Transverse magnetoresistance and thermopower in metals

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The transport equation is solved for a quantum-mechanical system in a transverse magnetic field. The solution gives the linear transport coefficients for electrostatic fields and temperature gradients for electrons in metals. In term of a small expansion parameter, the solution to lowest order confirms the existing semiclassical theory on electrical conductivity and thermal conductivity. We also found that the electron-phonon mass enhancement factor is present in the adiabatic thermopower. The next-higher-order approximation exhibits the anisotropic effect caused by the presence of magnetic fields and explains the experimentally observed linear magnetoresistance in simple metals under a high magnetic field.

I. INTRODUCTION

Historically, the linear response of a quantum system to external electric and thermal fields are usually calculated by using Kubo's formula.[1] However, because of the complexity in evaluating correlation functions, a complete quantum-mechanical treatment is still lacking when a magnetic field is present in the system. The existing semiclassical theory is unable to explain some experimental results. This paper develops a solution to this problem through the transport equation approach.

To provide some concrete examples, we mention the topics of magnetoresistance and magnetic thermopower. In general, simple metals can be classified into two categories, with or without open orbits on the electron Fermi surface. For a simple metal with no open orbits on the Fermi surface, semiclassical theory predicts that the magnetoresistance should saturate in the high magnetic field limit.[2] However, experimental measurements give a quite different picture. Even for simple metals such as K and Al, whose Fermi surfaces are believed to be well known, linear magnetoresistance has been observed without obvious saturation.[3-6] So far, no one has been able to unify experimental results in a more quantitative manner because the data from different experiments depend also on sample purity, annealing history, or even how the resistivity was measured.

Recently Mahan proposed that the increase in resistivity with field is caused by the anisotropy in the energy and scattering introduced by the field. This anisotropy, when introduced into the transport equation, caused symmetry breaking between the forward and backward scattering of the particles. This, in turn, introduces into the resistivity a dependence upon both particle lifetimes: the transport lifetime τ_{tr} and the time between scatterings τ. Mahan's calculation only applied to the longitudinal magnetoresistance. The present calculation extends the theory to the transverse magnetoresistance, which is mathematically more complicated.

As for the magnetic thermopower, it has long been suggested that the discrepancy between experiments and semiclassical calculation[7] can be eliminated by introducing the correction factor of electron-phonon mass enhancement. This was first shown by Opsal *et al.*[8] using a phenomenological derivation employing the quasiparticle approximation. Hänsch and Mahan confirmed this result by solving the transport equation in the high-field limit.[9] Here we solve this problem in the entire range of magnetic field. We found the phonon enhancement should be present for arbitrary magnetic fields. It has also been controversial as to whether this factor should exist in every component of the thermoelectric tensor (i.e., the Nernst-Ottingshausen coefficient) or not.[10] The answer from our result is positive and agrees with the experiments.[11]

Recently, it has been proposed that the conventional Kubo-type formula for thermal response is invalid if a magnetic field is present.[12] Jonson and Girvin showed explicitly that some assumptions used in deriving this formula were incorrect for systems in magnetic fields. It is then necessary to examine the general conclusions derived from the Kubo formula, one of which is the Onsager relation.[13] Our transport equation for thermal response did not use this assumption. We found terms in the equation which violate the Onsager reciprocal relation. However, this effect is negligible in the bulk properties of simple metals, as is shown in the solution scheme developed in this paper.

In this paper we develop a solution to the transport equation in a transverse magnetic field by using the quantum transport theory of Kadanoff and Baym[14] (referred to hereafter as KB). A general method of deriving a transport equation in the presence of external fields was developed by Mahan and Hänsch.[15] Mahan also calculated some general properties of the electron Green's function in a magnetic field.[16] These theories provide the important background for solving this problem.

This paper is organized as follows: Section I gives the definitions of transport coefficients, the distribution function, and the transport equation. Section II solves for the dc electrical conductivity. Section III discusses the linear transverse magnetoresistance in some simple metals. Section IV derives the thermal transport coefficients and the electron-phonon mass-enhancement factor in the adiabatic thermopower.

A. Definition of transport coefficients

It is well known that the existence of external fields will induce currents in the system. In linear response, one can write[13]

$$\vec{J}_i=\sum_j \overleftrightarrow{L}_{ij}\cdot\vec{X}_j\ , \tag{1.1}$$

where $\overleftrightarrow{L}_{ij}$ is a tensor defined as the transport coefficient which describes the linear response of current $\vec{J}_i$ to the external force $\vec{X}_j$. For a uniform system, in the presence of electric field $\vec{E}$ and temperature gradient, the particle current $\vec{J}$ and heat current $\vec{J}_Q$ can be written as

$$\vec{J}=\beta\overleftrightarrow{L}^{11}\cdot e\vec{E}+\overleftrightarrow{L}^{12}\cdot\vec{\nabla}\beta\ , \tag{1.2a}$$

$$\vec{J}_Q=\overleftrightarrow{L}^{21}\cdot e\vec{E}+\overleftrightarrow{L}^{22}\cdot\vec{\nabla}\beta\ , \tag{1.2b}$$

where $\beta=1/k_BT$ with k_B the Boltzmann constant and T the temperature. The electrical conductivity tensor $\overleftrightarrow{\sigma}$ and thermal conductivity tensor $\overleftrightarrow{K}$ are defined by

$$\vec{J}_e=e\vec{J}=\overleftrightarrow{\sigma}\cdot\vec{E}\,|_{\vec{\nabla}\beta=\vec{0}}\ , \tag{1.3}$$

$$\vec{J}_Q=-\overleftrightarrow{K}\cdot\vec{\nabla}T\,|_{\vec{J}=\vec{0}}\ . \tag{1.4}$$

Thus

$$\overleftrightarrow{\sigma}=e^2\beta\overleftrightarrow{L}^{11}\ , \tag{1.5}$$

$$\overleftrightarrow{K}=k_B\beta^2[\overleftrightarrow{L}^{22}-\overleftrightarrow{L}^{21}\cdot(\overleftrightarrow{L}^{11})^{-1}\cdot\overleftrightarrow{L}^{12}]\ . \tag{1.6}$$

Expressions for $\overleftrightarrow{L}^{ij}$ are to be derived from quantum theory.

Semiclassical theory gives $\overleftrightarrow{L}^{12}=\overleftrightarrow{L}^{21}$ which has been known as the Onsager reciprocal relation. This relation can also be derived from the Kubo expressions for $\overleftrightarrow{L}^{12}$ and $\overleftrightarrow{L}^{21}$.[13] However, Jonson and Girvin pointed out the assumption used to derive the Kubo formula for thermal response is no longer true when a magnetic field is present in the system.[12] The correction terms they proposed violate the above relation. Our transport equation also gives terms of this nature. But the effect is negligible for the simple metal systems in our consideration, as is shown in Sec. IV.

B. Quantum transport equation

We choose the unit for $\hbar=1$ and use the notation in KB. For fermion particles, the Green's functions are defined by[14]

$$g^>(X_1,X_1')=\frac{1}{i}\langle\psi(X_1)\psi^\dagger(X_1')\rangle\ , \tag{1.7a}$$

$$g^<(X_1,X_1')=-\frac{1}{i}\langle\psi^\dagger(X_1')\psi(X_1)\rangle\ , \tag{1.7b}$$

$$g^r(X_1,X_1')=\frac{1}{i}\langle T\psi(X_1)\psi^\dagger(X_1')\rangle\ , \tag{1.7c}$$

where the notation is $X=(\vec{r},t)$ and T is the time-ordering operator. Only two Green's functions are independent, and the third one can always be determined in terms of the other two.

According to KB, the Green's functions $G(X_1,X_1')$ and self-energies $\Sigma(X_1,X_1')$ are expressed by the following relative and center-of-mass coordinates:

$$\begin{aligned} x&=(\vec{r},t)=X_1-X_1'\ ,\\ X&=(\vec{R},T)=\tfrac{1}{2}(X_1+X_1')\ , \end{aligned} \tag{1.8}$$

so that

$$\begin{aligned} f(X_1,X_1')&=f(X+\tfrac{1}{2}x,X-\tfrac{1}{2}x)\to f(x,X)\\ &=f(\vec{r},t;\vec{R},T)\ . \end{aligned} \tag{1.9}$$

One can define Green's functions and self-energies with respect to the energy ω, momentum $\vec{p}$, position $\vec{R}$, and $\vec{T}$:

$$\left.\begin{matrix} g^>(\vec{p},\omega;\vec{R},T)\\ g^<(\vec{p},\omega;\vec{R},T)\end{matrix}\right\}=\pm i\int d^3r\int dt\,e^{-i\vec{p}\cdot\vec{r}+i\omega t}\times\left\{\begin{matrix} g^>(\vec{r},t;\vec{R},T)\ ,\\ g^<(\vec{r},t;\vec{R},T)\ ,\end{matrix}\right. \tag{1.10}$$

$$g^r(\vec{p},\omega;\vec{R},T)=\int d^3r\int dt\,e^{-i\vec{p}\cdot\vec{r}+i\omega t}g^r(\vec{r},t;\vec{R},T)\ , \tag{1.11}$$

$$\left.\begin{matrix} \Sigma^>(\vec{p},\omega;\vec{R},T)\\ \Sigma^<(\vec{p},\omega;\vec{R},T)\end{matrix}\right\}=\pm i\int d^3r\int dt\,e^{-i\vec{p}\cdot\vec{r}+i\omega t}\times\left\{\begin{matrix} \Sigma^>(\vec{r},t;\vec{R},T)\ ,\\ \Sigma^<(\vec{r},t;\vec{R},T)\ ,\end{matrix}\right. \tag{1.12}$$

$$\Sigma^r(\vec{p},\omega;\vec{R},T)=\int d^3r\int dt\,e^{-i\vec{p}\cdot\vec{r}+i\omega t}\Sigma^r(\vec{r},t;\vec{R},T)\ . \tag{1.13}$$

The distribution function in quantum theory is defined as $g^<(\vec{p},\omega;\vec{R},T)$ in Eq. (1.10) which is the Fourier transform of $-i\,g^<(\vec{r},t;\vec{R},T)$ with respect to the relative coordinates $(\vec{r},t)$. It can be verified that the particle current and heat current are given by

$$\vec{J}(\vec{R},T)=2\int\frac{d\omega}{2\pi}\times\int\frac{d^3p}{(2\pi)^3}\vec{v}(\vec{p},\omega;\vec{R},T)g^<(\vec{p},\omega;\vec{R},T)\ , \tag{1.14}$$

$$\vec{J}_Q(\vec{R},T)=2\int\frac{d\omega}{2\pi}(\omega-\mu)\times\int\frac{d^3p}{(2\pi)^3}\vec{v}(\vec{p},\omega;\vec{R},T)g^<(\vec{p}\cdot\omega;\vec{R},T)\ , \tag{1.15}$$

where $\vec{v}$ is the velocity of electron, μ is the chemical potential, and the factors of 2 account for the spin degeneracy of an electron.

The relation between the three Green's functions are expressed by the following identities:

$$a(\vec{p},\omega;\vec{R},T)\equiv-2\,\mathrm{Im}g^r(\vec{p},\omega;\vec{R},T)=g^>+g^<\ , \tag{1.16}$$

$$\Gamma_0(\vec{p},\omega;\vec{R},T)\equiv-2\,\mathrm{Im}\Sigma^r(\vec{p},\omega;\vec{R},T)=\Sigma^>+\Sigma^<\ , \tag{1.17}$$

where Im denotes "imaginary part." To solve $g^<(\vec{p},\omega;\vec{R},T)$ and $G^r(\vec{p},\omega;\vec{R},T)$, some approximation must be made beforehand. In the rest of our discussion, we use the slow-external-disturbance approximation of KB, which has been proved to be quite successful for solving constant and uniform external field problems.[15]

In calculating the linear response, we make an expansion of the distribution function

$$g^<(\vec{p},\omega;\vec{R},T)=G^<+\vec{g}\,{}^<_{1e}\cdot e\vec{E}+\vec{g}\,{}^<_{1\beta}\cdot\nabla\beta\ , \tag{1.18}$$

where $G^<\equiv g^<|_{\vec{E}=\vec{0},\ \nabla\beta=\vec{0}}$ and $\vec{g}\,{}^<_{1e}$ and $\vec{g}\,{}^<_{1\beta}$ are the linear-response coefficients to the electric field and temperature gradient, respectively. Therefore the transport tensors are

$$\overleftrightarrow{L}^{11}=\frac{2}{\beta}\int\frac{d\omega}{2\pi}\int\frac{d^3p}{(2\pi)^3}\vec{v}\,\vec{g}\,{}^<_{1e}\ , \tag{1.19a}$$

$$\overleftrightarrow{L}^{21}=\frac{2}{\beta}\int\frac{d\omega}{2\pi}(\omega-\mu)\int\frac{d^3p}{(2\pi)^3}\vec{v}\,\vec{g}\,{}^<_{1e}\ , \tag{1.19b}$$

$$\overleftrightarrow{L}^{12}=2\int\frac{d\omega}{2\pi}\int\frac{d^3p}{(2\pi)^3}\vec{v}\,\vec{g}\,{}^<_{1\beta}\ , \tag{1.19c}$$

$$\overleftrightarrow{L}^{22}=2\int\frac{d\omega}{2\pi}(\omega-\mu)\int\frac{d^3p}{(2\pi)^3}\vec{v}\,\vec{g}_{1\beta}\ . \tag{1.19d}$$

In the following sections, we assume the magnetic field lies in the z direction which has a twofold or higher symmetry. The electric field or temperature gradient lies in the x direction. Therefore only L_{xx}, L_{xy}, L_{yx}, and L_{yy} must be considered, and we can assume $L_{xx}=L_{yy}$ and $L_{yx}=-L_{xy}$ throughout.

