interactions" in a perturbation theory using electron propagators, and vice versa.)

The exchange interaction (c) is analogous to the "dipole-dipole interaction" responsible for the mobility of a Frenkel exciton in a molecular solid.[13] By contrast, the direct interaction (b) is the term responsible for electron-hole binding in the familiar hydrogenic Wannier exciton;[13] in the present case, however, the separation between the electron and hole is dependent on the overall wave vector $\vec{k}$ of the exciton. The exchange interaction (c) is absent in the case of a triplet exciton, so that the k dependence of the spectrum is completely determined by the direct interaction in that case.

In cases where there are several exciton branches, for a given m, which are not distinguished by symmetry considerations, the three terms (a), (b), and (c) are matrices connecting the various branches, and the sum of these matrices must be diagonalized to find the energy spectrum at

a given value of $\vec{k}$.

If the exchange terms (a) and (c) are ignored, and if any possible transitions of the electron and hole between different Landau levels are omitted, then the exciton problem is equivalent to the problem of two interacting charged particles of opposite sign, confined respectively to Landau levels n and $n'=n+m$, in two dimensions. This problem has been studied in some detail by Lerner and Lozovik,[14] in a somewhat different context—the excitons in their discussion are made of an electron and hole in the conduction and valence bands, respectively, of the host semiconductor.[15] (In that case, transitions between Landau levels are generally excluded in the strong-field limit by energy conservation, since the cyclotron frequencies of the conduction and valence bands will be generally incommensurate.)

The mathematical description of Lerner and Lozovik is directly applicable to our case, since the exciton wave functions are independent of the masses of the two particles involved. The form of the wave function, in the strong-field limit, is independent of the potential of interaction between the positive and negative particle, and is uniquely determined by the wave vector $\vec{k}$ and the Landau-level indices n and n'. The form of these wave functions and the associated binding energies will be reviewed in Sec. II below.

In Sec. III we introduce the Feynman diagram perturbation expansion for the density response function $\chi_\rho(\vec{k},\omega)$, and the spin-density response functions χ_{σ_z} and $\chi_{\sigma_\pm}$. We review the random-phase approximation and show how the results are modified when one includes the exchange self-energy and the ladder diagrams, which describe the electron-hole interaction calculated in Sec. II. We note that the density response function χ_ρ has been studied by previous authors in the random-phase approximation,[16] and also, in a calculation which includes some corrections to RPA, by Fukuyama, Kuramoto, and Platzman.[17] It is difficult for us to make a direct comparison with the work of Fukuyama *et al.*, however, because their calculations were restricted to the long-wavelength limit, at finite temperatures T and filling factor ν less than 1, and their formulas are not well defined in the limit $\nu\to 1$, $T\to 0$.

Results of our calculations for exciton dispersion curves in some particular cases of interest are presented in Sec. IV. Included are the spectrum of modes near the cyclotron frequency ($m=1$ modes) for several different filling factors, and the spin-wave dispersion relation ($m=0$) for the case of a single Landau level with one spin occupied. We also present an illustrative curve for the higher mode $m=2$, neglecting damping for the initial state of a single occupied Landau level.

A qualitative discussion of the applicability of our results to excitations from a partially filled Landau level, including a state of the fractional quantized Hall effect, will be given in Sec. V below. Calculations of some required matrix elements are presented in the Appendix.

It must be emphasized that the calculations of this paper are not directly applicable to current experiments, because the parameter $e^2/\epsilon l_0\omega_c$ is of order unity in the experiments of greatest interest, and because the effects of impurities cannot be completely neglected. However, we believe that the present description provides a useful conceptual framework, and perhaps a useful starting point for more accurate calculations when $e^2/\epsilon l_0\omega_c$ is of order unity.

We shall present in a separate publication calculations of the effects of a moderate amount of impurity scattering on the magnetoplasmon absorption near the cyclotron frequency, which we have carried out using a self-consistent Green's function approximation.

Bychkov, Iordanskii, and Éliashberg[18] have discussed the spin-wave spectrum ($m=0$) and magnetoplasmon dispersion ($m=1$) for the case of one spin component occupied in the first Landau level ($\nu_\downarrow=1$, $\nu_\uparrow=0$). Their calculation includes the exchange energies (a) and (c) as well as the direct Coulomb interaction (b), and agrees with our results, presented in Sec. IV, for this case. Details of the calculation were not given in Ref. 18, however.

II. EXCITON WAVE FUNCTIONS AND BINDING ENERGIES

In this section, we consider two particles of opposite charge, each with an effective mass m^*, and confined respectively to Landau levels n and $n'=n+m$, in two dimensions. (In GaAs, $m^*=0.067m_e$, where m_e is the mass of the electron.) We assume that these particles are described by the Hamiltonian

$$\mathcal{H}=\frac{1}{2m^*}[(\vec{p}_1-e\vec{A}_1/c)^2+(\vec{p}_2+e\vec{A}_2/c)^2]-u(\vec{r}_1-\vec{r}_2)\ , \tag{2.1}$$

where $\vec{p}_i$ is the momentum of the ith particle and the particles interact through the potential $u(\vec{r})$. In the case of interest $u(\vec{r})$ is the Coulomb potential $e^2/\epsilon r$, but we will let it be arbitrary for now. We choose the Landau gauge, so that the vector potential is $\vec{A}_i\equiv\vec{A}(\vec{r}_i)=Bx_i\hat{y}$. In this gauge, the single particle wave functions which describe an electron of charge $e<0$ in the nth Landau level are

$$\phi_{k,n}(\vec{r})=\frac{1}{(2\pi)^{1/2}(\pi^{1/2}2^n n!l_0)^{1/2}}e^{iky}\times\exp\left[-\frac{(x+kl_0^2)^2}{2l_0^2}\right]H_n((x+kl_0^2)/l_0)\ , \tag{2.2}$$

where H_n is a Hermite polynomial. The wave function which describes a positively charged particle (hole) with momentum k in the y direction, in the Landau level n' is

$$\bar{\phi}_{k,n'}(\vec{r})=\phi^*_{-k,n'}(\vec{r})\ . \tag{2.3}$$

The momentum operator $P_y=p_{1y}+p_{2y}$ commutes with the above Hamiltonian (2.1) in the Landau gauge, as can be seen by writing the Hamiltonian in relative and center-of-mass coordinates:

$$\mathscr{H}=\frac{1}{m^*}\left[\frac{P_x^2}{4}+\left[\frac{P_y}{2}-\frac{\Delta x}{2l_0^2}\right]^2+\Delta p_x^2 + \left[\Delta p_y-\frac{x}{l_0^2}\right]^2\right]-u(\Delta r)\,, \tag{2.4}$$

where $\vec{R}=(\vec{r}_1+\vec{r}_2)/2=(X,Y)$ and $\Delta\vec{r}=\vec{r}_1-\vec{r}_2=(\Delta x,\Delta y)$, while $\Delta\vec{p}=\vec{p}_1-\vec{p}_2$. Therefore the total momentum in the y direction is a good quantum number which we denote by k_y. The eigenstates of the Hamiltonian (2.1) can be labeled by k_y,n,m and an additional quantum number α and can be expanded in the single-particle eigenstates as follows,

$$\psi^{mn}_{\alpha k_y}(\vec{r}_1,\vec{r}_2)=\int dq\, C_{\alpha k_y q}\phi_{q,n+m}(\vec{r}_1)\bar{\phi}_{k_y-q,n}(\vec{r}_2)\,. \tag{2.5}$$

We wish to solve the eigenvalue equation

$$\langle\phi_{q,n'}\phi_{k_y-q,n}\,|\,\mathscr{H}\,|\,\psi^{mn}_{\alpha k_y}\rangle=[\Delta E^{mn}_{\alpha k_y}+(2n+m+1)\omega_c] \times\langle\phi_{q,n'}\bar{\phi}_{k_y-q,m}\,|\,\psi^{mn}_{\alpha k_y}\rangle\,, \tag{2.6}$$

which gives the following integral equation for the expansion coefficients $C_{\alpha k_y q}$:

$$\int dq' C_{\alpha k_y q'}[\Delta E^{mn}_{\alpha k_y}\delta(q-q')-F^{mn}_{k_y}(q-q')]=0\,, \tag{2.7}$$

where

$$F^{mn}_{k_y}(q)=\frac{e^{-q^2l_0^2/2}}{\pi l_0^2 2^{n+n'}n!n'!}\int d\Delta\vec{r}\,u(\Delta x+l_0^2k_y,\Delta y)e^{-iq\Delta y}e^{-\Delta x^2/2l_0^2} \times\int dX\,e^{-2X^2/l_0^2}H_{n'}\left[\frac{X+\Delta x/2-l_0^2q/2}{l_0}\right]H_{n'}\left[\frac{X+\Delta x/2+l_0^2q/2}{l_0}\right] \times H_n\left[\frac{X-\Delta x/2-l_0^2q/2}{l_0}\right]H_n\left[\frac{X-\Delta x/2+l_0^2q/2}{l_0}\right]. \tag{2.8}$$

Since the expression in square brackets in Eq. (2.7) depends only on the difference between q and q', the integral equation is diagonalized by a Fourier transformation. Thus we find solutions to Eq. (2.7) of the form $C_{\alpha k_y q}=e^{iq\alpha}$ and hence, from Eq. (2.5),

$$\psi^{mn}_{\alpha k_y}(\vec{r}_1,\vec{r}_2)=\int dq e^{iq\alpha}\phi_{q,n+m}(\vec{r}_1)\bar{\phi}_{k_y-q,n}(\vec{r}_2)\,. \tag{2.9}$$

Performing the q integral and defining $\alpha l_0^2\equiv k_x$, one finds[14]

$$\psi^{mn}_{\vec{k}}(\vec{R},\Delta\vec{r})=\frac{1}{2\pi}e^{i\vec{k}\cdot\vec{R}}e^{iX\Delta y/l_0^2}g_{mn}(\vec{\Delta}r-l_0^2\vec{k}\times\hat{z})\,, \tag{2.10}$$

where

$$g_{mn}(\vec{r})\equiv\frac{1}{[2^m l_0^2(n+m)!/n!]^{1/2}}L_n^m\left[\frac{r^2}{2l_0^2}\right] \times e^{-r^2/4l_0^2}\left[\frac{x+iy}{l_0}\right]^m, \tag{2.11}$$

and the energy spectrum depends on the Fourier transform $\tilde{F}^{mn}_{k_y}$ of $F^{mn}_{k_y}(q)$,

$$\Delta E^{mn}_{\vec{k}}=-\tilde{F}^{mn}_{k_y}(k_x l_0^2) =-\frac{n!}{2^m l_0^2(n+m)!}\int d\Delta\vec{r}\,u(\Delta\vec{r}+l_0^2\vec{k}\times\hat{z}) \times e^{-\Delta r^2/2l_0^2}\left[\frac{\Delta r}{l_0}\right]^{2m} \times\left[L_n^m\left[\frac{\Delta r^2}{2l_0^2}\right]\right]^2, \tag{2.12}$$

where L_n^m is a Laguerre polynomial.[14,19]

From Eq. (2.10) we see that the two particles form a bound state, which we will call a *magnetic exciton,* whose binding energy due to the direct Coulomb interaction between the two particles is given by Eq. (2.12). The exciton wave functions and eigenvalues are labeled by the continuous variables k_x and k_y. The vector $\vec{k}$ plays the role of the total momentum of the particles as can be seen by noting that the operator[20]

$$\vec{Q}=[\vec{P}-e(\vec{A}_1-\vec{A}_2)/c]+e\vec{B}\times\Delta\vec{r}/c\,, \tag{2.13}$$

which is equal to $\vec{P}-eB\,\Delta y\,\hat{x}/c$ in the Landau gauge, commutes with the two-particle Hamiltonian (2.1) and its components commute with each other. In addition, one has classically

$$\partial\mathscr{H}/\partial\vec{Q}=(1/2m^*)[\vec{P}-e(\vec{A}_1-\vec{A}_2)/c]=\vec{V}\,,$$

where $\vec{V}$ is the velocity of the center of mass. ($\vec{Q}$ is also the generator of infinitesimal translations to within a gauge transformation.) We will refer to $\vec{k}$ as the exciton momentum, noting that

$$\vec{Q}\psi^{mn}_{\vec{k}}=\vec{k}\psi^{mn}_{\vec{k}}\,. \tag{2.14}$$

One can define a conserved momentum operator $\vec{Q}_N$ for any system of N charges $\{q_i\}$ which is charge neutral, i.e., $\sum_i q_i=0$. The operator

$$\vec{Q}_N=\sum_i\left[\vec{p}_i-\frac{1}{c}q_i\vec{A}_i\right]-\frac{1}{c}\sum_i q_i\vec{b}\times\vec{r}_i\,, \tag{2.15}$$

plays the role of the total momentum of the particles, in a magnetic field $\vec{B}$, and is conserved by the Hamiltonian in

the same way as discussed for the two-particle system.

From Eq. (2.10) we see that, other than a phase factor which is gauge dependent, the exciton wave function is the direct product of a plane wave in the center-of-mass coordinates $\vec{R}$ and a function of the relative coordinates $g_{mn}(\Delta\vec{r}-l_0^2\vec{k}\times\hat{z})$, whose magnitude is spherically symmetric about the point $\Delta\vec{r}=l_0^2\vec{k}\times\hat{z}$. Therefore one can define a dipole moment of the exciton,

$$e\langle\psi_{\vec{k}}^{mn}|\Delta\vec{r}|\psi_{\vec{k}}^{mn}\rangle=el_0^2\vec{k}\times\hat{z}\ , \tag{2.16}$$

which is perpendicular to $\vec{k}$ and proportional to k, independent of m and n. This is what one would expect from the classical picture, where one finds that two particles of opposite charge in a magnetic field move parallel to one another with a constant linear velocity perpendicular to their separation (in contrast to the case of two *electrons* in a magnetic field, which orbit one another). The exciton momentum increases with increasing separation between the particles, although the velocity decreases.

For $u(r)=e^2/\epsilon r$, the asymptotic form of the exciton binding energy is

$$\Delta E_{\vec{k}}^{mn}=-\frac{e^2}{\epsilon k l_0^2}\quad\text{for } kl_0\gg 1\ , \tag{2.17}$$

as one would expect for point particles separated by a distance kl_0^2. It turns out that except for the case $m=n=0$, the dispersion $\Delta E_{\vec{k}}^{mn}$ is nonmonotonic at small k, because of the internal structure of the exciton wave function.

III. CALCULATION OF RESPONSE FUNCTIONS

In this section we calculate the electron charge density and spin-density response functions for a two-dimensional electron gas, at zero temperature, in a strong perpendicular magnetic field B, described by the Hamiltonian

$$\mathcal{H}=\frac{1}{2m^*}\sum_j(\vec{p}_j-e\vec{A}_j/c)^2+\sum_{\substack{i,j\\ i<j}}u(\vec{r}_i-\vec{r}_j)+|g\mu_B B|\sum_j S_{zj}\ . \tag{3.1}$$

where $\vec{A}_i=Bx_i\hat{y}$. We consider the case where the Fermi level lies between two Landau levels. The charge-density response function χ_ρ is related to the density-density correlation function by

$$\chi_\rho(\vec{k},\omega)=-i\int_0^\infty dt\, e^{i\omega t}\langle[\rho(\vec{k},t),\rho(-\vec{k},t)]\rangle\ , \tag{3.2}$$

where the density operator is

$$\rho(\vec{k},t)=e^{i\mathcal{H}t}\sum_j e^{i\vec{k}\cdot\vec{r}_j}e^{-i\mathcal{H}t}\ .$$

The spin-density response functions are

$$\chi_{\sigma_z}(\vec{k},\omega)=-i\int_0^\infty dt\, e^{i\omega t}\langle[\sigma_z(\vec{k},t),\sigma_z(-\vec{k},t)]\rangle\ , \tag{3.3}$$

$$\chi_{\sigma_\pm}(\vec{k},\omega)=-i\int_0^\infty dt\, e^{i\omega t}\langle[\sigma_\mp(\vec{k},t),\sigma_\pm(-\vec{k},t)]\rangle\ , \tag{3.4}$$

where the spin-density operators are

$$\vec{\sigma}(\vec{k},t)=2e^{i\mathcal{H}t}\sum_j e^{i\vec{k}\cdot\vec{r}_j}\vec{S}_j e^{-i\mathcal{H}t}\ ,$$

$\sigma_\pm\equiv(\sigma_x\pm i\sigma_y)/\sqrt{2}$ and $\vec{S}_j$ is the spin angular momentum operator for the jth electron.

These response functions have poles at the frequencies $\omega(k)$ corresponding to the charge neutral excitations of the system. [The imaginary part of $\omega(k)$ is the decay rate of the excitation.] The poles of χ_ρ are the magnetoplasma modes of the system, which may also be defined through the dielectric function

$$\epsilon(k,\omega)\equiv[1-\tilde{u}(k)\chi(k,\omega)]^{-1}\ ,$$

where $\tilde{u}(k)=2\pi e^2/\epsilon k$ is the two-dimensional Fourier transform of the Coulomb potential. The magnetoplasma frequencies $\mathrm{Re}[\omega(k)]$ satisfy the equation $\mathrm{Re}[\epsilon(k,\omega)]=0$.