II. LINEAR RESPONSE TO dc ELECTRIC FIELDS

The calculation of linear response to electric field gives expressions for L^{11} and L^{21}. To solve for the distribution function $g^<$, the usual technique is to first solve g^r. The result is then used to eliminate $g^>$ by Eq. (1.16), so $g^<$ can eventually be solved. We include a uniform transverse magnetic field throughout the discussion.

The electron Hamiltonian is

$$H(\vec{r},t,\vec{p})=\frac{1}{2m}\left[\vec{p}-\frac{e}{c}\vec{A}(\vec{r},t)\right]^2\ , \tag{2.1}$$

where the vector potential includes the presence of constant and uniform electromagnetic fields. We have

$$\vec{A}(\vec{r},t)=-c\vec{E}t+\tfrac{1}{2}\vec{B}\times\vec{r}\ . \tag{2.2}$$

Notice e is the electric charge of the particle which is negative for an electron. This is also implied in the electron's cyclotron frequency $\omega_c=eB/mc$. We ignored the coupling of electron spins to the magnetic field, because this will not affect the transport property.

The retarded Green's function satisfies the equation[9]

$$\left[\omega-\frac{Q^2}{2m}-\frac{1}{8m}\left[e\vec{E}\frac{\partial}{\partial\omega}+\vec{B}\times\vec{\nabla}_{\vec{Q}}\right]^2\right]g^r(\vec{Q},\omega)=1+\Sigma^r(\vec{Q},\omega)g^r(\vec{Q},\omega)\ , \tag{2.3}$$

where $\vec{Q}=\vec{p}+e\vec{E}T-e/(2c)\vec{B}\times\vec{R}$ is the renormalized wave vector. This equation has the solution, to terms linear in E,[16]

$$g^r=G^r+eEg_1^r\ , \tag{2.4}$$

with

$$G^r=2\sum_{n=0}^{\infty}\frac{(-1)^n e^{-\Delta}L_n(2\Delta)}{\omega-\epsilon_z-(n+\frac{1}{2})\omega_c-\Sigma^r}\ , \tag{2.5}$$

$$g_1^r=\frac{-2}{m\omega_c}Q_y\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}\left[\frac{L_n(2\Delta)}{[\omega-\epsilon_z-(n+\frac{1}{2})\omega_c-\Sigma^r]^2}+\frac{2}{\omega_c}\frac{L_n(2\Delta)+2L_{n-1}^1(2\Delta)}{\omega-\epsilon_z-(n+\frac{1}{2})\omega_c-\Sigma^r}\right]\ , \tag{2.6}$$

where $\Delta\equiv Q_\perp^2/m\omega_c\equiv(Q_x^2+Q_y^2)/m\omega_c$ and L_n^m are Laguerre polynomials. In the following discussion, we use the notation defined in Eq. (2.4) in the expansion for all Green's functions and self-energies. G and Σ_0 are Green's functions and self-energy in the absence of electric field and temperature gradient.

The transport equation for $g^<$ is

$$-\left[e\vec{E}\cdot\vec{\nabla}_{\vec{Q}}+e\vec{E}\cdot\frac{\vec{Q}}{m}\frac{\partial}{\partial\omega}-\frac{e\vec{B}}{mc}\cdot\vec{Q}\times\vec{\nabla}_{\vec{Q}}\right]g^{<}=\Sigma^{<}g^{>}-\Sigma^{<}g^{>}+e\vec{E}\cdot\left[\left[\frac{\partial R_e g^r}{\partial\omega}\vec{\nabla}_{\vec{Q}}\Sigma^{<}-\frac{\partial\Sigma^{<}}{\partial\omega}\vec{\nabla}_{\vec{Q}}\mathrm{Re}g^r\right]-\left[\frac{\partial\mathrm{Re}\Sigma^r}{\partial\omega}\vec{\nabla}_{\vec{Q}}g^{<}-\frac{\partial g^{<}}{\partial\omega}\vec{\nabla}_{\vec{Q}}\mathrm{Re}\Sigma^r\right]\right]+\frac{e\vec{B}}{c}\cdot(\vec{\nabla}_{\vec{Q}}\Sigma^r\times\vec{\nabla}_{\vec{Q}}\mathrm{Re}g^r-\vec{\nabla}_{\vec{Q}}g^{<}\times\vec{\nabla}_{\vec{Q}}\mathrm{Re}\Sigma^r)\,. \tag{2.7}$$

Without electric fields, we have

$$\omega_c\left[Q_x\frac{\partial}{\partial Q_y}-Q_y\frac{\partial}{\partial Q_x}\right]G^{<}=\Sigma_0^{>}G^{<}-\Sigma_0^{<}G^{>}+m\omega_c\left[\frac{\partial\mathrm{Re}G^r}{\partial Q_x}\frac{\partial\Sigma_0^{<}}{\partial Q_y}-\frac{\partial\Sigma_0^{<}}{\partial Q_x}\frac{\partial\mathrm{Re}G^r}{\partial Q_y}-\frac{\partial\mathrm{Re}\Sigma_0^r}{\partial Q_x}\frac{\partial G^{<}}{\partial Q_y}+\frac{\partial G^{<}}{\partial Q_x}\frac{\partial\mathrm{Re}\Sigma_0^r}{\partial Q_y}\right], \tag{2.8}$$

which reduces to

$$\Sigma_0^{>}G^{<}-\Sigma_0^{<}G^{>}=0 \tag{2.9}$$

if the system has a twofold symmetry in z direction. The solution of Eq. (2.9) has the same symbolic form as the solution in the absence of magnetic field, although $A=-2\,\mathrm{Im}G^r$ and $\Gamma_0=-2\,\mathrm{Im}\Sigma_0^r$ depend on the magnetic field through Eq. (2.5). We have

$$\begin{aligned}G^{<}(\vec{Q},\omega)&=n(\omega)A(\vec{Q},\omega)\,,\\ G^{>}(\vec{Q},\omega)&=[1-n(\omega)]A(\vec{Q},\omega)\,,\\ \Sigma_0^{<}(\vec{Q},\omega)&\equiv n(\omega)\Gamma_0(\vec{Q},\omega)\,,\\ \Sigma_0^{>}(\vec{Q},\omega)&=[1-n(\omega)]\Gamma_0(\vec{Q},\omega)\,,\end{aligned} \tag{2.10}$$

where $n(\omega)=\{\exp[\beta(\omega-\mu)]+1\}^{-1}$ is the Fermion occupation number.

To the first order in electric field, every variable in Eq. (2.7) can be expanded by

$$g=G+eEg_1\,. \tag{2.11}$$

All zeroth-order quantities canceled in Eq. (2.7) so that the first-order terms satisfy

$$\begin{aligned}-\frac{\partial G^{<}}{\partial Q_x}-\frac{Q_x}{m}\frac{\partial G^{<}}{\partial\omega}+\omega_c\left[Q_x\frac{\partial}{\partial Q_y}-Q_y\frac{\partial}{\partial Q_x}\right]g_1^{<}
&=\Sigma_0^{>}g_1^{<}+\Sigma_1^{>}G^{<}-\Sigma_0^{<}g_1^{>}-\Sigma_1^{<}G^{>}\\
&\quad+\left[\frac{\partial\mathrm{Re}G^r}{\partial\omega}\frac{\partial\Sigma_0^{<}}{\partial Q_x}-\frac{\partial\Sigma_0^{<}}{\partial\omega}\frac{\partial\mathrm{Re}G^r}{\partial Q_x}-\frac{\partial\mathrm{Re}\Sigma_0^r}{\partial\omega}\frac{\partial G^{<}}{\partial Q_x}+\frac{\partial G^{<}}{\partial\omega}\frac{\partial\mathrm{Re}\Sigma_0^r}{\partial Q_x}\right]\\
&\quad+m\omega_c\left[\left[\frac{\partial\mathrm{Re}G^r}{\partial Q_x}\frac{\partial\Sigma_1^{<}}{\partial Q_y}+\frac{\partial\mathrm{Re}g_1^r}{\partial Q_x}\frac{\partial\Sigma_0^r}{\partial Q_y}-\frac{\partial\Sigma_1^{<}}{\partial Q_x}\frac{\partial\mathrm{Re}G^r}{\partial Q_y}-\frac{\partial\Sigma_0}{\partial Q_x}\frac{\partial\mathrm{Re}g_1^r}{\partial Q_y}\right]\right.\\
&\quad\left.-\left[\frac{\partial\mathrm{Re}\Sigma_1^r}{\partial Q_x}\frac{\partial G^{<}}{\partial Q_y}+\frac{\partial\mathrm{Re}\Sigma_0^r}{\partial Q_x}\frac{\partial g_1^{<}}{\partial Q_y}-\frac{\partial g_1^{<}}{\partial Q_x}\frac{\partial\mathrm{Re}\Sigma_0^r}{\partial Q_y}-\frac{\partial G^{<}}{\partial Q_x}\frac{\partial\mathrm{Re}\Sigma_1^r}{\partial Q_y}\right]\right]. \end{aligned}\tag{2.12}$$

The calculations for the longitudinal and transverse magnetic field differ at this point. For $\vec{B}\|\vec{E}$ the retarded function have no linear term in E. They do for $\vec{B}\perp\vec{E}$, because of the Hall field, which increases the complexity of the calculation. We can reformulate some terms by using Eqs. (1.16), (1.17), and (2.10),

$$\Sigma_0^{>}g_1^{<}+\Sigma_1^{>}G^{<}-\Sigma_0^{<}g_1^{>}-\Sigma_1^{<}G^{>}=\Gamma_0g_1^{<}+nA\Gamma_1-n\Gamma_0a_1-A\Sigma_1^{<}\,, \tag{2.13}$$

$$\frac{\partial \mathrm{Re}G^r}{\partial\omega}\frac{\partial \Sigma_0^<}{\partial Q_x}-\frac{\partial \Sigma_0^<}{\partial\omega}\frac{\partial \mathrm{Re}G^r}{\partial Q_x}-\frac{\partial \mathrm{Re}\Sigma_0^r}{\partial Q_x}\frac{\partial G^<}{\partial\omega}-\frac{\partial \mathrm{Re}\Sigma_0^r}{\partial\omega}\frac{\partial G^<}{\partial Q_x}$$
$$=n\left[\frac{\partial\Gamma_0}{\partial Q_x}\frac{\partial \mathrm{Re}G^r}{\partial\omega}-\frac{\partial\Gamma_0}{\partial\omega}\frac{\partial \mathrm{Re}G^r}{\partial Q_x}-\frac{\partial\sigma_0}{\partial\omega}\frac{\partial A}{\partial Q_x}+\frac{\partial A}{\partial\omega}\frac{\partial\sigma_0}{\partial Q_x}\right]-\frac{\partial n}{\partial\omega}\left[\Gamma_0\frac{\partial \mathrm{Re}G^r}{\partial Q_x}-A\frac{\partial\sigma_0}{\partial Q_x}\right], \tag{2.14}$$

where we started to use the notation $\sigma\equiv\mathrm{Re}\Sigma^r$. Since all zeroth-order quantities are invariant with respect to rotations around the z axis, they depend on $\vec{Q}$ only through $Q_\perp$ and Q_z. Thus

$$\frac{\partial f_0}{\partial Q_x}=\frac{Q_x}{m}\frac{\partial f_0}{\partial\epsilon_\perp} \tag{2.15}$$

with $\epsilon_\perp\equiv Q_\perp^2/2m$. Equation (2.12) can be rewritten as

$$-n\frac{Q_x}{m}\left[\frac{\partial A}{\partial\epsilon_\perp}\left[1-\frac{\partial\sigma_0}{\partial\omega}\right]+\frac{\partial A}{\partial\omega}\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]+\frac{\partial\Gamma_0}{\partial\epsilon_\perp}\frac{\partial \mathrm{Re}G^r}{\partial\omega}-\frac{\partial\Gamma_0}{\partial\omega}\frac{\partial \mathrm{Re}G^r}{\partial\epsilon_\perp}\right]$$
$$-\frac{\partial n}{\partial\omega}\frac{Q_x}{m}\left[A\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]-\Gamma_0\frac{\partial \mathrm{Re}G^r}{\partial\epsilon_\perp}\right]+\omega_c\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]\left[Q_x\frac{\partial}{\partial Q_y}-Q_y\frac{\partial}{\partial Q_x}\right]g_1^<$$
$$=\Gamma_0 g_1^<+nA\Gamma_1-n\Gamma_0 a_1-A\Sigma_1^<$$
$$+\omega_c\left[\frac{\partial \mathrm{Re}G^r}{\partial\epsilon_\perp}\left[Q_x\frac{\partial}{\partial Q_y}-Q_y\frac{\partial}{\partial Q_x}\right]\Sigma_1^<+n\frac{\partial\Gamma_0}{\partial\epsilon_\perp}\left[Q_x\frac{\partial}{\partial Q_y}-Q_y\frac{\partial}{\partial Q_x}\right]\mathrm{Re}g_1^r-n\frac{\partial A}{\partial\epsilon_\perp}\left[Q_x\frac{\partial}{\partial Q_y}-Q_y\frac{\partial}{\partial Q_x}\right]\sigma_1\right]. \tag{2.16}$$

This is an integral equation for $g_1^<(\vec{Q},\omega)$ since $\Sigma_1^<$, Γ_1, and σ_1 are integrals of $g_1^<$. Generally, $g_1^<$ has the following form:

$$g_1^<(\vec{Q},\omega)=Q_xS_x(Q_\perp,Q_z,\omega)+Q_yS_y(Q_\perp,Q_z,\omega)\ , \tag{2.17}$$

which satisfies

$$\left[Q_x\frac{\partial}{\partial Q_y}-Q_y\frac{\partial}{\partial Q_x}\right]g_1^<=-Q_yS_x+Q_xS_y\ . \tag{2.18}$$

From Eq. (2.17), we note that $\Sigma_1^<$, Γ_1, and σ_1 must be of the form

$$\Sigma_1=\Sigma_{1x}+\Sigma_{1y} \tag{2.19}$$

which means Σ_{1i}/Q_i depends only on $Q_\perp$, Q_z, and ω, and therefore commutes with the operator $Q_x\partial/\partial Q_y-Q_y\partial/\partial Q_x$.

The operator $Q_x\partial/\partial Q_y-Q_y\partial/\partial Q_x$ can be eliminated in Eq. (2.16):

$$-n\frac{Q_x}{m}\left[\frac{\partial A}{\partial\epsilon_\perp}\left[1-\frac{\partial\sigma_0}{\partial\omega}\right]+\frac{\partial A}{\partial\omega}\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]+\frac{\partial\Gamma_0}{\partial\epsilon}\frac{\partial \mathrm{Re}G^r}{\partial\omega}\right.$$
$$\left.-\frac{\partial\Gamma_0}{\partial\omega}\frac{\partial \mathrm{Re}G^r}{\partial\epsilon_\perp}\right]-\frac{\partial n}{\partial\omega}\frac{Q_x}{m}\left[A\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]-\Gamma_0\frac{\partial \mathrm{Re}G^r}{\partial\epsilon_\perp}\right]+\omega_c\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right](-Q_yS_x+Q_xS_y)$$
$$=\Gamma_0(Q_xS_x+Q_yS_y)+nA(\Gamma_{1x}+\Gamma_{1y})-n\Gamma_0 a_1-A(\Sigma_{1x}^<+\Sigma_{1y}^<)$$
$$+\omega_c\left[\frac{\partial \mathrm{Re}G^r}{\partial\epsilon_\perp}\left[Q_x\frac{\Sigma_{1y}^<}{Q_y}-Q_y\frac{\Sigma_{1x}^<}{Q_x}\right]+n\frac{\partial\Gamma_0}{\partial\epsilon_\perp}Q_x\frac{\mathrm{Re}g_1^r}{Q_y}-n\frac{\partial A}{\partial\epsilon_\perp}\left[Q_x\frac{\sigma_{1y}}{Q_y}-Q_y\frac{\sigma_{1x}}{Q_x}\right]\right]. \tag{2.20}$$

We can group together terms with a factor of Q_x or Q_y to reduce Eq. (2.20) into a couple of equations which are invariant under rotations around the z axis. We have

$$\Gamma_0 S_x - \omega_c\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]S_y = A\frac{\Sigma_{1x}^<}{Q_x} - nA\frac{\Gamma_{1x}}{Q_x} - \omega_c\frac{\partial \mathrm{Re}G^r}{\partial\epsilon_\perp}\frac{\Sigma_{1y}^<}{Q_y} + \omega_c n\frac{\partial A}{\partial\epsilon_\perp}\frac{\sigma_{1y}}{Q_y} - \omega_c n\frac{\partial A}{\partial\epsilon_\perp}\frac{\mathrm{Re}g_1^r}{Q_y}$$
$$-\frac{1}{m}\frac{\partial n}{\partial\omega}\left[A\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]-\Gamma_0\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\right]$$
$$-\frac{n}{m}\left[\frac{\partial A}{\partial\epsilon_\perp}\left[1-\frac{\partial\sigma_0}{\partial\omega}\right]+\frac{\partial A}{\partial\omega}\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]+\frac{\partial\Gamma_0}{\partial\epsilon_\perp}\frac{\partial\mathrm{Re}G^r}{\partial\omega}-\frac{\partial\Gamma_0}{\partial\omega}\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\right], \qquad (2.21a)$$

$$\omega_c\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]S_x+\Gamma_0 S_y = A\frac{\Sigma_{1y}^<}{Q_y} - nA\frac{\Gamma_{1y}}{Q_y} + \omega_c\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\frac{\Sigma_{1x}^<}{Q_x} - \omega_c n\frac{\partial A}{\partial\epsilon_\perp}\frac{\sigma_{1x}}{Q_x} + n\Gamma_0\frac{a_1}{Q_y}\,. \qquad (2.21b)$$

S_x and S_y can be superficially solved as

$$S_x = \frac{1}{\Gamma_0^2+\omega_c^2\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]^2}\left\{-\frac{\Gamma_0}{m}\frac{\partial n}{\partial\omega}\left[A\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]-\Gamma_0\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\right]\right.$$
$$-\frac{\Gamma_0}{m}n\left[\frac{\partial A}{\partial\epsilon_\perp}\left[1-\frac{\partial\sigma_0}{\partial\omega}\right]+\frac{\partial A}{\partial\omega}\left[1+\frac{\partial\sigma_0}{\partial\epsilon\perp}\right]+\frac{\partial\Gamma_0}{\partial\epsilon_\perp}\frac{\partial\mathrm{Re}G^r}{\partial\omega}-\frac{\partial\Gamma_0}{\partial\omega}\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\right]$$
$$+\omega_c\Gamma_0 n\left[\frac{a_1}{Q_y}-\frac{\partial\Gamma_0}{\partial\epsilon_\perp}\frac{\mathrm{Re}g_1^r}{Q_y}\right]+\omega_c n\frac{\partial A}{\partial\epsilon_\perp}\left[\Gamma_0\frac{\sigma_{1y}}{Q_y}-\omega_c\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]\frac{\sigma_{1x}}{Q_x}\right]$$
$$-nA\left[\omega_c\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]\frac{\Gamma_{1y}}{Q_y}+\Gamma_0\frac{\Gamma_{1x}}{Q_x}\right]+\omega_c\left[A\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]-\Gamma_0\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\right]\frac{\Sigma_{1y}^<}{Q_y}$$
$$\left.+\left[\Gamma_0 A+\omega_c^2\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\right]\frac{\Sigma_{1x}^<}{Q_x}\right\}, \qquad (2.22a)$$

$$S_y = \frac{1}{\Gamma_0^2+\omega_c^2\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]^2}\left\{\frac{\omega_c}{m}\frac{\partial n}{\partial\omega}\left[A\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]-\Gamma_0\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\right]\right.$$
$$+\frac{\omega_c}{m}n\left[\frac{\partial A}{\partial\epsilon_\perp}\left[1-\frac{\partial\sigma_0}{\partial\omega}\right]+\frac{\partial A}{\partial\omega}\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]+\frac{\partial\Gamma_0}{\partial\epsilon_\perp}\frac{\partial\mathrm{Re}G^r}{\partial\omega}-\frac{\partial\Gamma_0}{\partial\omega}\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\right]$$
$$+n\left[\Gamma_0^2\frac{a_1}{Q_y}+\omega_c^2\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]\frac{\partial\Gamma_0}{\partial\epsilon_\perp}\frac{\mathrm{Re}g_1^r}{Q_y}\right]-\omega_c n\frac{\partial A}{\partial\epsilon_\perp}\left[\omega_c\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]\frac{\sigma_{1y}}{Q_y}+\Gamma_0\frac{\sigma_{1x}}{Q_x}\right]$$
$$+nA\left[\omega_c\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]\frac{\Gamma_{1x}}{Q_x}-\Gamma_0\frac{\Gamma_{1y}}{Q_y}\right]+\left[\Gamma_0 A+\omega_c^2\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\right]\frac{\Sigma_{1y}^<}{Q_y}$$
$$\left.-\,\omega_c\left[A\left[1+\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]-\Gamma_0\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\right]\frac{\Sigma_{1x}^<}{Q_x}\right\}. \qquad (2.22b)$$

The transport coefficients are expressed in terms of S_x and S_y,

$$
\begin{aligned}
L_{xx}^{11}&=\frac{2}{\beta}\int\frac{d\omega}{2\pi}\int\frac{d^3Q}{(2\pi)^3}\frac{Q_x}{m}g_1^<=\frac{1}{m\beta}\int\frac{d\omega}{2\pi}\int\frac{d^3Q}{(2\pi)^3}Q_\perp^2S_x(Q_\perp,Q_z,\omega)\ ,\\
L_{xy}^{11}&=\frac{1}{m\beta}\int\frac{d\omega}{2\pi}\int\frac{d^3Q}{(2\pi)^3}Q_\perp^2S_y(Q_\perp,Q_z,\omega)\ ,\\
L_{xx}^{21}&=\frac{1}{m\beta}\int\frac{d\omega}{2\pi}(\omega-\mu)\int\frac{d^3Q}{(2\pi)^3}Q_\perp^2S_x(Q_\perp,Q_z,\omega)\ ,\\
L_{xy}^{21}&=\frac{1}{m\beta}\int\frac{d\omega}{2\pi}(\omega-\mu)\int\frac{d^3Q}{(2\pi)^3}Q_\perp^2S_y(Q_\perp,Q_z,\omega)\ .
\end{aligned}
\tag{2.23}
$$

Some quantities in (2.28) should be examined closely. Defining $\phi_n=\omega-\sigma_0-\epsilon_z-(n+\frac{1}{2})\omega_c$, Eqs. (2.5) and (2.6) can be written

$$
G^r=2\sum_{n=0}^{\infty}\frac{(-1)^n e^{-\Delta}L_n(2\Delta)}{\phi_n+i\Gamma_0/2}\ ,
\tag{2.24}
$$

$$
g_1^r=\frac{-2Q_y}{m\omega_c}\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}\left[\frac{L_n(2\Delta)}{(\phi_n+i\Gamma_0/2)^2}+\frac{2}{\omega_c}\frac{L_n(2\Delta)+2L_{n-1}^1(2\Delta)}{\phi_n+i\Gamma_0/2}\right].
\tag{2.25}
$$

By definition,

$$
A=2\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}L_n(2\Delta)\frac{\Gamma_0}{\phi_n^2+\Gamma_0^2/4}\ ,
\tag{2.26}
$$

$$
\mathrm{Re}G^r=2\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}L_n(2\Delta)\frac{\phi_n}{\phi_n^2+\Gamma_0^2/4}\ ,
\tag{2.27}
$$

$$
\frac{\partial A}{\partial\omega}=2\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}L_n(2\Delta)\frac{1}{(\phi_n^2+\Gamma_0^2/4)^2}\left[\frac{\partial\Gamma_0}{\partial\omega}\left[\phi_n^2-\frac{\Gamma_0^2}{4}\right]-2\Gamma_0\phi_n\left[1-\frac{\partial\sigma_0}{\partial\omega}\right]\right],
\tag{2.28}
$$

$$
\frac{\partial\mathrm{Re}G^r}{\partial\omega}=-2\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}L_n(2\Delta)\frac{1}{(\phi_n^2+\Gamma_0^2/4)^2}\left[\left[1-\frac{\partial\sigma_0}{\partial\omega}\right]\left[\phi_n^2-\frac{\Gamma_0^2}{4}\right]-\frac{\phi_n\Gamma_0}{2}\frac{\partial\Gamma_0}{\partial\omega}\right],
\tag{2.29}
$$

$$
\frac{\partial A}{\partial\epsilon_\perp}=2\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}\left\{\frac{L_n(2\Delta)}{(\phi_n^2+\Gamma_0^2/4)^2}\left[\frac{\partial\Gamma_0}{\partial\epsilon_\perp}\left[\phi_n^2-\frac{\Gamma_0^2}{4}\right]-2\Gamma_0\phi_n\frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]-\frac{\Gamma_0}{\phi_n^2+\Gamma_0^2/4}\frac{2}{\omega_c}[L_n(2\Delta)+2L_{n-1}^1(2\Delta)]\right\},
\tag{2.30}
$$

$$
\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}=2\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}\left\{\frac{L_n(2\Delta)}{(\phi_n^2+\Gamma_0^2/4)^2}\left[\left[\phi_n^2-\frac{\Gamma_0^2}{4}\right]^2\frac{\partial\sigma_0}{\partial\epsilon_\perp}-\frac{\phi_n\Gamma_0}{2}\frac{\partial\Gamma_0}{\partial\epsilon_\perp}\right]-\frac{\phi_n}{\phi_n^2+\Gamma_0^2/4}\frac{2}{\omega_c}[L_n(2\Delta)+2L_{n-1}^1(2\Delta)]\right\},
\tag{2.31}
$$

where we have used the relations

$$
\frac{d}{dx}L_n(x)=-L_{n-1}^1(x)\ ,
\tag{2.32}
$$

$$
\frac{\partial\Delta}{\partial\epsilon_\perp}=\frac{2}{\omega_c}\ ,
\tag{2.33}
$$

in taking derivatives.