A. Diagrammatic formalism

The charge density and spin density response function can be written in terms of the single-particle Green's function and the vertex parts Γ_ρ and Γ_σ as

$$\chi_A(\vec{k},\omega)=\sum_{\alpha,\beta}\int\frac{d\vec{k}'}{(2\pi)^2}\int dq_1\int dq_2\langle q_1\alpha|e^{i\vec{k}\cdot\vec{r}}\theta_a^\dagger|q_2\beta\rangle\int\frac{d\omega'}{2\pi}G_\alpha(\omega-\omega')G_\beta(\omega')\Gamma_{A\alpha\beta}(q_1,q_2;\vec{k}',\omega)\ , \tag{3.5}$$

where $A=\rho$, σ_z, or $\sigma_\pm$; and $\theta_\rho=1$, $\theta_{\sigma_z}=2S_z$, and $\theta_{\sigma_\pm}=2S_\pm$. The labels α and β represent both the Landau level index n and the spin index $s=\pm\frac{1}{2}$, e.g., $\alpha=(n_\alpha,s_\alpha)$, so that $\langle\vec{r}|q,\alpha\rangle$ is a single-particle eigenstate, given by Eq. (2.2), multiplied by a spin state ($\uparrow$ or $\downarrow$), and $G_\alpha(\omega)$ is the electron Green's function which is independent of q.

We assume the Coulomb energy $e^2/\epsilon l_0$ is smaller than the cyclotron energy ω_c, and we calculate the excitation spectra to lowest order in $(e^2/\epsilon l_0)/\omega_c$. This means that we keep only the terms in χ_A which correspond to a single exciton present at all times and neglect terms with two or more excitons present. The diagrammatic representation of this approximation, which we refer to as the strong-field approximation, is given in Fig. 1. Examples of some omitted diagrams are given in Fig. 2.

The single-particle Green's function is

$$G_\alpha(\omega)=\frac{1}{\omega-(n_\alpha+\frac{1}{2})\omega_c-|g\mu_B B|s_\alpha-\Sigma_\alpha+i\epsilon_\alpha}\ , \tag{3.6}$$

where $\epsilon_\alpha=0^+$ for $n_\alpha<(\nu s_\alpha)$, $\epsilon_\alpha=0^-$ for $n_\alpha\geq\nu(s_\alpha)$, and $\nu(s_\alpha)$ is the lowest unoccupied Landau level with spin s_α [i.e., $\nu(\frac{1}{2})=\nu_\uparrow$, $\nu(-\frac{1}{2})=\nu_\downarrow$]. In the strong-field approximation, the self-energy is

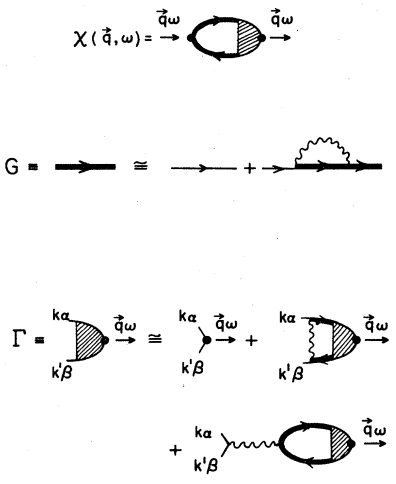

FIG. 1. Strong-field approximation for the response functions $\chi_\rho(\vec{q},\omega)$ and $\chi_\sigma(\vec{q},\omega)$ is shown. $\Gamma_{\alpha\beta}(k,k';\vec{q},\omega)$ is the vertex part and $G_\alpha(\omega)$ (thick line) is the single-particle Green's function. The thin lines with arrows represent the noninteracting single-particle Green's function $G^0_\alpha(\omega)$ and the wiggly lines represent the bare (unscreened) electron-electron interaction. As usual, the direct (Hartree) self-energy is cancelled by the uniform positive background.

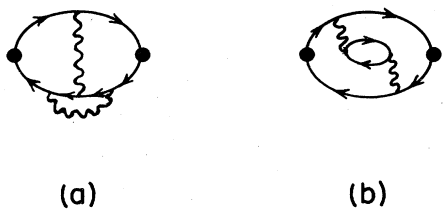

FIG. 2. Shown are two examples of diagrams which are not included in the strong-field approximation for the response functions. For any choice of Landau-level indices on the propagators in (a), there is always a propagator off the energy shell. (We are only considering integer filling factors.) Therefore this diagram will contribute to the exciton spectrum only in order $(e^2/\epsilon l_0)^2/\omega_c$ and not order $e^2/\epsilon l_0$. The same is true of (b) for $\omega<2\omega_c$. However, for $\omega\geq 2\omega_c$, one can choose the Landau-level indices such that all propagators are on the energy shell. Such a choice, corresponds to the decay of an exciton (with energy near $2\omega_c$, for example) into two excitons (each with energy near ω_c). In this case, (b) will contribute to order $e^2/\epsilon l_0$ in the exciton energy shift and decay rate. Since we neglect such diagrams, our calculation is exact, in order $e^2/\epsilon l_0$ only for $\omega<2\omega_c$ (i.e., $m\leq 1$).

$$\Sigma_\alpha(q,\omega)=-i\lim_{\delta\to 0^+}\sum_\beta\int dq_1\int dq_2\int\frac{d\omega'}{2\pi}V_{\alpha\beta\alpha\beta}(q,q_2,q_1,q_2)G_\beta(\omega-\omega')e^{i(\omega-\omega')\delta}\,, \tag{3.7}$$

where

$$V_{\alpha\beta\lambda\mu}(q_1,q_2,q_3,q_4)=\delta_{s_\alpha,s_\mu}\delta_{s_\beta,s_\lambda}\int d\vec{r}_1\int d\vec{r}_2 u(\vec{r}_1-\vec{r}_2)\langle q_1\alpha\,|\,\vec{r}_1\rangle\langle q_2\beta\,|\,\vec{r}_2\rangle\langle\vec{r}_2\,|\,q_3\lambda\rangle\langle\vec{r}_1\,|\,q_4\mu\rangle\,, \tag{3.8a}$$

$$\equiv\delta_{s_\alpha,s_\mu}\delta_{s_\beta,s_\lambda}V_{n_\alpha n_\beta n_\lambda n_\mu}(q_1,q_2,q_3,q_4)\,. \tag{3.8b}$$

We shall see that the self-energy is real and independent of q as well as ω. The vertex part satisfies the following integral equation,

$$\Gamma_{A\alpha\beta}(q_1,q_2;\vec{k}',\omega)=\langle q_2\beta\,|\,e^{-i\vec{k}'\cdot\vec{r}}\theta_A\,|\,q_1\alpha\rangle-i\sum_{\lambda,\mu}\int dq_3\int dq_4 V_{\alpha\lambda\beta\mu}(q_1,q_4,q_2,q_3)$$
$$\times\int\frac{d\omega'}{2\pi}G_\lambda(\omega-\omega')G_\mu(\omega')\Gamma_{A\lambda\mu}(q_4,q_3;\vec{k}',\omega)$$
$$+i\sum_{\lambda,\mu}\int dq_3\int dq_4 V_{\alpha\lambda\mu\beta}(q_1,q_4,q_3,q_2)\int\frac{d\omega'}{2\pi}G_\lambda(\omega-\omega')G_\mu(\omega')\Gamma_{A\lambda\mu}(q_4,q_3;\vec{k}',\omega)\,. \tag{3.9}$$

The above equations will be solved in Sec. III D, but first we consider separately the two contributions to the energy shift $\Delta E_m(k)$ which arise from exchange terms: the exchange energy which is calculated in the RPA and the constant exchange self-energy difference, as explained in the Introduction.

B. Random-phase approximation

Before solving for the response functions in the strong-field approximation, we review the RPA,[16] which is one contribution in the strong-field limit. In the RPA only the bubble diagrams are kept. These diagrams correspond to the electron and hole annihilating at one point and an electron-hole pair exciting simultaneously at some other point. In this way, the exciton moves within the plane. When the initial state is spin symmetric, the bubble diagrams contribute only to the charge-density response function χ_ρ, and not to the spin-density response functions. (An electron-hole pair with parallel spins cannot recombine through the Coulomb potential.)

The RPA density response function is

$$\chi_{\rm RPA}(k,\omega)=\frac{\chi_0(k,\omega)}{1+\tilde{u}(k)\chi_0(k,\omega)}\,, \tag{3.10}$$

where χ_0 is the response function in the *absence* of Coulomb interactions,

$$\chi_0(k,\omega)=\sum_{\alpha,\beta}\int\frac{d\vec{k}'}{(2\pi)^2}\int dq_1\int dq_2\langle q_1\alpha|e^{i\vec{k}\cdot\vec{r}}|q_2\beta\rangle\langle q_2\beta|e^{-i\vec{k}'\cdot\vec{r}}|q_1\alpha\rangle\int\frac{d\omega'}{2\pi}G^0_\alpha(\omega-\omega')G^0_\beta(\omega')\,, \tag{3.11}$$

and

$$[G^0_\alpha(\omega)]^{-1}=\omega-(n_\alpha+\tfrac{1}{2})\omega_c-|g\mu_B B|s_\alpha+i\epsilon_\alpha\,.$$

Evaluating χ_0, one finds

$$\chi_0(k,\omega)=\sum_{s=\pm1/2}\sum_{m=1}^{\infty}{\sum_n}'\frac{n!}{2^m(n+m)!}\frac{(kl_0)^{2m}}{2\pi l_0^2}\frac{e^{-k^2l_0^2/2}[L_n^m(k^2l_0^2/2)]^2}{(\omega-m\omega_c+i\epsilon)}+(\omega\to-\omega)\,, \tag{3.12}$$

$$\equiv\sum_{s=\pm1/2}\sum_m{\sum_n}'V_{mn}(k)\left[\frac{1}{\omega-m\omega_c+i\epsilon}-\frac{1}{\omega+m\omega_c+i\epsilon}\right], \tag{3.13}$$

where the sum $\sum'$ is over the range $\nu(s)-m\le n\le\nu(s)-1$.

The RPA response function has poles at the solutions to

$$1+\tilde{u}(k)\sum_{s=\pm1/2}\sum_m{\sum_n}'V_{mn}(k)\left[\frac{1}{\omega-m\omega_c+i\epsilon}-\frac{1}{\omega+m\omega_c+i\epsilon}\right]=0\,. \tag{3.14}$$

For $\omega>0$ there is a solution near each harmonic $m\omega_c$ for $m\ge1$. In the strong-field limit [i.e., neglecting terms of order $(e^2/\epsilon l_0)^2/\omega_c$] these solutions are

$$E_m^{\rm RPA}(k)=m\omega_c+\tilde{u}(k)\sum_{s=\pm1/2}{\sum_n}'V_{mn}(k)\,. \tag{3.15}$$

The energy $\Delta E_m^{\rm RPA}\equiv E_m^{\rm RPA}(k)-m\omega_c$ is always positive [for $\tilde{u}(k)>0$] since it corresponds to an exchange interaction between electron and holes. Using Eqs. (2.10) and (2.11), we see the the matrix element is $V_{mn}(k)=2\pi|\psi^{mn}_{\vec{k}}(\vec{R},\Delta\vec{r}=\vec{0})|^2$, which makes it clear that this interaction is a point interaction; the electron and hole can annihilate only if their wave functions overlap. Therefore $\Delta E_m^{\rm RPA}(k)$ goes to zero rapidly for large k. The asymptotic form for small k is

$$\Delta E_m^{\rm RPA}(k)=\frac{e^2}{\epsilon l_0}\left[\frac{(n+m)!}{2^m(m!)^2n!}(kl_0)^{2m-1}+O((kl_0)^{2m+1})\right], \tag{3.16}$$

which is linear in k for $m=1$ and $kl_0\ll1$.

C. Exchange self-energy

There is a self-energy contribution to the exciton energy spectrum due to the difference of the exchange self-energy of an electron in the excited Landau level $n_\beta=n_\alpha+m$ and the self-energy in the Landau level n_α from which the electron is removed. From Eq. (3.7), this energy difference is

$$E^{\rm exch}_{\beta\alpha}(q,\omega)\equiv\Sigma_\beta-\Sigma_\alpha$$

$$=-i\lim_{\delta\to0^+}\sum_\lambda\int dq_1\int dq_2\int\frac{d\omega'}{2\pi}G_\lambda(\omega-\omega')e^{i(\omega-\omega')\delta}[V_{\beta\lambda\beta\lambda}(q,q_2,q_1,q_2)-V_{\alpha\lambda\alpha\lambda}(q,q_2,q_1,q_2)] \tag{3.17}$$

$$=\int dq_1\int dq_2\left[\sum_{l<\nu(s_\beta)}V_{n_\beta l n_\beta l}(q,q_2,q_1,q_2)-\sum_{l<\nu(s_\alpha)}V_{n_\alpha l n_\alpha l}(q,q_2,q_1,q_2)\right], \tag{3.18}$$

where $\alpha=(n_\alpha,s_\alpha)$.

The matrix elements V are defined in Eq. (2.7) and evaluated in the Appendix. The exchange energy can be written as [see Eq. (A12)]

$$E^{\rm exch}_{\beta\alpha}=\int\frac{d\vec{r}}{2\pi l_0^2}u(r)e^{-r^2/2l_0^2}\times\left[L^1_{\nu(s_\alpha)-1}\left[\frac{r^2}{2l_0^2}\right]L^0_{n_\beta}\left[\frac{r^2}{2l_0^2}\right]-L^1_{\nu(s_\alpha)-1}\left[\frac{r^2}{2l_0^2}\right]L^0_{n_\alpha}\left[\frac{r^2}{2l_0^2}\right]\right], \tag{3.19}$$

independent of q and ω. This energy is positive for $u(r)>0$; there is a cost in exchange energy to excite an electron to a higher Landau level.

D. Inclusion of ladder diagrams

In this section we solve for the response functions in the strong-field approximation, which is defined by Eqs. (3.5)—(3.9) and is shown diagrammatically in Fig. 1. That is, we now include the direct interaction between the electron and hole, i.e., the ladder diagrams, as well as the exchange terms considered in Secs. III B and III C. If one ignores possible transitions of the electron and hole be-

tween different Landau levels, then summing the ladder diagrams is equivalent to the two-particle problem considered in Sec. II. In certain cases—for example, when only the lowest Landau level is occupied or when the two spin states of each Landau level are equally occupied and $m=1$—there are no possible transitions of the electron and hole between different Landau levels in the strong-field limit. We shall see that, in these cases, the total exciton energy is just the sum of the three contributions which we have already calculated—the particle-hole binding energy plus the RPA energy (which vanishes for a spin triplet exciton) plus the difference in exchange self-energy. In general, these different terms are matrices which we need to diagonalize to obtain the exciton energy spectrum. We now treat the general case.

The integral equation for the vertex part $\Gamma_{A\alpha\beta}(q_1,q_2;\vec{k},\omega)$, Eq. (3.9), is diagonalized by the same transformation used in the two-particle problem of Sec. II. That is, we change to the variables $q\equiv(q_1+q_2)/2$ and $\Delta q\equiv q_1-q_2$ and then take a partial Fourier transformation of Γ with respect to q, which we define as

$$\tilde{\Gamma}_{A\alpha\beta}(p,\Delta q;\vec{k},\omega)=\int dq e^{ipql_0^2}\Gamma_{A\alpha\beta}(q,\Delta q;\vec{k},\omega) .$$

One finds, from Eq. (3.9), that

$$\tilde{\Gamma}_{A\alpha\beta}(p,\Delta q;\vec{k},\omega)=\delta(p-k_x)\delta(\Delta q+k_y)\Gamma'_{A\alpha\beta}(\vec{k},\omega) ,$$

where Γ' satisfies the matrix equation,

$$\Gamma'_{A\alpha\beta}(\vec{k},\omega)=M^*_{A\alpha\beta}(\vec{k})+\sum_{\lambda,\mu}[\tilde{V}^{(1)}_{\alpha\lambda\beta\mu}(\vec{k})-\tilde{V}^{(2)}_{\alpha\lambda\mu\beta}(\vec{k})]\times D_{\mu\lambda}(\omega)\Gamma'_{A\mu\lambda}(\vec{k},\omega) . \quad (3.20)$$

The matrix elements $\tilde{V}^{(i)}$ are defined in the Appendix; the matrix element M_A is defined as

$$M_{A\alpha\beta}(\vec{k})\delta(p-k_x)\delta(\Delta q+k_y)=\int dq e^{ipql_0^2}\langle q_1\alpha|e^{i\vec{k}\cdot\vec{r}}\theta_A|q_2\beta\rangle ,$$

so that

$$M_{A\alpha\beta}(\vec{k})=\left[\frac{2^{n_\beta}n_\beta!}{2^{n_\alpha}n_\alpha!}\right]^{1/2}e^{-k^2l_0^2/4}[l_0(k_y-ik_x)]^{n_\alpha-n_\beta}L_{n_\beta}^{n_\alpha-n_\beta}\left[\frac{k^2l_0^2}{2}\right]\langle s_\beta|\theta_A|s_\alpha\rangle , \quad (3.21)$$

and $D_{\alpha\beta}$ is the two-particle propagator,

$$D_{\alpha\beta}(\omega)=\int\frac{d\omega'}{2\pi}G_\alpha(\omega+\omega')G_\beta(\omega') \quad (3.22a)$$

$$=\left[\frac{f_\alpha(1-f_\beta)}{\omega-(n_\beta-n_\alpha)\omega_c-|g\mu_BB|(s_\beta-s_\alpha)-\Sigma_\beta+\Sigma_\alpha+i\epsilon}-\frac{f_\beta(1-f_\alpha)}{\omega-(n_\beta-n_\alpha)\omega_c-|g\mu_BB|(s_\beta-s_\alpha)-\Sigma_\beta+\Sigma_\alpha-i\epsilon}\right] . \quad (3.22b)$$

The response functions from Eq. (3.5) are then given by

$$\chi_A(\vec{k},\omega)=\sum_{\alpha,\beta}M_{A\alpha\beta}(\vec{k})D_{\alpha\beta}(\omega)\Gamma'_{A\alpha\beta}(\vec{k},\omega) . \quad (3.23)$$