We write explicitly,

$$\frac{\partial A}{\partial \epsilon_\perp}\left[1-\frac{\partial \sigma_0}{\partial \omega}\right]+\frac{\partial A}{\partial \omega}\left[1+\frac{\partial \sigma_0}{\partial \epsilon_\perp}\right]+\frac{\partial \Gamma_0}{\partial \epsilon_\perp}\frac{\partial \mathrm{Re}G^r}{\partial \omega}-\frac{\partial \Gamma_0}{\partial \omega}\frac{\partial \mathrm{Re}G^r}{\partial \epsilon_\perp}$$

$$=-2\left[1-\frac{\partial \sigma_0}{\partial \omega}\right]\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}\left[L_n(2\Delta)\frac{2\phi_n\Gamma_0}{(\phi_n^2+\Gamma_0^2/4)^2}+\frac{\Gamma_0}{\phi_n^2+\Gamma_0^2/4}\frac{2}{\omega_c}[L_n(2\Delta)+2L_{n-1}^1(2\Delta)]\right]$$

$$+2\frac{\partial \Gamma_0}{\partial \omega}\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}\left[L_n(2\Delta)\frac{\phi_n^2-\Gamma_0^2/4}{(\phi_n^2+\Gamma_0^2/4)^2}+\frac{2}{\omega_c}[L_n(2\Delta)+2L_{n-1}^1(2\Delta)]\frac{\phi_n}{\phi_n^2+\Gamma_0^2/4}\right], \quad (2.34)$$

$$\mathrm{Re}g_1^r=\frac{-Q_y}{m\omega_c}2\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}\left[L_n(2\Delta)\frac{\phi_n^2-\Gamma_0^2/4}{(\phi_n^2+\Gamma_0^2/4)^2}+\frac{2}{\omega_c}[L_n(2\Delta)+2L_{n-1}^1(2\Delta)]\frac{\phi_n}{\phi_n^2+\Gamma_0^2/4}\right], \quad (2.35)$$

$$a_1=\frac{-Q_y}{m\omega_c}2\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}\left[L_n(2\Delta)\frac{2\phi_n\Gamma_0}{(\phi_n^2+\Gamma_0^2/4)^2}+\frac{2}{\omega_c}[L_n(2\Delta)+2L_{n-1}^1(2\Delta)]\frac{\Gamma_0}{\phi_n^2+\Gamma_0^2/4}\right]. \quad (2.36)$$

These three equations can be reformulated by using the identity of Laguerre polynomial

$$L_n(x)=L_n^1(x)-L_{n-1}^1(x)\ . \quad (2.37)$$

We define two auxiliary functions

$$U_1(\omega)=2\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}L_n^1(2\Delta)\frac{\Gamma_0}{\phi_n^2+\Gamma_0^2/4}\ , \quad (2.38)$$

$$U_2(\omega)=2\sum_{n=0}^{\infty}(-1)^n e^{-\Delta}L_n^1(2\Delta)\frac{\phi_n}{\phi_n^2+\Gamma_0^2/4}\ , \quad (2.39)$$

so that Eqs. (2.34)—(2.36) become

$$\frac{\partial A}{\partial \epsilon_\perp}\left[1-\frac{\partial \sigma_0}{\partial \omega}\right]+\frac{\partial A}{\partial \omega}\left[1+\frac{\partial \sigma_0}{\partial \epsilon_\perp}\right]+\frac{\partial \Gamma_0}{\partial \epsilon_\perp}\frac{\partial \mathrm{Re}G^r}{\partial \omega}-\frac{\partial \Gamma_0}{\partial \omega}\frac{\partial \mathrm{Re}G^r}{\partial \epsilon_\perp}$$

$$=-\left[1-\frac{\partial \sigma_0}{\partial \omega}\right]\left[\frac{2}{\omega_c}[U_1(\omega)-U_0(\omega-\omega_c)]-\frac{\partial}{\partial \omega}[U_1(\omega)+U_1(\omega-\omega_c)]\right]$$

$$+\frac{\partial \Gamma_0}{\partial \omega}\left[\frac{2}{\omega_c}[U_2(\omega)-U_2(\omega-\omega_c)]-\frac{\partial}{\partial \omega}[U_2(\omega)+U_2(\omega-\omega_c)]\right], \quad (2.40)$$

$$\mathrm{Re}g_1^r=-\frac{Q_y}{m\omega_c}\left[\frac{2}{\omega_c}[U_2(\omega)-U_2(\omega-\omega_c)]-\frac{\partial}{\partial \omega}[U_2(\omega)+U_2(\omega-\omega_c)]\right], \quad (2.41)$$

$$a_1=-\frac{Q_y}{m\omega_c}\left[\frac{2}{\omega_c}[U_1(\omega)-U_1(\omega-\omega_c)]-\frac{\partial}{\partial \omega}[U_1(\omega)+U_1(\omega-\omega_c)]\right], \quad (2.42)$$

where the ω dependence in U refers only to the explicit ω in ϕ_n. All three equations above vanish without magnetic field. This can also be seen from the zero magnetic field solution

$$g^r=\frac{1}{\omega-\epsilon_Q-\Sigma^r} \quad (2.43)$$

to Eq. (2.3). When $|\omega_c|$ is "small," Eqs. (2.40)—(2.42) are $O(\omega_c^2)$. The question is by what scale $|\omega_c|$ could be small, which we answer below.

Consider a system of electron gas in a metal. For magnetic fields in experimentally interesting ranges the electron cyclotron frequency ω_c is much smaller than the electron Fermi energy E_F. The ratio $|\omega_c|/E_F$ is less than 10^{-4}; we make this an expansion parameter in the solution to Eq. (2.22). The rest of this section derives the lowest-order solution of Eq. (2.22), while Sec. III takes into account the effect of next-higher-order terms. Obviously the lowest-order solution is different from the solution at zero magnetic field. Because of the existence of another energy quantity Γ_0, we cannot simply set $\omega_c=0$ in Eq. (2.22), since $|\omega_c|$ is certainly not small compared to Γ_0.

To see how an expansion in ω_c/E_F is possible, we notice that all inhomogeneous terms in Eq. (2.22) such as A,$\mathrm{Re}G^r$ are infinite series summing over Landau levels, which appear quite different from the corresponding terms in zero magnetic field. However, the transport coefficients in Eqs. (2.23) depend only on their integrations over $\int d^3Q/[(2\pi)^3]Q_\perp^2$ or $\int d^3Q'/[(2\pi)^3]V_{QQ'}^2$. These integrals are related to the corresponding integrals at zero magnetic field in a way described below.

By interchanging the order of summation over Landau levels and wave-vector integration, we write a typical integral as

$$I(\omega_c) \propto \sum_{n=0}^{\infty} \int \frac{d^3Q}{(2\pi)^3} = \sum_{n=0}^{\infty} f(n\omega_c)\omega_c \tag{2.44}$$

where we have omitted the ω and, in some cases, $\vec{Q}$ dependence in I, to make the notations simple. It is certainly reasonable to expect that the integral of the zero magnetic field quantity corresponds to

$$I(\omega_c=0) = \lim_{\omega_c \to 0} I(\omega_c) = \int_0^\infty dx\, f(x) . \tag{2.45}$$

Apart from the oscillation term which has null average effect,[16] the summation in (2.44) and integral in (2.45) are related by the Euler-MacLaurin formula:

$$\int_0^\infty dx\, f(x) = \sum_{n=0}^{\infty} \int_{(n-1/2\omega_c)}^{(n+1/2\omega_c)} f(x)dx - \int_{-(1/2)\omega_c}^{0} dx\, f(x)$$

$$= \omega_c \sum_{n=0}^{\infty} f(n\omega_c) - \frac{\omega_c}{2} f(0) + \sum_{k=1}^{\infty} \frac{[1+(-1)^k]}{(k+1)!} \left[\frac{\omega_c}{2}\right]^{k+1} \sum_{n=0}^{\infty} f^{(k)}(n\omega_c) - \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{(k+1)!} \left[\frac{\omega_c}{2}\right]^{k+1} . \tag{2.46}$$

In metals, $I(\omega_c)$ varies considerably with respect to ω only on a scale of E_F. This means

$$\omega_c \sum_{n=0}^{\infty} f^{(k)}(n\omega_c) \sim \frac{1}{E_F} \omega_c \sum_{n=0}^{\infty} f^{(k-1)}(n\omega_c) \tag{2.47}$$

and

$$\omega_c \sum_{n=0}^{\infty} f(n\omega_c) = \int_0^\infty dx\, f(x) + \frac{\omega_c}{2} f(0) + O\left[\frac{\omega_c}{E_F}\right]^2 . \tag{2.48}$$

It was shown[16] that only the first $n_0 \sim |E_F/\omega_c|$ terms are important in the summation of (2.44); the terms with $n > n_0$ diminish rapidly with increasing n. This gives a rough scale on $f(0)$

$$f(0) \sim \frac{\omega_c}{E_F} \sum_{n=0}^{\infty} f(n\omega_c) , \tag{2.49}$$

and thus

$$I(\omega_c) = I(\omega_c=0) + O\left[\frac{\omega_c}{E_F}\right] \tag{2.50}$$

from Eq. (2.48).

As an illustration of what is meant in the above argument, we integrate the spectral function $A(\vec{Q},\omega,\omega_c)$ by $\int d^3Q'/[(2\pi)^3] V_{QQ'}^2$ to get the electron self-energy $\Gamma_0 = -2\,\mathrm{Im}\Sigma_0^r$, which is a slowly-varying quantity with respect to ω (Ref. 13) as we assumed for $I(\omega_c)$ in Eq. (2.44). In elastic scattering,

$$\Gamma_0(\vec{Q},\omega,\omega_c) = \int \frac{d^3Q'}{(2\pi)^3} V_{QQ'}^2 A(\vec{Q}',\omega,\omega_c) \tag{2.51}$$

and

$$\Gamma_0(\vec{Q},\omega,\omega_c=0) = \int \frac{d^3Q}{(2\pi)^3} V_{QQ'}^2 A(\vec{Q}',\omega,\omega_c=0) , \tag{2.52}$$

from Eq. (2.43),

$$A(\vec{Q},\omega,\omega_c=0) = \frac{\Gamma_0}{(\omega-\epsilon_Q-\sigma_0)^2+\Gamma_0^2/4} . \tag{2.53}$$

Equation (2.48) means

$$\sum_{n=0}^{\infty} \int \frac{d^3Q'}{(2\pi)^3} V_{QQ'}^2 \frac{(-1)^n e^{-\Delta'} L_n(2\Delta')\Gamma_0}{(\phi_n^2+\Gamma_0^2/4)^2} = \int \frac{d^3Q'}{(2\pi)^3} V_{QQ'}^2 \frac{\Gamma_0}{(\omega-\epsilon_Q'-\sigma_0)^2+\Gamma_0^2/4} + \tfrac{1}{2} \int \frac{d^3Q'}{(2\pi)^3} V_{QQ'}^2 \frac{e^{-\Delta'}\Gamma_0}{\phi_n^2+\Gamma_0^2/4} + O\left[\frac{\omega_c}{E_F}\right]^2 , \tag{2.54}$$

where the second term on the right side is of the order of ω_c/E_F comparing to the first, so that

$$\Gamma_0(\vec{Q},\omega,\omega_c) = \Gamma_0(\vec{Q},\omega,\omega_c=0) + O(\omega_c/E_F) , \tag{2.55}$$

as is the meaning of Eq. (2.50).

We point out that the terms in Eqs. (2.34)–(2.36) are negligible to the transport equation (2.22) because the contributions are of $(\omega_c/E_F)^2$ higher than the lowest-order terms. This is because

$$U_1(\omega) + U_1(\omega-\omega_c) = A(\vec{Q},\omega,\omega_c) , \tag{2.56}$$

$$U_2(\omega) + U_2(\omega-\omega_c) = \mathrm{Re}G^r , \tag{2.57}$$

so that the integrals

$$\int \frac{d^3Q'}{(2\pi)^3} V_{QQ'}^2 U_1(\vec{Q}',\omega,\omega_c) = \tfrac{1}{2}\Gamma_0 + O(\omega_c/E_F) , \tag{2.58}$$

$$\int \frac{d^3Q'}{(2\pi)^3} V_{QQ'}^2 U_2(\vec{Q}',\omega,\omega_c) = \tfrac{1}{2}\sigma_0 + O(\omega_c/E_f) \tag{2.59}$$

are slowly varying with respect to ω. Thus

$$\tfrac{1}{2}\left[\frac{2}{\omega_c}[\Gamma_0(\omega)-\Gamma_0(\omega-\omega_c)] - \frac{\partial}{\partial\omega}\Gamma_0(\omega) + [\Gamma_0(\omega-\omega_c)]\right] \approx \tfrac{1}{6}\omega_c^2 \frac{\partial^3\Gamma_0}{\partial\omega^3} \sim \left[\frac{\omega_c}{E_F}\right]^2 \frac{\Gamma_0}{E_F} . \tag{2.60}$$

As a next simplification, in Eq. (2.22) we ignore the $\vec{Q}$ dependence of Γ_0 and σ_0 so that $\partial\Gamma_0/\partial\epsilon_\perp$ and $\partial\sigma_0/\partial\epsilon_\perp$ are dropped out; this is usually done in the calculation of metals. After all these considerations, we obtain

$$S_x = \frac{1}{\Gamma_0^2+\omega_c^2}\left[\frac{\Gamma_0}{m}\left[-\frac{\partial n}{\partial\omega}\right]\left[A-\Gamma_0\frac{\partial \mathrm{Re}G^r}{\partial\epsilon_\perp}\right] + \left[\Gamma_0 A + \omega_c^2\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\right]\frac{\Sigma_{1x}^<}{Q_x} - n\Gamma_0 A\frac{\Gamma_{1x}}{Q_x} + \omega_c\left[A-\Gamma_0\frac{\partial\mathrm{Re}G^r}{\partial\epsilon_\perp}\right]\frac{\Sigma_{1x}^<}{Q_y} - nA\omega_c\frac{\Gamma_{1y}}{Q_y} + \omega_c n\frac{\partial A}{\partial\epsilon_\perp}\left[\Gamma_0\frac{\sigma_{1y}}{Q_y} - \omega_c\frac{\sigma_{1x}}{Q_x}\right]\right] , \tag{2.61a}$$

$$S_y=\frac{1}{\Gamma_0^2+\omega_c^2}\left[\frac{\omega_c}{m}\frac{\partial n}{\partial\omega}\left[A-\Gamma_0\frac{\partial \mathrm{Re}G^r}{\partial\epsilon_\perp}\right]-\omega_c\left[A-\Gamma_0\frac{\partial \mathrm{Re}G^r}{\partial\epsilon_\perp}\right]\frac{\Sigma_{1x}^<}{Q_x}+\omega_c nA\frac{\Gamma_{1x}}{Q_x}+\left[\Gamma_0 A+\omega_c^2\frac{\partial \mathrm{Re}G^r}{\partial\epsilon_\perp}\right]\frac{\Sigma_{1y}^<}{Q_y}\right.$$
$$\left.-n\Gamma_0 A\frac{\Gamma_{1y}}{Q_y}-\omega_c n\frac{\partial A}{\partial\epsilon_\perp}\left[\omega_c\frac{\sigma_{1y}}{Q_y}+\Gamma_0\frac{\sigma_{1x}}{Q_x}\right]\right] . \tag{2.61b}$$

We reemphasize that the simplification from Eqs. (2.22) to Eqs. (2.61) are based on ignoring terms of the relative size $O(\omega_c/E_F)^2$. This is certainly a good approximation for electrons in metals where the electron density is high. In fact, this approximation also indicates the range of validity of the Kubo formula, depending on whether the diathermal current term[12] is negligible or not. We will return to this point in Sec. IV.