If we define

$$\Pi_{A\alpha\beta}(\vec{k},\omega)\equiv D_{\alpha\beta}(\omega)\Gamma'_{A\alpha\beta}(\vec{k},\omega) ,$$

then the response functions can be written as

$$\chi_A(\vec{k},\omega)=\sum_{\alpha,\beta}M_{A\alpha\beta}(\vec{k})\Pi_{A\alpha\beta}(\vec{k},\omega) , \quad (3.24)$$

where Π_A satisfies the matrix equation

$$\sum_{\lambda,\mu}\{\delta_{\alpha,\lambda}\delta_{\beta,\mu}[D(\omega)]^{-1}_{\alpha\beta}-\tilde{V}^{(1)}_{\alpha\mu\beta\lambda}(\vec{k})+\tilde{V}^{(2)}_{\alpha\mu\lambda\beta}(\vec{k})\}\times\Pi_{A\lambda\mu}(\vec{k},\omega)=M^*_{A\alpha\beta}(\vec{k}) . \quad (3.25)$$

Equation (3.25) must be satisfied for all (α,β) such that one member of the pair is occupied and the other unoccupied in the ground state; the polarization $\Pi_{A\alpha\beta}$ is zero otherwise. The poles of the response functions, $\omega(k)$, are the solutions to

$$\sum_{\lambda,\mu}\{\delta_{\alpha,\lambda}\delta_{\beta,\mu}[D(\omega)]^{-1}_{\alpha\beta}-\tilde{V}^{(1)}_{\alpha\mu\beta\lambda}(\vec{k})+\tilde{V}^{(2)}_{\alpha\mu\lambda\beta}(\vec{k})\}B_{\lambda\mu}=0 . \quad (3.26)$$

The spin structure of the eigenvector $B_{\lambda\mu}$ determines which response function the pole appears in.

IV. DISPERSION CURVES

The dispersion curves, which correspond to poles in the response functions, can be labeled by the index m and an additional label μ, when necessary, as explained previously. In addition, these curves depend on which Landau levels are initially occupied. In this section we give explicit expressions for the dispersion curves in the cases of greatest interest—small filling factors and small m.

To lowest order in $e^2/\epsilon l_0\omega_c$, we may restrict the pairs of indices (α,β) and (λ,μ) in Eq. (3.26) to values such that $|n_\alpha-n_\beta|=|n_\lambda-n_\mu|=m$, and such that one member of each pair is occupied and the other is empty in the initial state. Thus Eq. (3.26) becomes a finite-dimensional matrix equation. The dimensionality of the matrices are further reduced when we take into account the spin symmetry of the system and the conservation of spin in the matrix elements $\tilde{V}^{(i)}$.

In the *simplest* case, the matrix can be reduced to a single element and the dispersion curve obtained directly

from Eq. (3.26). For example, if there is just a single mode contributing to $\chi_\rho(k,\omega)$ in the vicinity of $m\omega_c$, its energy will be given by

$$E_m(\vec{k})=m\omega_c+\gamma_n\tilde{V}^{(2)}_{n'nn'n}(\vec{k})-\tilde{V}^{(1)}_{n'nnn'}(\vec{k})+E^{\rm exch}_{n'n}\ , \tag{4.1}$$

or, explicitly,

$$\begin{aligned}E_m(\vec{k})=m\omega_c&+\frac{n!}{2^m(n+m)!}\gamma_n\frac{\tilde{u}(k)}{2\pi l_0^2}(kl_0)^{2m}\left[L_n^m\left[\frac{k^2l_0^2}{2}\right]\right]^2e^{-k^2l_0^2/2}\\&-\frac{n!}{2^m(n+m)!}\int\frac{d\vec{r}}{2\pi l_0^2}u(\vec{r}-l_0^2\vec{k}\times\hat{z})\left[\frac{r}{l_0}\right]^{2m}\left[L_n^m\left[\frac{r^2}{2l_0^2}\right]\right]^2e^{-r^2/2l_0^2}\\&+\int\frac{d\vec{r}}{2\pi l_0^2}u(\vec{r})L_n^1\left[\frac{r^2}{2l_0^2}\right]\left[L_n^0\left[\frac{r^2}{2l_0^2}\right]-L^0_{n+m}\left[\frac{r^2}{2l_0^2}\right]\right]e^{-r^2/2l_0^2},\end{aligned} \tag{4.2}$$

where $n'\equiv n+m$, n is the highest occupied Landau level, $\gamma_n=2$ if both spin states of the nth Landau level are occupied, and $\gamma_n=1$ otherwise. In fact, the above expression applies for two cases: (i) $m=1$, $\gamma_n=2$, with arbitrary n; and (ii) $n=1$, $\gamma_n=1$ or 2, and arbitrary m. In these cases, there are no possible transitions of the excited electron and hole between different Landau levels in the strong-field limit, and therefore the total exciton energy is just the sum of the three contributions calculated in Secs. II, III B, and III C, as can be seen by comparing Eq. (4.2) with the sum of Eqs. (2.12), (3.15), and (3.19). (For $m\geq 2$, however, we have made an uncontrolled approximation in neglecting multiexciton states.)

There are two checks that we can perform on Eq. (4.2). It follows from Kohn's theorem that the $m=1$ exciton energy $E_1(\vec{k})$ must approach ω_c as $k\to 0$. Since $\tilde{V}^{(2)}_{n+1,n,n+1,n}(0)=0$ and $\tilde{V}^{(1)}_{n+1,n,n+1,n}(0)=E^{\rm exch}_{n+1,n}$, Kohn's theorem is clearly satisfied by Eq. (4.2). The second check that we can perform on Eq. (4.2) is to consider a δ-function potential between electrons, with a spin-polarized initial state, i.e., $u(\vec{r})=c\delta(\vec{r})$ and $\gamma_n=1$. Since electrons in the same spin state are never in the same place, the exciton energy in this case must be the same as that in the absence of electron-electron interactions, $\Delta E_m(\vec{k})=0$. For $u(\vec{r})=c\delta(\vec{r})$, one has $E^{\rm exch}_{n'n}=0$ and $\tilde{V}^{(2)}_{n'nn'n}(\vec{k})=\tilde{V}^{(1)}_{n'nnn'}(\vec{k})$, so that Eq. (4.2) gives $E_m(\vec{k})=m\omega_c$ as required. We now consider some specific cases of small filling factors and small m.

A. Excitations near ω_c ($m=1$)

We first consider the excitations with $m=1$. There are three types of initial states to consider: (i) a completely spin-polarized sample (here, $\nu_\downarrow=1$ is the only physically realizable occupation for GaAs since the Zeeman energy is small compared to the cyclotron energy); (ii) equal occupation of the two spin states; and (iii) both spins present, but in unequal occupations (only $\nu_\downarrow=\nu_\uparrow+1$ is physically realizable in GaAs).

The first two cases (i) and (ii) are included in Eqs. (4.1) and (4.2). For a completely spin-polarized sample with $\nu_\downarrow=1$ and $\nu_\uparrow=0$, the dispersion curve, from Eq. (4.1), is $E_1^\rho(\vec{k})=\omega_c+\Delta E_1^\rho(\vec{k})$, with

$$\Delta E_1^\rho(\vec{k})=\tilde{V}^{(2)}_{1010}(\vec{k})-\tilde{V}^{(1)}_{1001}(\vec{k})+E^{\rm exch}_{10}\ . \tag{4.3}$$

Performing the integrals of Eq. (4.2), we find, in agreement with Ref. 18,

$$\Delta E_1^\rho(\vec{k})=\frac{e^2}{\epsilon l_0}\frac{1}{2}\left[\frac{\pi}{2}\right]^{1/2}\left\{1-e^{-k^2l_0^2/4}\left[(1+k^2l_0^2/2)I_0\left[\frac{k^2l_0^2}{4}\right]-\frac{k^2l_0^2}{2}I_1\left[\frac{k^2l_0^2}{4}\right]\right]+\left[\frac{2}{\pi}\right]^{1/2}kl_0e^{-k^2l_0^2/2}\right\}. \tag{4.4}$$

where I_n is a modified Bessel function of the first kind. This spectrum corresponds to a pole in the density response function χ_ρ. There is also a pole in the spin-density response function χ_{σ_+} at $E_1^{\sigma_+}(\vec{k})=\omega_c+|g\mu_BB|+\Delta E_1^{\sigma_+}(\vec{k})$, with

$$\Delta E_1^{\sigma_+}(\vec{k})=-\tilde{V}^{(1)}_{1001}(\vec{k})+E^{\rm exch}_{10} \tag{4.5a}$$

$$=\frac{e^2}{\epsilon l_0}\frac{1}{2}\left[\frac{\pi}{2}\right]^{1/2}\left\{1-e^{-k^2l_0^2/4}\left[(1+k^2l_0^2/2)I_0\left[\frac{k^2l_0^2}{4}\right]-\frac{k^2l_0^2}{2}I_1\left[\frac{k^2l_0^2}{4}\right]\right]\right\}. \tag{4.5b}$$