In Eqs. (2.61), variables labeled by x are integrals of S_x and those by y are integrals of S_y. All these are assumed to vary slowly with respect to $\vec{Q}$. This assumption is consistent with the solution we find. Since

$$\frac{\partial A}{\partial\epsilon_\perp}\approx-\frac{2}{\omega_c}[U_1(\omega)-U_1(\omega-\omega_c)] , \tag{2.62}$$

$$\frac{\partial \mathrm{Re}G^r}{\partial\epsilon_\perp}\approx-\frac{2}{\omega_c}[U_2(\omega)-U_2(\omega-\omega_c)] , \tag{2.63}$$

all terms with factors of $\partial A/\partial\epsilon_\perp$ or $\partial \mathrm{Re}G^r/\partial\epsilon_\perp$ can be ignored because their contributions are at least an order of ω_c/E_F higher than the A terms. Thus,

$$S_x=\frac{A}{\Gamma_0^2+\omega_c^2}\left[-\frac{\partial n}{\partial\omega}\frac{\Gamma_0}{m}+\omega_c\left[\frac{\Sigma_{1y}^<}{Q_y}-n\frac{\Gamma_{1y}}{Q_y}\right]\right.$$
$$\left.+\Gamma_0\left[\frac{\Sigma_{1x}^<}{Q_x}-n\frac{\Gamma_{1x}}{Q_x}\right]\right] , \tag{2.64a}$$

$$S_y=\frac{A}{\Gamma_0^2+\omega_c^2}\left[\frac{\omega_c}{m}\frac{\partial n}{\partial\omega}+\Gamma_0\left[\frac{\Sigma_{1y}^<}{Q_y}-n\frac{\Gamma_{1y}}{Q_y}\right]\right.$$
$$\left.-\omega_c\left[\frac{\Sigma_{1x}^<}{Q_x}-n\frac{\Gamma_{1x}}{Q_x}\right]\right] . \tag{2.64b}$$

The expressions for Γ_1, σ_1, and $\Sigma_1^{>,<}$ are[15]

$$\Sigma_1^<(\vec{Q},\omega)=\int\frac{d^3q}{(2\pi)^3}V_q^2\times\{[1+n_B(\omega_q)]g_1^<(\vec{Q}+\vec{q},\omega+\omega_q)+n_\beta(\omega_q)g_1^<(\vec{Q}+\vec{q},\omega-\omega_q)\} , \tag{2.65}$$

$$\Sigma_1^>(\vec{Q},\omega)=\int\frac{d^3q}{(2\pi)^3}V_q^2\{n_B(\omega_q)g_1^>(\vec{Q}+\vec{q},\omega+\omega_q)+[1+n_B(\omega_q)]g_1^>(\vec{Q}+\vec{q},\omega-\omega_q)\}, \tag{2.66}$$

$$\Gamma_1=\Sigma_1^>+\Sigma_1^<=\int\frac{d^3q}{(2\pi)^3}V_q^2[g_1^<(\vec{Q}+\vec{q},\omega+\omega_q)-g_1^<(\vec{Q}+\vec{q},\omega-\omega_q)]+O\left[\frac{\omega_c}{E_f}\right]^2 , \tag{2.67}$$

$$\sigma_1(\vec{Q},\omega)=\mathrm{P}\int\frac{d\omega'}{2\pi}\frac{\Gamma_1(\vec{Q},\omega')}{\omega-\omega'} , \tag{2.68}$$

where we used the identity

$$g_1^>+g_1^<=a_1=O((\omega_c/E_F)^2) \tag{2.69}$$

from Eq. (1.16).

Although Eqs. (2.65)–(2.68) are written in the form of an electron-phonon interaction, they can be used for other types of interaction as well. For instance, we should set the phonon occupation number $n_B=0$ for impurity scattering and $\omega_q=0$ if the scattering is elastic.

Define

$$S_i=\frac{1}{m}\left[-\frac{\partial n}{\partial\omega}\right]f_i(\vec{Q},\omega) , \quad i=x,y . \tag{2.70}$$

We can write

$$\frac{\Sigma_{1i}^<}{Q_i}-n(\omega)\frac{\Gamma_{1i}}{Q_i}=\frac{1}{m}\left[-\frac{\partial n}{\partial\omega}\right]\int\frac{d^3q}{(2\pi)^3}V_q^1\frac{\vec{Q}\cdot(\vec{Q}+\vec{q})}{Q^2}\{[n_B(\omega_q)+n(\omega+\omega_q)]f_i(\vec{Q}+\vec{q},\omega+\omega_q)$$
$$+[1+n_B(\omega_q)-n(\omega-\omega_q)]f_i(\vec{Q}+\vec{q},\omega-\omega_q)\} , \tag{2.71}$$

where we have used the identity

$$\int\frac{d^3q}{(2\pi)^3}V_q^2\frac{Q_i+q_i}{Q_i}=\int\frac{d^3q}{(2\pi)^3}V_q^2\frac{\vec{Q}\cdot(\vec{Q}+\vec{q})}{Q^2} \tag{2.72}$$

for isotropic integrands. Eliminating the factor of $1/m(-\partial n/\partial\omega)$, Eqs. (2.64) become

$$f_x=\frac{A}{\Gamma_0^2+\omega_c^2}\left[\Gamma_0+\Gamma_0\int\frac{d^3q}{(2\pi)^3}V_q^2\frac{\vec{Q}\cdot(\vec{Q}+\vec{q})}{Q^2}\{[n_B(\omega_q)+n(\omega+\omega_q)]f_x(\vec{Q}+\vec{q},\omega+\omega_q)\right.$$
$$+[1+n_B(\omega_q)-n(\omega-\omega_q)]f_x(\vec{Q}+\vec{q},\omega-\omega_q)\}$$
$$+\omega_c\int\frac{d^3q}{(2\pi)^3}V_q^2\frac{\vec{Q}\cdot(\vec{Q}+\vec{q})}{Q^2}\{[n_B(\omega_q)+n(\omega+\omega_q)]f_y(\vec{Q}+\vec{q},\omega+\omega_q)$$
$$\left.+[1+n_B(\omega_q)-n(\omega-\omega_q)]f_y(\vec{Q}+\vec{q},\omega-\omega_q)\}\right], \tag{2.73a}$$

$$f_y=\frac{A}{\Gamma_0^2+\omega_c^2}\left[-\omega_c-\omega_c\int\frac{d^3q}{(2\pi)^3}V_q^2\frac{\vec{Q}\cdot(\vec{Q}+\vec{q})}{Q^2}\{[n_B(\omega_q)+n(\omega+\omega_q)]f_x(\vec{Q}+\vec{q},\omega+\omega_q)\right.$$
$$+[1+n_B(\omega_q)-n(\omega-\omega_q)]f_x(\vec{Q}+\vec{q},\omega-\omega_q)\}$$
$$+\Gamma_0\int\frac{d^3q}{(2\pi)^3}V_q^2\frac{\vec{Q}\cdot(\vec{Q}+\vec{q})}{Q^2}\{[(n_B(\omega_q)+n_B(\omega+\omega_q)]f_y(\vec{Q}+\vec{q},\omega+\omega_q)$$
$$\left.+[1+n_B(\omega_q)-n(\omega-\omega_q)]f_y(\vec{Q}+\vec{q},\omega-\omega_q)\}\right] \tag{2.73b}$$

because of the factors $-(\partial n/\partial\omega)$ in S_i and $A(\vec{Q},\omega)$ in f_i which are sharply peaked at $\omega=\mu=E_F$ and $Q^2/2m=Q_F^2/2m=\omega-\sigma_0$, respectively, we only care for the value of f_i near these regions. Therefore we take $\Gamma_0=\Gamma_0(Q_F,E_F)$ as a constant, and define

$$\int_0^\infty\frac{d\epsilon}{2\pi}f_i(\epsilon_Q,\omega)\approx\int_{-\infty}^\infty\frac{d\epsilon}{2\pi}f_i(\epsilon,\omega)=F_i(\omega)\ . \tag{2.74}$$

For spherical Fermi surfaces, the integral can be done by

$$\int\frac{d^3Q}{(2\pi)^3}Q_1^2f_i(\epsilon_Q,\omega)\approx\frac{2mQ_F^3}{3\pi}\int_0^\infty\frac{d\epsilon}{2\pi}f_i(\epsilon,\omega)=\frac{2mQ_F^3}{3\pi}F_i(\omega)\ , \tag{2.75}$$

where $Q_F=mv_F=[2m(\omega-\sigma_0)]^{1/2}$. The wave-vector integral in Eqs. (2.73) can be approximated by

$$\int\frac{d^3q}{(2\pi)^3}V_q^2\frac{\vec{Q}\cdot\vec{Q}+\vec{q}}{Q^2}\approx\int\frac{d^2q}{(2\pi)^2V_F}V_q^2\left[1-\frac{q^2}{2Q_F^2}\right]\int\frac{d\epsilon_{Q+q}}{2\pi} \tag{2.76}$$

for Q, $|\vec{Q}+\vec{q}|\sim Q_F$; and $1-q^2/2Q_F^2$ is the cosine of the scattering angle between $\vec{Q}$ and $\vec{Q}+\vec{q}$.

Finally, we make use of two McMillan functions,[13]

$$\alpha^2F(u)=\int\frac{d^2q}{(2\pi)^2V_F}V_q^2\delta(\omega_q-u)\ , \tag{2.77}$$

$$\alpha_{tr}^2F(u)=\int\frac{d^2q}{(2\pi)^2V_F}V_q^2\delta(\omega_q-u)\frac{q^2}{2A_F^2}\ , \tag{2.78}$$

so Eqs. (2.73) can be transformed into two coupled integral equations for $F_i(\omega)$, with $i=x,y$,

$$F_x=\frac{1}{\Gamma_0^2+\omega_c^2}\left[\Gamma_0+\int_0^{\omega_D}du[\alpha^2F(u)-\alpha_{tr}^2F(u)]\{[n_B(u)+n(\omega+u)][\Gamma_0F_x(\omega+u)+\omega_cF_y(\omega+u)]\right.$$
$$\left.+[1+n_B(u)-n(\omega-u)][\Gamma_0F_x(\omega-u)+\omega_cF_y(\omega-u)\}\right], \tag{2.79a}$$

$$F_y=\frac{1}{\Gamma_0^2+\omega_c^2}\left[-\omega_c+\int_0^{\omega_D}du[\alpha^2F(u)-\alpha_{tr}^2F(u)]\{[n_B(u)+n(\omega+u)](-\omega_cF_x+\Gamma_0F_y)(\omega+u)\right.$$
$$\left.+[1+n_B(u)-n(\omega-u)](-\omega_cF_x+\Gamma_0F_y)(\omega-u)\}\right]. \tag{2.79b}$$

These two equations can be solved numerically for electrical conductivity. At $\omega_c=0$, (2.79a) reduces to Holstein's equation[17] for dc conductivity, which has been used in quite a number of calculations.