The asymptotic forms of these spectra are

$$\Delta E_1^\rho(k)=\begin{cases}\dfrac{e^2}{\epsilon l_0}\dfrac{kl_0}{2} & \text{for } kl_0\ll 1\ , \quad (4.6a)\\[2ex] \dfrac{e^2}{\epsilon l_0}\left[\dfrac{1}{2}\left[\dfrac{\pi}{2}\right]^{1/2}-\dfrac{1}{kl_0}\right] & \text{for } kl_0\gg 1\ , \quad (4.6b)\end{cases}$$

$$\Delta E_1^{\sigma_+}(k)=\begin{cases}-\dfrac{e^2}{\epsilon l_0}\left[\dfrac{\pi}{2}\right]^{1/2}\left[\dfrac{kl_0}{2}\right]^2 & \text{for } kl_0\ll 1\ , \quad (4.7a)\\[2ex] \dfrac{e^2}{\epsilon l_0}\left[\dfrac{1}{2}\left[\dfrac{\pi}{2}\right]^{1/2}-\dfrac{1}{kl_0}\right] & \text{for } kl_0\gg 1\ . \quad (4.7b)\end{cases}$$

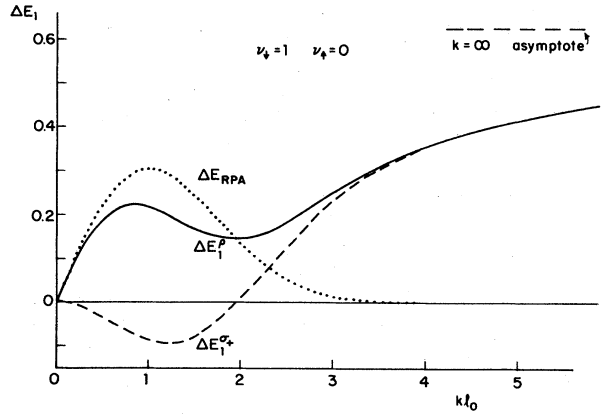

FIG. 3. Energy shifts for a spin-polarized sample with only the lowest Landau level filled, $\nu_\downarrow=1$ and $\nu_\uparrow=0$, and $m=1$ are shown. The energy scale is in units of $e^2/\epsilon l_0$. The solid curve denotes $\Delta E_1^\rho=E_1^\rho(k)-\omega_c$, where $\omega=E_1^\rho(k)$ is the pole in the density response function χ_ρ. The same pole appears in χ_{σ_z} also. The dashed curve denotes $\Delta E_1^{\sigma+}=E_1^{\sigma+}-\omega_c-|g\mu_B B|$, where $\omega=E_1^{\sigma+}$ is the pole in the spin-response function χ_{σ_+}. The RPA energy shift, $E_{\rm RPA}-\omega_c$, is denoted by the dotted curve. The $k=\infty$ asymptote for ΔE_1^ρ and $\Delta E_1^{\sigma+}$ is also indicated.

These dispersion curves are shown in Fig. 3. The RPA energy is also shown in this figure. The branch ΔE_1^ρ, which is the sum of $\Delta E_{\rm RPA}$ and $\Delta E_1^{\sigma+}$, has a maximum at $kl_0\approx 0.9$ and a minimum at $kl_0\approx 2$. The RPA contribution to the shift is more important for $kl_0\ll 1$, the RPA and ladder contributions are comparable for $kl_0\approx 1$, and the ladder diagrams are more important for $kl_0\gg 1$.

If the two spin states are initially equally occupied, the dispersion curve for $m=1$, which corresponds to a pole in χ_ρ, is given by Eq. (4.1), which now reads $E_1^s(\vec{k})=\omega_c+\Delta E_1^s(\vec{k})$, with

$$\Delta E_1^s(\vec{k})=2\tilde{V}^{(2)}_{n+1,n,n+1,n}(\vec{k})-\tilde{V}^{(1)}_{n+1,n,n,n+1}(\vec{k})+E^{\rm exch}_{n+1,n}\ . \tag{4.8}$$

This pole corresponds to a spin singlet exciton. There are also poles, corresponding to spin triplet excitons, in the three spin-density response functions. These occur at

$$\omega=\omega_c+\Delta E_1^t(k)+|g\mu_B B|\,\delta S_z$$

where $\delta S_z=0,\pm 1$, and

$$\Delta E_1^t(\vec{k})=-\tilde{V}^{(1)}_{n+1,n,n,n+1}(\vec{k})+E^{\rm exch}_{n+1,n}\ . \tag{4.9}$$

The curves ΔE_1^s and ΔE_1^+ are shown in Fig. 4 for $n=0$, 1, and 2. One can calculate analytic expressions for these curves, involving modified Bessel functions, polynomials, and exponentials, for all n. However, they become increasingly complex and we will not display the expressions here.

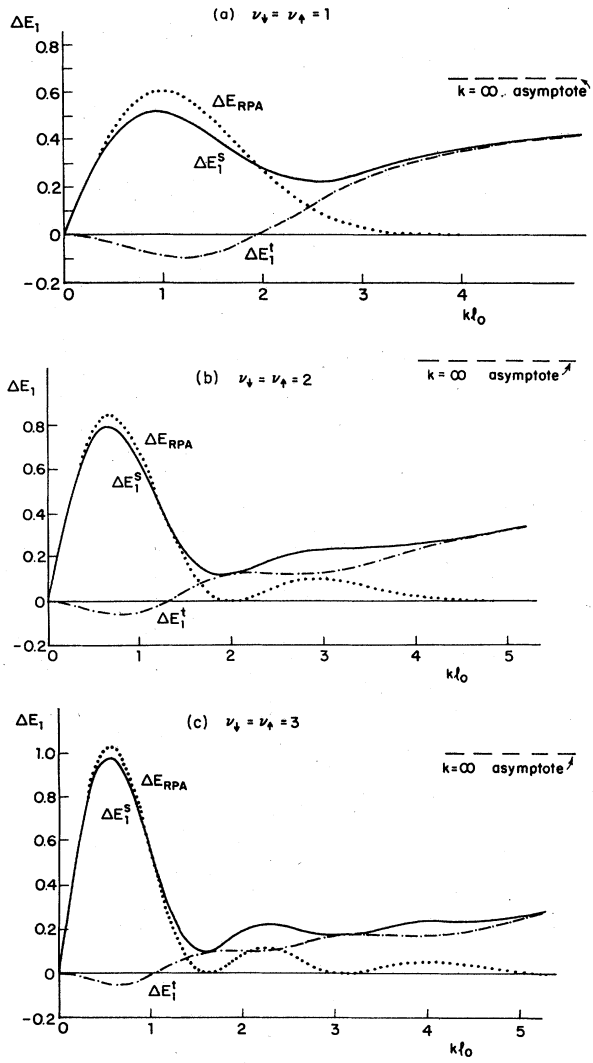

FIG. 4. Energy shifts are shown for $m=1$ and equal occupation of the spin states with initial filling factors (a) $\nu_\downarrow=\nu_\uparrow=1$, (b) $\nu_\downarrow=\nu_\uparrow=2$, and (c) $\nu_\downarrow=\nu_\uparrow=3$. The energy scales are in units of $e^2/\epsilon l_0$. The energy shifts of the singlet excitons, $\Delta E_1^s=E_1^s-\omega_c$, are denoted by solid curves and those of the triplet excitons, $\Delta E_1^t=E_1^t-\omega_c$, by dashed curves. There is a pole in the density response function χ_ρ at $\omega=E_1^s$ and in the spin-density response functions χ_{σ_z} at $\omega=E_1^t$ and $\chi_{\sigma_\pm}$ at $\omega=E_1^t\pm|g\mu_B B|$. The RPA energy shifts, $\Delta E_{\rm RPA}=E_{\rm RPA}-\omega_c$, are denoted by dotted curves. The $k=\infty$ asymptotes for ΔE_1^s and ΔE_1^t are also indicated.

For the third case, $m=1$ and $\nu_\downarrow=\nu_\uparrow+1$, Eq. (3.26) is a 2×2 matrix equation (to order $e^2/\epsilon l_0$) which we need to diagonalize. One finds two poles in χ_ρ at $\omega=\omega_c+\Delta E_1^\mu(\vec{k})$, with

$$\Delta E_1^\mu(\vec{k})=\frac{E_{n+1}(\vec{k})+E_n(\vec{k})}{2}\pm\left[\left[\frac{E_{n+1}(\vec{k})+E_n(\vec{k})}{2}\right]^2+\tilde{V}^{(2)}_{n+2,n+1,n+1,n}(\vec{k})\tilde{V}^{(2)}_{n+1,n,n+2,n+1}(\vec{k})\right]^{1/2}, \tag{4.10}$$

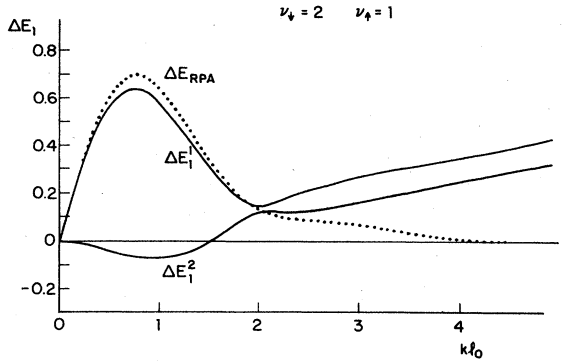

FIG. 5. Energy shifts are shown for $\nu_\downarrow=2$, $\nu_\uparrow=1$, and $m=1$. There are two poles in χ_ρ, at $\omega=E_1^1=\Delta E_1^1+\omega_c$ and $\omega=E_1^2=\Delta E_1^2+\omega_c$. There are also poles in χ_{σ_+} at $\omega=E_1^1(k)+|g\mu_B B|$ and $\omega=E_1^2(k)+|g\mu_B B|$. The RPA energy shift, $E_{\rm RPA}-\omega_c$, is denoted by the dotted curve. All energies are in units of $e^2/\epsilon l_0$.

where $\mu=1,2$, while $n=\nu_\downarrow-1$ and

$$E_n(\vec{k})\equiv\tilde{V}^{(2)}_{n+1,n,n+1,n}(\vec{k})-\tilde{V}^{(1)}_{n+1,n,n,n+1}(\vec{k})\ .$$

There are also poles in χ_{σ_+} at $\omega=\omega_c+\Delta E_1^\mu(\vec{k})+|g\mu_B B|$. The shifts ΔE_1^μ are shown in Fig. 5.

B. Spin waves ($m=0$)

The case $m=0$ only occurs when the two spin states are unequally occupied, and corresponds to spin waves. We consider the physically relevant case $\nu_\downarrow=\nu_\uparrow+1$. The dispersion relation is

$$E_0(\vec{k})=|g\mu_B B|-\Sigma_n-\tilde{V}^{(1)}_{nnnn}(\vec{k})\ , \tag{4.11}$$

where $n=\nu_\uparrow$. For $n=0$, we have, in agreement with Ref. 18,

$$E_0(\vec{k})-|g\mu_B B| = \frac{e^2}{\epsilon l_0}\left[\frac{\pi}{2}\right]^{1/2}\left[1-e^{-k^2l_0^2/4}I_0\left[\frac{k^2l_0^2}{4}\right]\right], \tag{4.12}$$

which is shown in Fig. 6. Note that the energy shift tends

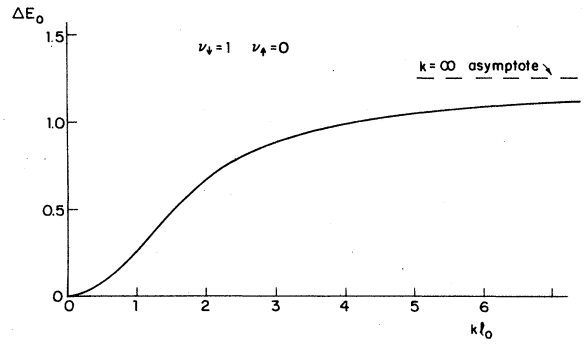

FIG. 6. Spin-wave spectrum ($m=0$) is shown for $\nu_\downarrow=1$ and $\nu_\uparrow=0$, with the Zeeman energy omitted. There is a pole in χ_{σ_+} at $\omega=\Delta E_0(k)+|g\mu_B B|$.

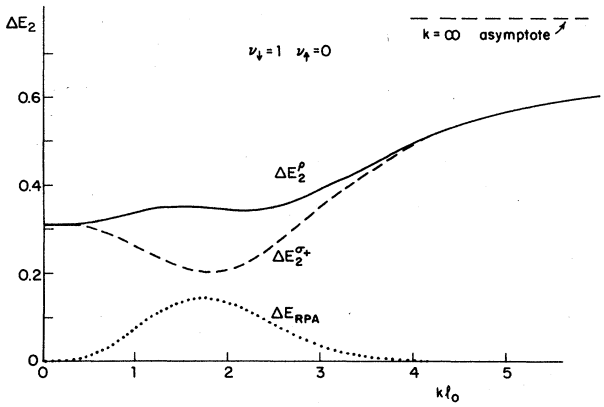

FIG. 7. Energy shifts are shown for the case where only one spin state of the lowest Landau level is occupied, $\nu_\downarrow=1$ and $\nu_\uparrow=0$, and $m=2$. These energy shifts were calculated, neglecting the possible decay of the exciton into two excitons. The solid curve denotes $\Delta E_2^\rho=E_2^\rho-2\omega_c$, which corresponds to a pole in χ_ρ and χ_{σ_z}. The dashed curve denotes $\Delta E_2^{\sigma+}=E_2^{\sigma+}-2\omega_c-|g\mu_B B|$, where there is a pole in χ_{σ_+} at $\omega=E_2^{\sigma+}$. The RPA energy shift, $E_{\rm RPA}-2\omega_c$, is denoted by the dotted curve.

to zero in the limit $k\to0$, and the spin-wave energy approaches the unshifted Zeeman energy, as required by Larmor's theorem.

C. Excitations near $m\omega_c$ for $m\geq2$

For the case $m\geq2$, we consider only the simplest example where $\nu_\downarrow=1$ and there are no spins with $s_z=\frac{1}{2}$. Then the dispersion relation, from Eq. (4.1), is

$$E_m^\rho(\vec{k})=m\omega_c+E_{m0}^{\rm exch}-\tilde{V}^{(1)}_{m00m}(\vec{k})+\tilde{V}^{(2)}_{m0m0}(\vec{k})\ , \tag{4.13}$$

corresponding to a pole in χ_ρ, and

$$E_m^{\sigma+}(\vec{k})=m\omega_c+|g\mu_B B|+E_{m0}^{\rm exch}-\tilde{V}^{(1)}_{m00m}(\vec{k})\ , \tag{4.14}$$

corresponding to a pole in χ_{σ_+}. These curves are shown in Fig. 7 for $m=2$.

V. PARTIALLY FILLED LANDAU LEVEL

Although the discussion above has been restricted to the case of an integral number of filled Landau levels, there are at least some qualitative extensions that can be made to the more general case, where the uppermost Landau level is only partially filled.

A. Quasiexcitons in a fractional quantized Hall state

A particularly interesting problem is the low-energy excitation spectrum when the initial state has a stable rational filling factor ν associated with the fractional quantized Hall effect.[3,21] The observed stable values of $\nu_\downarrow$ can be written in the form $\nu_\downarrow=l/p$, where l and p are integers, with p odd.[2,22–24] (We assume $\nu_\downarrow<1$, and $\nu_\uparrow=0$.) As noted by Laughlin, our understanding of the fractional quantized Hall effect requires that the elementary charged excitations be quasiparticles and quasiholes with fraction-

al charge $\pm qe$, where q^{-1} is equal to p, the denominator of the fraction $\nu_\downarrow$. We may therefore expect that the lowest-lying neutral excitations may be described as *quasiexcitons*—a bound state of a quasiparticle and quasihole.

A quantitative analysis of the quasiexciton spectrum would require detailed knowledge of the microscopic wave functions for the quasiparticle and quasihole, which is not available. Certain qualitative features are evident from our earlier discussion, however, For these neutral excitations, the wave vector $\vec{k}$ is a good quantum number, and the lowest branch of the excitation spectrum should be characterized by a dispersion relation $E_0(\vec{k})$. For large values of kl_0, the quasiexciton consists of a quasiparticle and quasihole separated by a large distance, $|\Delta\vec{r}| = kl_0^2/q$. (Note that the effective magnetic length for a particle of charge qe is equal to $l_0 q^{-1/2}$.) Then for large values of k we may write

$$E_0(k)=\Delta-\frac{q^3e^2}{k\epsilon l_0^2}\ , \tag{5.1}$$

where Δ is the energy gap corresponding to the creation of one quasiparticle and one quasihole, infinitely far apart. The value of Δ has been estimated by Laughlin[22] to be $\approx 0.056e^2/\epsilon l_0$, for the case of the $\nu=\frac{1}{3}$ state, in the strong magnetic field limit. A molecular-dynamics evaluation of a specific trial wave function for the quasiparticle, carried out by Morf, gives a somewhat higher estimate, $\Delta\approx 0.1e^2/\epsilon l_0$.[25]

For values of $kl_0 \lesssim 1$, the interaction between the quasiparticle and quasihole is undoubtedly more complicated than the Coulomb interaction between two point charges of magnitude $|qe|$. In the limit $k\to 0$, we expect that $E_0(k)$ will approach a constant $E_0(0)>0$, with corrections of order k^2.

The quasiexciton should appear as a pole in the density response function $\chi_\rho(k,\omega)$ for $k\neq 0$. The weight of the pole must tend to zero faster than k^2, for $k\to 0$, however, in order for Kohn's theorem to be satisfied—at $k=0$ the dipole matrix element can only connect the ground state with an excitation of energy ω_c. It follows from this that the exchange interaction between the quasiparticle and quasihole will not give a contribution to $E_0(k)$ that is linear in k for small k, such as was found for the $m=1$ magnetoplasma mode, in the preceding sections. A sketch of a possible form of $E_0(k)$ for the quasiexciton, consistent with the above considerations, is given in Fig. 8.