The simplest example is the case of elastic scattering, where we can gain more insight into the problem. The lifetime τ_0 and the transport lifetime τ_{tr} are defined by

$$\frac{1}{\tau_0}=\Gamma_0=\int\frac{d^2q}{(2\pi)^2V_F}V_q^2\ , \tag{2.80}$$

$$\frac{1}{\tau_{tr}}=\Gamma_{tr}=\int\frac{d^2q}{(2\pi)^2V_F}V_q^2\frac{q^2}{2Q_F^2}\ . \tag{2.81}$$

Setting $\omega_q=0$ and $n_B=0$, Eqs. (2.79) reduce to a couple of algebraic equations for F_x and F_y:

$$F_x=\frac{1}{\Gamma_0^2+\omega_c^2}[\Gamma_0+(\Gamma_0-\Gamma_{tr})(\Gamma_0F_x+\omega_cF_y)]\ , \tag{2.82a}$$

$$F_y=\frac{1}{\Gamma_0^2+\omega_c^2}[-\omega_c+(\Gamma_0-\Gamma_{tr})(-\omega_cF_x+\Gamma_0F_y)]\ . \tag{2.82b}$$

This 2×2 matrix equation has the solution of

$$F_x=\frac{\Gamma_{tr}}{\omega_c^2+\Gamma_{tr}^2}=\frac{\tau_{tr}}{1+\omega_c^2\tau_{tr}^2}\ , \tag{2.83a}$$

$$F_y=\frac{-\omega_c}{\omega_c^2+\Gamma_{tr}^2}=\frac{-\omega_c\tau_{tr}^2}{1+\omega_c^2\tau_{tr}^2}\ . \tag{2.83b}$$

If we identify the density of electrons to be

$$n_0=\left.\frac{Q_F^3}{3\pi^2}\right|_{\omega=E_F}\ , \tag{2.84}$$

the conductivity components are

$$\sigma_{xx}=\frac{n_0e^2}{m}\frac{\tau_{tr}}{1+\omega_c^2\tau_{tr}^2}\ , \tag{2.85a}$$

$$\sigma_{xy}=-\frac{n_0e^2}{m}\frac{\omega_c\tau_{tr}^2}{1+\omega_c^2\tau_{tr}^2}\ , \tag{2.85b}$$

which is regarded as a simple solution to Boltzmann's equation in semiclassical theory.[2]

III. LINEAR MAGNETORESISTANCE AND HALL COEFFICIENT

The semiclassical galvanomagnetic theory predicts the transverse magnetoresistance should saturate at high magnetic field for simple metals with no open orbits on the electron Fermi surface.[2] However, experiments indicate an obvious discrepancy from the theory. Quite a number of metals exhibit linear-increasing magnetoresistance without saturating at high field,[3–6] a well-investigated example being potassium.[3,4] Generally, potassium is regarded as one of the simplest metal systems for applying semiclassical theory, because the electron Fermi surface is measured to be spherical with less than 10^{-3} deviation and the conduction electrons are believed to be nearly free. It has also been reported that the Hall coefficient of potassium has no significant dependence on the magnetic field.[18,19] This gives a gauge on the validity of possible theoretical explanations to the linear magnetoresistance.

This section proposes an explanation to the linear transverse magnetoresistance of simple metals. This is based on taking into account of the anisotropic effect[20] caused by the magnetic field. Phenomenologically, the linear increase is expressed by a Kohler slope S defined as

$$\Delta\rho/\rho=S\,|\,\omega_c\tau_{tr}\,|\ , \tag{3.1}$$

where τ_{tr} is the transport lifetime. In typical experimental situations, $|\,\omega_c\tau_{tr}\,|\gtrsim100$ can be reached so the linear effect would be very obvious if values of S can be as large as 10^{-2}–10^{-3}. To explain this effect, the theory should be able to give the values of S in that range.

The existence of a magnetic field indicates a special direction in a three-dimensional space, so anisotropic effects should exist in the transport equation. However, this effect is of the order of magnitude of ω_c/E_F. It was ignored in Sec. II, where the summation of Landau levels was replaced with energy integrals. To include the anisotropic terms, we use the next high-order approximation in Eq. (2.48), where the second term on the right is anisotropic. As a result, the spectral function in Eq. (2.26) is equivalent to its zero magnetic field form in Eq. (2.53) plus a correction term

$$\frac{1}{2}\frac{e^{-\Delta}\Gamma_0}{(\omega-\epsilon_z-\sigma_0)^2+\Gamma_0^2/4}\ . \tag{3.2}$$

For $\omega\approx E_F$ and $Q^2/2m\approx E_F$, this term has a bigger magnitude along the z direction. For a qualitative discussion, we suppose that this gives rise to an anisotropic dispersion relation for E_Q in the spectral function

$$E_Q=E_Q^0[1-P_2(\hat{Q}\cdot\hat{B})d]\ , \tag{3.3}$$

where P_2 is the second-order Legendre polynomial, and d is a positive number with the order of magnitude of $|\,\omega_c\,|/E_F$.

We assume the solution to transport equation is affected by

$$f_i(E_Q,\omega)=f_i^0(E_Q^0,\omega)[1+b_i'+b_iP_2(\hat{Q}\cdot\hat{B})]\ , \tag{3.4}$$

$$F_i(\hat{Q},\omega)=F_i^0(\omega)[1+b_i'+b_iP_2(\hat{Q}\cdot\hat{B})]\ , \tag{3.5}$$

and the components of conductivity are changed by

$$\sigma_{xx}=\sigma_{xx}^0(1+C_x)\ ,\qquad \sigma_{xy}=\sigma_{xy}^0(1+C_y)\ , \tag{3.6}$$

where C_i is proportional to d and indicates linear dependence in magnetic field.

In the case of a spherical Fermi surface, the integral in Eq. (2.75) is

$$\int\frac{d^3Q}{(2\pi)^3}Q_\perp^2f_i=\int_0^\infty\frac{dQ}{2\pi^2}Q^4f_i^0\times\int\frac{d\Omega}{4\pi}\sin^2\theta[1+b_i'+b_iP_2(\cos\theta)]\approx\frac{mQ_F^3}{3\pi^2}F_i^0(1+b_i'-b_i/5)\ . \tag{3.7}$$

This gives

$$C_i = b_i' - b_i/5 . \tag{3.8}$$

The next step is to solve b_i and b_i' in terms of d.

From Eq. (3.3), the spectral function satisfies

$$\int \frac{dE_Q^0}{2\pi} A = \frac{1}{1-dP_2} . \tag{3.9}$$

In the case of elastic scattering, the following equations can be derived from Eq. (2.64):

$$(1+b_x'+b_xP_2)F_x^0 = \frac{1}{1-dP_2}\frac{1}{\Gamma_0^2+\omega_c^2}\left[\Gamma_0\int\frac{d^3Q'}{(2\pi)^3}\frac{Q'\sin\theta'\cos\phi'}{Q\sin\theta\cos\phi}T_{QQ'}[1+b_x'+b_xP_2(\hat{Q}'\cdot\hat{B})]f_x^0(E_{Q'},\omega)\right.$$
$$\left.+\omega_c\int\frac{d^3Q'}{(2\pi)^3}\frac{Q'\sin\theta'\cos\phi'}{Q\sin\theta\cos\phi}T_{QQ'}[1+b_y'+b_yP_2(\hat{Q}'\cdot\hat{B})]f_y^0(E_Q^0,\omega)+\Gamma_0\right] , \tag{3.10a}$$

$$(1+b_y'+b_yP_2)F_y^0 = \frac{1}{1-dP_2}\frac{1}{\Gamma_0^2+\omega_c^2}\left[-\omega_c\int\frac{d^3Q'}{(2\pi)^3}\frac{Q'\sin\theta'\cos\phi'}{Q\sin\theta\cos\phi}T_{QQ'}[1+b_x'+b_xP_2(\hat{Q}'\cdot\hat{B})]f_x^0(E_{Q'},\omega)\right.$$
$$\left.+\Gamma_0\int\frac{d^3Q'}{(2\pi)^3}\frac{Q'\sin\theta'\cos\phi'}{Q\sin\theta\cos\phi}T_{QQ'}[1+b_y'+b_yP_2(\hat{Q}'\cdot\hat{B})]f_y^0(E_{Q'},\omega)-\omega_c\right] . \tag{3.10b}$$

Some remarks should be given at this point. First, we still ignore the $\partial A/\partial\epsilon_1$ terms in Eqs. (2.61) because these terms are isotropic and not interesting to this problem. Secondly, we did not include the anisotropic effect in other quantities such as Γ_0 in writing Eqs. (3.10). It can be shown that this negligence would not change the answer because the solution in (2.83) does not depend on Γ_0.

The scattering probability $T_{QQ'}$ is expanded by

$$T_{QQ'} = \sum_{l=0}^{\infty} T_l(Q')P(\hat{Q}\cdot\hat{Q}') \tag{3.11}$$

and $l \geq 3$ terms are neglected. The wave-vector integral is done as

$$\int d^3Q'(2\pi)^3\frac{Q'\sin\theta'\cos\phi'}{Q\sin\theta\cos\phi}T_{QQ'}[1+b_i'+b_iP_2(\hat{Q}'\cdot\hat{B})]f_i^0(E_{Q'},\omega)$$
$$=\frac{mQ_F}{4\pi}F_i^0\sum_l T_l(Q_F)\frac{P_l^1(\cos\theta)}{\sin\theta}\int_0^{\pi}d\theta'\sin^2\theta' P_l^1(\cos\theta')[1+b_i'+b_iP_2(\cos\theta')]\approx F_i^0(1+b_i'-b_i/5)(\Gamma_0-\Gamma_{\rm tr}) , \tag{3.12}$$

where we have used the identities

$$\int_{-1}^{1} dxP_1^1(x)P_l^1(x) = \delta_{l1}\tfrac{4}{3} , \tag{3.13}$$

$$\int_{-1}^{1} dxP_1^1(x)P_1(x)P_2(x) = -\tfrac{4}{15} , \tag{3.14}$$

and

$$\frac{mQ_F}{3\pi}T_1 = \Gamma_0-\Gamma_{\rm tr} . \tag{3.15}$$

All these turn Eqs. (3.10) into two linear matrix equations

$$(1+b_x'+b_xP_2)(1-dP_2)F_x^0 = \frac{1}{\Gamma_0^2+\omega_c^2}[\Gamma_0(\Gamma_0-\Gamma_{\rm tr})(1+b_x'-b_x/5)F_1^0+\omega_c(\Gamma_0-\Gamma_{\rm tr})(1+b_y'-b_y/5)F_2^0-\Gamma_0] , \tag{3.16a}$$

$$(1+b_y'+b_yP_2)(1-dP_2)F_y^0 = \frac{1}{\Gamma_0^2+\omega_c^2}[-\omega_c(\Gamma_0-\Gamma_{\rm tr})(1+b_x'-b_x/5)F_x^0+\Gamma_0(\Gamma_0-\Gamma_{\rm tr})(1+b_y'-b_y/5)F_y^0-\omega_c] , \tag{3.16b}$$

which have the solution of

$$b_x = b_y = d , \tag{3.17}$$

$$b_x' = -\frac{d}{5}\frac{\Gamma_0-\Gamma_{\rm tr}}{\omega_c^2+\Gamma_{\rm tr}^2}\left[\Gamma_{\rm tr}+\omega_c\frac{F_y^0}{F_x^0}\right] , \tag{3.18}$$

$$b_y' = -\frac{d}{5}\frac{\Gamma_0-\Gamma_{\rm tr}}{\omega_c^2+\Gamma_{\rm tr}^2}\left[-\omega_c\frac{F_x^0}{F_y^0}+\Gamma_{\rm tr}\frac{F_y^0}{F_x^0}\right] . \tag{3.19}$$

By substituting from Eqs. (2.83),

$$C_x = \frac{d}{5}\frac{\Gamma_0\omega_c^2+2\Gamma_{\rm tr}^3-\Gamma_0\Gamma_{\rm tr}^2}{\Gamma_{\rm tr}(\omega_c^2+\Gamma_{\rm tr}^2)} , \tag{3.20}$$

$$C_y = -\frac{d}{5}\frac{\omega_c^2 + 2\Gamma_0\Gamma_{tr} - \Gamma_{tr}^2}{\omega_c^2 + \Gamma_{tr}^2} . \quad (3.21)$$

In the high-field range $|\omega_c\tau_{tr}| \gg 1$, $|\sigma_{xy}| \gg \sigma_{xx}$,

$$C_x \approx \frac{d}{5}\frac{\Gamma_0}{\Gamma_{tr}} \sim \frac{\Gamma_0}{E_F}|\omega_c\tau_{tr}| , \quad (3.22a)$$

$$C_y \approx \frac{d}{5} \sim -\frac{\Gamma_{tr}}{E_F}|\omega_c\tau_{tr}| , \quad (3.22b)$$

$$\rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2} \approx \frac{\sigma_{xx}}{\sigma_{xy}^2} , \quad (3.23a)$$

$$\rho_{xy} = \frac{\sigma_{yx}}{\sigma_{xx}^2 + \sigma_{xy}^2} \approx -\frac{1}{\sigma_{xy}} . \quad (3.23b)$$

We argue that C_x has a much larger magnitude than C_y in the temperature range of a few kelvins. This is because, for metals, the electron lifetime τ_0 can be two or three orders of magnitude smaller than the electron transport lifetime τ_{tr}. As a numerical example, we again consider the data of potassium. Most experiments on linear magnetoresistances were carried at the helium temperature 4.2 K. A typical sample may have residual resistivity ratio equal to $\rho(293\text{ K})/\rho(4.2\text{ K}) = 5000$. By using the standard data of $\rho(293\text{ K}) = 7.19\times10^{-8}$ Ω m and $n_0 = 1.4\times10^{22}/\text{cm}^3$,[21] the transport lifetime can be estimated from

$$\tau_{tr} = \frac{m}{n_0 e^2 \rho(4.2\text{ K})} \approx 3\times10^{-11}\text{ sec} . \quad (3.24)$$

The Fermi energy of potassium is around 2 eV so the ratio Γ_{tr}/E_F is as small as 10^{-5} which essentially makes the effect of C_y negligible. On the other hand, $\Gamma_0 = 1/\tau_0$ can be determined from the measurement of the Dingle temperature[22]

$$T_D = \frac{\Gamma_0}{2\pi k_B} . \quad (3.25)$$

For very pure samples, $T_D = 0.3$ K for potassium at the temperature of 1.3 K.[23] For a rough estimation, we can assume Γ_0 has a temperature dependence of T^3.[20] The quantity Γ_0/E_F has a value of 3×10^{-3} at 4.2 K. This agrees qualitatively with the experimentally measured Kohler's slope for the transverse magnetoresistance of potassium. More quantitative discussion is not significant because both S and T_D depend greatly on sample purity and other external conditions. But we suspect the small E_F might be the reason why the linear magnetoresistance is large in potassium. On summarizing Eqs. (2.85), (3.6), (3.22), and (3.23), we conclude that the resistivity may exhibit linear magnetoresistance of the form

$$\rho_{xx} \approx \frac{m}{n_0 e^2 \tau_{tr}}(1 + S|\omega_c\tau_{tr}|) , \quad (3.26)$$

which gives an estimate on the Kohler slope $S \sim \Gamma_0/E_F$ in qualitative agreement with experimental observations, while the Hall coefficient

$$R = \frac{\rho_{yx}}{B} \quad (3.27)$$

shows no notable field dependence within the same range. Both of these results are in accord with experimental results. Therefore, we regard this theory as an acceptable explanation to this phenomenon.

The above results contain a parameter d, which we have not succeeded in calculating from first principles. Since it was introduced for describing the relative weight of the lowest Landau level in the electron's spectral function, we expect that d is basically temperature-independent. One way to check the correctness of our explanation is to see the temperature dependence of the magnetoresistance of simple metals. When the scattering is phonon limited, τ_{tr} roughly has a temperature dependence of T^{-5}, which indicates a T^{-2} dependence of the magnetoresistance at a fixed magnetic field. This estimation is too simple to apply quantitatively, but it does agree qualitatively with the experiments on copper and gold[24] in the temperature range of a few kelvins and up. For potassium, there exist accurate calculations on the temperature dependence of phonon-limited resistivity[25,26] which have an excellent agreement with the experiments.[27,28] However, we are not aware of any detailed experimental data on the temperature dependence of the magnetoresistance of potassium. Under ideal conditions, we expect that the magnetoresistance should be proportional to the ratio of Dingle temperature and the electrical resistivity. This would be a crucial test to our explanation based on the anisotropic effect.

IV. THERMAL CONDUCTIVITY AND THERMOELECTRIC EFFECTS

In this section we give the solution to the transport equation in a uniform temperature gradient field. We assume the temperature gradient points along the x axis, which is perpendicular to the magnetic field in the z direction. As was mentioned before, we only have to consider L_{xx}^{12}, L_{xy}^{12}, L_{xx}^{22}, and L_{xy}^{22}.

In semiclassical theory, both electric field and temperature gradient appear as the driving terms to Boltzmann equation,[2] so that the two effects can be treated in a similar fashion. In quantum theory, the situation is somehow different because temperature is a macroscopic quantity which cannot appear in the Hamiltonian as does the electric field. We related this difference to the diathermal current effect proposed by Jonson and Girvin.[12]

The electric Hamiltonian is

$$H(\vec{r},t,\vec{p}) = \frac{1}{2m}\left[\vec{p} - \frac{e}{2c}\vec{B}\times\vec{r}\right]^2 , \quad (4.1)$$

which contains no terms relating to temperature. Quite generally, the effect of local temperature dependence gives the following functional form for every term in the transport equation

$$f(\vec{p},\omega;\vec{R},T) = f(\vec{Q},\omega,\beta(\vec{R})) , \quad (4.2)$$

where $\vec{Q} = \vec{p} - (e/2c)\vec{B}\times\vec{R}$ is the renormalized wave vector.

In the sense of linear response, the gradient expansion of KB can be written

$$\vec{\nabla}_{\vec{R}} = \vec{\nabla}\beta \cdot \frac{\partial}{\partial\beta} + \frac{e}{2c}\vec{B}\times\vec{\nabla}_{\vec{Q}} \;. \tag{4.3}$$

The equation for the retarded Green's function is

$$\left[\omega - \frac{Q^2}{2m} + \frac{1}{8m}(\vec{B}\times\vec{\nabla}_{\vec{Q}})^2\right] g^r(\vec{Q},\omega) = 1 + \Sigma^r(\vec{Q},\omega) g^r(\vec{Q},\omega) \;, \tag{4.4}$$

which can be solved as in Eq. (2.5):

$$g^r = G^r = 2\sum_{n=0}^{\infty} \frac{(-1)^n e^{-\Delta} L_n(2\Delta)}{\omega - \epsilon_z - (n+\frac{1}{2})\omega_c - \Sigma^r} \;. \tag{4.5}$$

The transport equation is

$$\left[-\frac{\vec{Q}}{m}\cdot\vec{\nabla}\beta\frac{\partial}{\partial\beta} + \frac{e\vec{\beta}}{mC}\cdot\vec{Q}\times\vec{\nabla}_{\vec{Q}}\right] g^< = \Sigma^> g^< - \Sigma^< g^> + \vec{\nabla}\beta\left[\frac{\partial \mathrm{Re} g^r}{\partial\beta}\vec{\nabla}_{\vec{Q}}\Sigma^< - \frac{\partial\Sigma^<}{\partial\beta}\vec{\nabla}_{\vec{Q}}\mathrm{Re} g^r - \frac{\partial \mathrm{Re}\Sigma^r}{\partial\beta}\vec{\nabla}_{\vec{Q}} g^< + \frac{\partial g^<}{\partial\beta}\vec{\nabla}_{\vec{Q}}\mathrm{Re}\Sigma^r\right] + \frac{e\vec{B}}{c}\cdot(\vec{\nabla}_{\vec{Q}}\Sigma^r\times\vec{\nabla}_{\vec{Q}}\mathrm{Re} g^r - \vec{\nabla}_{\vec{Q}} g^< \times\vec{\nabla}_{\vec{Q}}\mathrm{Re}\Sigma^r) \;. \tag{4.6}$$

By expansions of the form

$$g^< = G^< + g_1^< \frac{\partial\beta}{\partial X} \;, \tag{4.7}$$

we can derive an integral equation similar to Eq. (2.16):

$$-\frac{\partial n}{\partial\beta}\left[A\left[\frac{Q_x}{m} + \frac{\partial\sigma_0}{\partial Q_x}\right] - \Gamma_0\frac{\partial \mathrm{Re} G^r}{\partial Q_x}\right] + \omega_c\left[1 + \frac{\partial\sigma_0}{\partial\epsilon_\perp}\right]\left[Q_x\frac{\partial}{\partial Q_y} - Q_y\frac{\partial}{\partial Q_x}\right] g_1^< = \Gamma_0 g_1^< + nA\Gamma_1 - A\Sigma_1^< - \omega_c\frac{\partial \mathrm{Re} G^r}{\partial\epsilon_\perp}\left[Q_x\frac{\partial}{\partial Q_y} - Q_y\frac{\partial}{\partial Q_x}\right]\Sigma_1^< - n\frac{\partial A}{\partial\epsilon_\perp}\left[Q_x\frac{\partial}{\partial Q_y} - Q_y\frac{\partial}{\partial Q_x}\right]\sigma_1 \;. \tag{4.8}$$

This equation has fewer terms than Eq. (2.20) because $g_1^r = 0$ and G^r is independent of β. By definition in Eq. (4.7), variables like $g_1^<$ should have been labeled by β since they are not the variables with the same notations in Sec. II.

Define

$$g_1^< = Q_x S_x^\beta(Q_\perp, Q_z, \omega) + Q_y S_y^\beta(Q_\perp, Q_z, \omega) \;, \tag{4.9}$$

so that

$$\Gamma_0 S_x^\beta - \omega_c\left[1 + \frac{\partial\sigma_0}{\partial\epsilon_\perp}\right] S_y^\beta = A\frac{\Sigma_{1x}^<}{Q_x} - nA\frac{\Gamma_{1x}}{Q_x} - \omega_c\frac{\partial \mathrm{Re} G^r}{\partial\epsilon_\perp}\frac{\Sigma_{1y}^<}{Q_y} + \omega_c n\frac{\partial A}{\partial\epsilon_\perp}\frac{\sigma_{1y}}{Q_y} - \frac{1}{m}\frac{\partial n}{\partial\beta}\left[A\left[1 + \frac{\partial\sigma_0}{\delta\epsilon_\perp}\right] - \Gamma_0\frac{\partial \mathrm{Re} G^r}{\partial\epsilon_\perp}\right], \tag{4.10a}$$

$$\omega_c\left[1 + \frac{\partial\sigma_0}{\partial\epsilon_\perp}\right] S_x^\beta + \Gamma_0 S_y^\beta = A\frac{\Sigma_{1y}^<}{Q_y} - nA\frac{\Gamma_{1y}}{Q_y} + \omega_c\frac{\partial \mathrm{Re} G^r}{\partial\epsilon_\perp}\frac{\Sigma_{1x}^<}{Q_x} - \omega_c n\frac{\partial A}{\partial\epsilon_\perp}\frac{\sigma_{1x}}{Q_x} \;. \tag{4.10b}$$

After dropping the derivatives of self-energies, as was done in obtaining (2.52),

$$S_x^\beta = \frac{1}{\Gamma_0^2 + \omega_c^2}\left[-\frac{\partial n}{\partial\beta}\frac{\Gamma_0}{m}\left[A - \Gamma_0\frac{\partial \mathrm{Re} G^r}{\partial\epsilon_\perp}\right] + \left[\Gamma_0 A + \omega_c^2\frac{\partial \mathrm{Re} G^r}{\partial\epsilon_\perp}\right]\frac{\Sigma_{1x}^<}{Q_x} - n\Gamma_0 A\frac{\Gamma_{1x}}{Q_x} + \omega_c\left[A - \Gamma_0\frac{\partial \mathrm{Re} G^r}{\partial\epsilon_\perp}\right]\frac{\Sigma_{1y}^<}{Q_y} - nA\omega_c\frac{\Gamma_{1y}}{Q_y} + n\omega_c\frac{\partial A}{\partial\epsilon_\perp}\left[\Gamma_0\frac{\sigma_{1y}}{Q_y} - \omega_c\frac{\sigma_{1x}}{Q_x}\right]\right] , \tag{4.11a}$$

$$S_y^\beta = \frac{1}{\Gamma_0^2 + \omega_c^2}\left[\frac{\partial n}{\partial\beta}\frac{\omega_c}{m}\left[A - \Gamma_0\frac{\partial \mathrm{Re} G^r}{\partial\epsilon_\perp}\right] - \omega_c\left[A - \Gamma_0\frac{\partial \mathrm{Re} G^r}{\partial\epsilon_\perp}\right]\frac{\Sigma_{1x}^<}{Q_x} + \omega_c nA\frac{\Gamma_{1x}}{Q_x} + \left[\Gamma_0 A + \omega_c^2\frac{\partial \mathrm{Re} G^r}{\partial\epsilon_\perp}\right]\frac{\Sigma_{1y}^<}{Q_y} - n\Gamma_0 A\frac{\Gamma_{1y}}{Q_y} - \omega_c n\frac{\partial A}{\partial\epsilon_\perp}\left[\omega_c\frac{\sigma_{1y}}{Q_y} + \Gamma_0\frac{\sigma_{1x}}{Q_x}\right]\right] . \tag{4.11b}$$

These two equations have a term-to-term correspondence with the two equations in (2.61). The only difference is $\partial n/\partial\beta$ in place of $\partial n/\partial\omega$. Since

$$\frac{\partial n}{\partial\beta}=\frac{\omega-\mu}{\beta}\frac{\partial n}{\partial\omega}\ , \tag{4.12}$$

it is easy to see

$$S_i^\beta=\frac{\omega-\mu}{\beta}S_i=-\frac{1}{m}\frac{\partial n}{\partial\beta}f_i\ , \tag{4.13}$$

where f_i is defined in Eq. (2.70).

There is no need to solve (4.11). All four transport tensors can be discussed using the solution in Sec. II. For $i=x,y$,

$$L_{xi}^{11}=\frac{1}{\beta m^2}\int\frac{d\omega}{2\pi}\left[-\frac{\partial n}{\partial\omega}\right]\int\frac{d^3Q}{(2\pi)^3}Q_\perp^2 f_i(\vec{Q},\omega)\ , \tag{4.14}$$

$$L_{xi}^{12}=L_{xi}^{21}=\frac{1}{\beta m^2}\int\frac{d\omega}{2\pi}\left[-\frac{\partial n}{\partial\omega}\right](\omega-E_F)\times\int\frac{d^3Q}{(2\pi)^3}Q_\perp^2 f_i(\vec{Q},\omega)\ , \tag{4.15}$$

$$L_{xi}^{22}=\frac{1}{\beta m^2}\int\frac{d\omega}{2\pi}\left[-\frac{\partial n}{\partial\omega}\right](\omega-E_F)^2\int\frac{d^3Q}{(2\pi)^3}Q_\perp^2 f_i(\vec{Q},\omega). \tag{4.16}$$

Equation (4.15) is the Onsager relation which has been well known in semiclassical theory. In quantum theory, this relation is also implied in the Kubo expressions for $\overleftrightarrow{L}^{12}$ and $\overleftrightarrow{L}^{21}$.[13] However, the Kubo formula for thermal response were derived by Luttinger[29] using some assumptions which are not valid if a magnetic field is present. This was discovered by Jonson and Girvin[12] recently. They derived some correction terms to $\overleftrightarrow{L}^{12}$ and $\overleftrightarrow{L}^{21}$ which violate the Onsager relation. They suggested that this effect may be important to systems of two dimensional semiconductors in extremely high fields, where the free electron density is very small compared to usual metals. By using the Euler-MacLaurin formula in Eq. (2.48), it is easy to see their correction term is of the order of magnitude of $O((\omega_c/E_F)^2)$. Our transport equation was derived without using Luttinger's assumption, and we found the Onsager relation is true only if the terms of the relative size $O((\omega_c/E_F)^2)$ are negligible [under the approximation scheme introduced in deriving Eqs. (2.61)]. For the systems of metals in our consideration, this is certainly a good approximation within the entire range of magnetic fields.