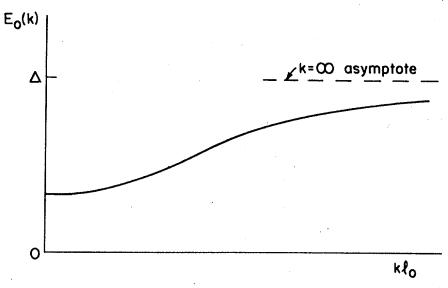

FIG. 8. Expected qualitative behavior of the quasiexciton spectrum for a partially filled level, when the ground state is a stable state associated with the fractional quantized Hall effect. Asymptotic energy Δ is the energy necessary to create a separated quasiparticle and quasihole.

B. $m=1$ magnetoplasma mode

We may also enquire about the behavior of the $m=1$ magnetoplasmon branch for the case of a partially filled Landau level, $\nu_\downarrow<1$. As a first approximation we may construct a dispersion curve qualitatively similar to that in Fig. 3, for a filled spin-polarized Landau level ($\nu_\downarrow=1$, $\nu_\uparrow=0$). The RPA energy is reduced by the factor $\nu_\downarrow$, in the partly filled case, while the attractive interaction between the electron and hole remains unchanged at large separations (large $\vec{k}$). As the background electron density is smaller than for a filled Landau level, the area occupied by a hole of charge $|e|$ must be larger by a factor $\approx\nu_\downarrow^{-1}$, so we might expect the attractive interaction $-e^2/(\epsilon kl_0^2)$ to be cut off at a constant value for $kl_0 \lesssim \text{const}\nu_\downarrow^{-1/2}$. At the same time, the positive exchange energy contribution $\Sigma_1-\Sigma_0$ will be reduced by a factor which might be of order $\nu_\downarrow^{1/2}$, if it scales inversely as the mean separation between electrons. Since Kohn's theorem still applies, the $m=1$ magnetoplasmon mode still has energy $E_1^\rho(k)\to\omega_c$, in the limit $k\to 0$. Thus, the net effect of the partial filling is to reduce the energy shift $\Delta E_1^\rho(k)$ relative to that shown in Fig. 3.

Partial filling of the Landau level will of course have a major effect on the damping of the magnetoplasma oscillations. At $T=0$, for sufficiently long wavelengths, the $m=1$ mode should still have very long lifetime, as it can only decay by conversion to a state containing a large number of $m=0$ quasiexcitons, in the strong filled limit. For larger values of k, however, the $m=1$ exciton state can decay more readily. For example, it should be possible to scatter the $m=1$ exciton to a small value of k, emitting a quasiexciton to conserve energy and wave vector.

Another possibility for a large-wave-vector $m=1$ exciton is that the hole in the partially filled Landau level will break up into several fractionally charged quasiholes, while the excited electron remains in the higher Landau level. This process is presumably forbidden near $k=0$, as the close proximity of the electron in that case should make it energetically unfavorable for the hole to break up.

At finite temperatures, there will be a variety of thermal excitations present in the partially filled Landau level—spin waves, quasiexcitons with small values of kl_0, and free quasiparticles and quasiholes, or large-wave-vector quasiexcitons.

All of these have energies small compared to the cyclotron energy, in the strong-field limit, and all of these should lead to increased scattering of the $m=1$ magnetoplasmon, reducing its lifetime and increasing the width of the $m=1$ peak in the response function $\chi_\rho(k,\omega)$.

We may also consider a situation where we have one or more completely filled Landau levels, in addition to a partially filled Landau level. For concreteness, suppose we have two filled Landau levels for spin $\downarrow$, while for spin $\uparrow$ we have the lowest level completely filled, and the second level partially filled, i.e., we have $\nu_\downarrow=2$, $\nu_\uparrow=1+x$, with $0<x<1$. If damping could be neglected, we might expect

to find two $m=1$ peaks in the response function $\chi_\rho(k,\omega)$, just as in the case $\nu_\downarrow=2$, $\nu_\uparrow=1$ illustrated in Fig. 5. As x increases to 1, the intensity of one of the branches should disappear, so that at $x=1$, we have only a single peak in $\chi_\rho(k,\omega)$, corresponding to the solid curve in Fig. 4(b).

For $T=0$, and x sufficiently small, there will be little damping, and there will be two well-defined magnetoplasmon branches, just as in Fig. 5. There may be considerable damping at finite temperatures, however, even at $k=0$, due to scattering by spin waves and other excitations.

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APPENDIX: MATRIX ELEMENTS

In this appendix we evaluate the relevant matrix elements V which are defined in Eq. (3.8). By writing the single-particle wave functions in center-of-mass and relative coordinates, these matrix elements can be written as

$$V_{n_1n_2n_3n_4}(q_1,q_2,q_3q_4)=2\pi\delta(q_1+q_2-q_3-q_4)V'_{n_1n_2n_3n_4}(q_1-q_3,q_2-q_3)\,, \tag{A1}$$

where

$$\begin{aligned}V'_{n_1n_2n_3n_4}(q,q')=&\frac{e^{-q'^2l_0^2/2}}{\pi(2^{n_1+n_2+n_3+n_4}n_1!n_2!n_3!n_4!)^{1/2}}\\&\times\int\frac{d\vec{\Delta r}}{2\pi l_0^2}u(\Delta r)e^{-iq'\Delta y}e^{-(\Delta x+l_0^2q)^2/2l_0^2}\\&\times\int dX\,e^{-2X^2/l_0^2}H_{n_1}\left[\frac{X+\frac{1}{2}\Delta x-l_0^2(q-q')/2}{l_0}\right]H_{n_2}\left[\frac{X-\frac{1}{2}\Delta x+l_0^2(q-q')/2}{l_0}\right]\\&\times H_{n_3}\left[\frac{X-\frac{1}{2}\Delta x+l_0^2(q+q')/2}{l_0}\right]H_{n_4}\left[\frac{X+\frac{1}{2}\Delta x-l_0^2(q+q')/2}{l_0}\right].\end{aligned} \tag{A2}$$

We will actually need the partial Fourier transforms of V' defined as follows:

$$\tilde{V}^{(1)}_{n_1n_2n_3n_4}(\vec{k})\equiv\int dq'e^{ik_xq'l_0^2}V'_{n_1n_2n_3n_4}(k_y,q')\,, \tag{A3}$$

$$\tilde{V}^{(2)}_{n_1n_2n_3n_4}(\vec{k})\equiv\int dq'e^{ik_xq'l_0^2}V'_{n_1n_2n_3n_4}(q',k_y)\,. \tag{A4}$$

Performing the q' and X integrals, one finds[19]

$$\begin{aligned}\tilde{V}^{(1)}_{n_1n_2n_3n_4}(\vec{k})=&\left[\frac{2^{n_2}2^{n_3}n_2!n_3!}{2^{n_1}2^{n_4}n_1!n_4!}\right]^{1/2}\int\frac{d\vec{\Delta}r}{2\pi l_0^2}u(\vec{\Delta}r-l_0^2\vec{k}\times\hat{z})e^{-\Delta r^2/2l_0^2}\left[\frac{\Delta x+i\Delta y}{l_0}\right]^{n_1-n_3}\left[\frac{\Delta x-i\Delta y}{l_0}\right]^{n_4-n_2}\\&\times L_{n_3}^{n_1-n_3}\left[\frac{\Delta r^2}{2l_0^2}\right]L_{n_2}^{n_4-n_2}\left[\frac{\Delta r^2}{2l_0^2}\right],\end{aligned} \tag{A5}$$

and

$$\begin{aligned}\tilde{V}^{(2)}_{n_1n_2n_3n_4}(\vec{k})=&\left[\frac{2^{n_2}2^{n_4}n_2!n_4!}{2^{n_1}2^{n_3}n_1!n_3!}\right]^{1/2}\frac{\tilde{u}(k)}{2\pi l_0^2}l_0^{n_1-n_4+n_3-n_2}(k_x+ik_y)^{n_1-n_4}(k_x-ik_y)^{n_3-n_2}\\&\times L_{n_4}^{n_1-n_4}\left[\frac{k^2l_0^2}{2}\right]L_{n_2}^{n_3-n_2}\left[\frac{k^2l_0^2}{2}\right]e^{-k^2l_0^2/2},\end{aligned} \tag{A6}$$

for $n_2 \leq n_4$ and $n_3 \leq n_1$. If $n_2 > n_4$, then n_2 and n_4 are interchanged in the above expressions; and if $n_3 > n_1$, then n_1 and n_3 are interchanged. For the case $n_1 = n_3$ and $n_2 = n_4$, $\tilde{V}^{(2)}$ is the matrix element which appears in the RPA energy [see Eq. (3.12)]:

$$\tilde{V}^{(2)}_{n'nn'n}(\vec{k}) = V_{mn}(\vec{k}) , \tag{A7}$$

where $m \equiv n' - n$.

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