Other relations between $\overleftrightarrow{L}^{ij}$ are obtained by the following approximation:[21]

$$\begin{aligned}\int d\omega\left[-\frac{\partial n}{\partial\omega}\right]f(\omega)&=\int d\omega\left[-\frac{\partial n}{\partial\omega}\right]\times[f(E_F)+f'(E_F)(\omega-E_F)+\tfrac{1}{2}f''(E_F)(\omega-E_F)^2+\cdots]\\&\approx f(E_F)+\tfrac{1}{2}I_2 f''(E_F)+O((k_BT/E_F)^4)\ ,\end{aligned} \tag{4.17}$$

where

$$I_2=\int_{-\infty}^{\infty}d\omega\left[-\frac{\partial n}{\partial\omega}\right](\omega-E_F)^2=\frac{\pi^2}{3\beta^2}\ . \tag{4.18}$$

If we write

$$\overleftrightarrow{L}^{11}=\int d\omega\left[-\frac{\partial n}{\partial\omega}\right]\overleftrightarrow{L}^{11}(\omega)\approx\overleftrightarrow{L}^{11}(\omega=E_F)\ , \tag{4.19}$$

which gives the definition of $\overleftrightarrow{L}^{11}(\omega)$, we can see

$$\overleftrightarrow{L}^{21}=\overleftrightarrow{L}^{12}=\frac{\pi^2}{3\beta^2}\frac{\partial\overleftrightarrow{L}^{11}(\omega)}{\partial\omega}\bigg|_{\omega=E_F}=\frac{\pi^2}{3e^2\beta^3}\frac{\partial\overleftrightarrow{\sigma}(\omega)}{\partial\omega}\bigg|_{\omega=E_F} \tag{4.20}$$

and

$$\overleftrightarrow{L}^{22}=\frac{\pi^2}{3\beta^2}\overleftrightarrow{L}^{11}\ . \tag{4.21}$$

In Eq. (1.6), the thermal conductivity can be approximated by

$$\overleftrightarrow{K}=k_B\beta^2\overleftrightarrow{L}^{22}+O(k_BT/E_F)^2)\ . \tag{4.22}$$

To that accuracy, the following relation holds:

$$\overleftrightarrow{K}=\frac{\pi^2k_B^2T}{3e^2}\overleftrightarrow{\sigma} \tag{4.23}$$

which is Wiedemann-Franz Law.

Another quantity which can be derived from the transport coefficients is the thermoelectric power tensor. If there exists no electric current in the metal, any temperature gradient has to be canceled away by an electric field. By definition,

$$-\vec{E}=\overleftrightarrow{S}\cdot\vec{\nabla}T\ , \tag{4.24}$$

$$\overleftrightarrow{S}=-\frac{k_B\beta}{e}(\overleftrightarrow{L}^{11})^{-1}\cdot\overleftrightarrow{L}^{12}=-ek_B\beta^2\overleftrightarrow{\rho}\cdot\overleftrightarrow{L}^{12}\ , \tag{4.25}$$

where $\overleftrightarrow{\rho}=\overleftrightarrow{\sigma}^{-1}$ is the resistivity tensor. Eliminating $\overleftrightarrow{L}^{12}$ by Eq. (4.20), we can write the components explicitly:

$$\begin{aligned}S_{xx}&=\frac{-e}{k_BT^2}(\rho_{xx}L_{xx}^{12}+\rho_{xy}L_{yx}^{12})\\&=-\tfrac{1}{2}eL_0T\frac{\partial}{\partial\omega}\ln(\sigma_{xx}^2+\sigma_{yy}^2)\bigg|_{\omega=E_F}\ ,\end{aligned} \tag{4.26}$$

$$\begin{aligned}S_{xy}&=-\frac{e}{k_BT^2}(\rho_{xx}L_{xy}^{12}+\rho_{xy}L_{yy}^{12})\\&=-eL_0T\frac{1}{\sigma_{xx}^2+\sigma_{xy}^2}\left[\sigma_{xx}\frac{\partial\sigma_{xy}}{\partial\omega}-\sigma_{xy}\frac{\partial\sigma_{yy}}{\partial\omega}\right]\bigg|_{\omega=E_F}\ ,\end{aligned} \tag{4.27}$$

where $L_0=\pi^2k_B^2/3e^2$ is the Lorentz number.

Experiments are usually performed under the adiabatic boundary condition. There is no heat flow along y direction, which is perpendicular to the electric field. This means

$$\frac{\partial T}{\partial y}=-\frac{K_{yx}}{K_{yy}}\frac{\partial T}{\partial x}=-\frac{\sigma_{yx}}{\sigma_{xx}}\frac{\partial T}{\partial x}, \tag{4.28}$$

$$-E_x=\left[S_{xx}-S_{xy}\frac{K_{yx}}{K_{xx}}\right]\frac{\partial T}{\partial x}. \tag{4.29}$$

The adiabatic thermopower S is defined by

$$S=\frac{-E_x}{\nabla_x T}=S_{xx}-S_{xy}\frac{K_{yx}}{K_{xx}}\approx S_{xx}-S_{xy}\frac{\sigma_{yx}}{\sigma_{xx}}. \tag{4.30}$$

From Eqs. (4.26) and (4.27),

$$S=-eL_0T\left[\frac{1}{2}\frac{\partial}{\partial\omega}\ln(\sigma_{xx}^2+\sigma_{xy}^2)+\frac{\sigma_{xy}}{\sigma_{xx}^2+\sigma_{xy}^2}\frac{\partial\sigma_{xy}}{\partial\omega}-\frac{\sigma_{xy}^2}{\sigma_{xx}^2+\sigma_{xy}^2}\frac{\partial\ln\sigma_{xx}}{\partial\omega}\bigg|_{\omega=E_F}\right]. \tag{4.31}$$

In high magnetic field limit $\omega_c\tau_{\rm tr}\ll 1$, a simplified expression,

$$S=-eL_0T\left[2\frac{\partial\ln|\sigma_{xy}|}{\partial\omega}-\frac{\partial\ln\sigma_{xx}}{\partial\omega}\right]\bigg|_{\omega=E_F}, \tag{4.32}$$

is usually used.[7]

Although Eq. (4.32) is of the same form as derived from semiclassical theory, its meaning in quantum theory is quite different. The importance of quantum effects can be seen from the following discussion on the electron-phonon mass enhancement effect.

The electron-phonon mass enhancement factor is defined by[13]

$$1+\lambda=1-\frac{\partial\sigma_0}{\partial\omega}\bigg|_{\omega=E_F}. \tag{4.33}$$

Calculations on this quantity have been done for quite a number of metals.[30] However, it is not always possible to see this effect by comparing calculations to experimental data or thermopower for several reasons. Firstly, the experimentally measured thermopower consists of two components, one from electron diffusion and another from the phonon drag effect; the latter is not related to the electron mass enhancement.[31] Secondly, the energy-dependent conductivity may have significant dependence on the intrinsic properties of individual samples. Thirdly, theoretical calculations based on the detailed Fermi-surface geometry have not been done for all metals so it is not always possible to make quantitative comparisons between theory and experiments.

The ambiguities were resolved in the experiment by Averback and Wagner.[7] They measured both the temperature and magnetic field dependences of the thermopower on a number of Al specimens. The data can be fitted into an equation

$$S(B)=a(B)T+b(B)T^3. \tag{4.34}$$

According to this temperature dependence, the first term is identified as the electron diffusion component S_d by the Nordheim-Gorter rule.[32] The observed $a(B)$ saturates at high magnetic field ($\omega_c\tau_{\rm tr}\geq 10$) so that $S_d(B\to\infty)$ can be well defined. Although $S_d(B)$ varies from sample to sample, the quantity $\Delta S_d=S_d(B\to\infty)-S_d(B=0)$ is approximately the same for all specimens.

$$S_d=(2.2\pm0.2)\times10^{-8}T\ \text{V/K}^2. \tag{4.35}$$

The Fermi surface of Al is well known. As an uncompensated metal with no open orbits, the high-field off-diagonal component of the conductivity tensor is[7]

$$\sigma_{xy}(\omega)=\frac{(n_e-n_h)ec}{B}+O\left[\frac{1}{B^2}\right], \tag{4.36}$$

where n_e and n_h are densities of electrons and holes, respectively.

From Eqs. (4.32) and (4.36), Averback and Wagner developed an expression for (4.35),

$$\Delta S_d=\frac{2\gamma T}{n_e-n_h}+eL_0T\frac{\partial\ln\sigma_{xx}(B)\sigma_{xx}(B=0)}{\partial\omega}\bigg|_{\omega=E_F}, \tag{4.37}$$

where γ is the electron specific-heat constant. This expression can be calculated numerically from semiclassical theory; they derived a numerical expression

$$\Delta S_d=1.6\times10^{-8}\ \text{V/K}^2, \tag{4.38}$$

which is 30% smaller than the experimental value.

Opsal *et al.* first explained this discrepancy in terms of the electron-phonon mass enhancement effect.[8] They used the momentum-dependent relaxation-time assumption from semiclassical theory, and semiempirically argued that the electron velocity and relaxation time should be appropriately normalized by the enhancement factor. As a result, both the energy-dependent conductivity and the specific-heat constant γ should have a normalization factor $1+\lambda$. Equation (4.37) can be explained with the value of $1+\lambda=1.45$, which agrees very well with previous calculations[33–35] as well as the experimental measurement on the effective mass;[36] the results all lie between 1.4 and 1.5. Since then, there have been many discussions on this topic, but not all of them are consistent with the real situation. Most calculations were done under the quasiparticle approximation and did not take into account the magnetic field.[10,37–40] There had been some ambiguity about whether all components of $\overleftrightarrow{L}^{12}$ should have the same enhancement factor or not. Lyo predicted that there should be an enhancement factor to the diagonal components of $\overleftrightarrow{L}^{12}$, but not to the nondiagonal components.[10] This conclusion does not agree with the later experiment of Thaler *et al.*,[11] where the components of $\overleftrightarrow{L}^{12}$ were measured directly. They observed a same enhancement factor to the semiclassical expression[41]

$$L_{xy}^{12}=\frac{\pi^2k_B^2N(E_F)}{3eBT}. \tag{4.39}$$

A treatment from quantum theory was given by Hänsch and Mahan;[9] they solved the electrical conductivity in the high magnetic field limit and used Eq. (4.32) to show the presence of the phonon-enhancement factor. In this work, we derived, for the first time, the thermal transport equation, and solved it in the entire range of magnetic fields. In our solution scheme, we rederived the well-known relations between thermal transport and electric transport. $\overleftrightarrow{L}^{12}$ can be solved directly in terms of the results in Sec. II. The effect of mass enhancement can be discussed as follows. For a magnetic field of arbitrary magnitude, the spectral function is peaked at $E_0=\omega-\sigma_0$, which is different from the δ function at $E_Q=\omega$ as is the semiclassical result.

Under the approximation of (2.75), the only notable difference from semiclassical theory is to substitute $\omega-\sigma_0$ for ω, as is seen from $Q_F=[2m(\omega-\sigma_0)^{1/2}]$. This means using

$$\overleftrightarrow{\sigma}(\omega)\,|_{\text{quantum}}=\overleftrightarrow{\sigma}(\omega-\sigma_0)\,|_{\text{semiclassical}} \tag{4.40}$$

so that

$$\overleftrightarrow{L}^{12}\,|_{\text{quantum}}=(1+\lambda)\overleftrightarrow{L}^{12}\,|_{\text{semiclassical}} \tag{4.41}$$

in the derivation from (4.24) to (4.32). This effect is negligible in electrical conductivity $\overleftrightarrow{\sigma}$ and thermoconductivity $\overleftrightarrow{K}$ since $\sigma_0 \ll E_F$. However, in taking the energy derivatives of (4.31) and (4.32), there should be a correction factor of

$$1+\lambda=1-\left.\frac{\partial\sigma_0}{\partial\omega}\right|_{\omega=E_F}$$

to the semiclassical results. This effect should exist not only in the thermopower S, but also in all components of $\overleftrightarrow{L}^{12}$. This explains the experimental result of Thaler *et al.*[11]

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[1]R. Kubo, J. Phys. Soc. Jpn. **12**, 570 (1957).

[2]J. M. Ziman, *Principles of the Theory of Solids* (Cambridge University Press, Cambridge, 1960).

[3]J. S. Lass, J. Phys. C **3**, 1926 (1970).

[4]A. M. Simpson, J. Phys. F. **3**, 1471 (1973).

[5]F. R. Fickett, Phys. Rev. B **3**, 1941 (1971).

[6]J. E. Huffman, M. L. Snodgrass, and F. J. Blatt, Phys. Rev. B **23**, 483 (1981).

[7]R. S. Averback and D. K. Wagner, Solid State Commun. **11**, 1109 (1972).

[8]J. L. Opsal, B. J. Thaler, and J. Bass, Phys. Rev. Lett. **36**, 1211 (1976).

[9]W. Hänsch and G. D. Mahan, Phys. Rev. B **28**, 1886 (1983).

[10]S. K. Lyo, Phys. Rev. Lett. **39**, 363 (1977).

[11]B. J. Thaler, R. Fletcher, and J. Bass, J. Phys. F **8**, 131 (1978).

[12]M. Jonson and S. M. Girvin, Phys. Rev. B **29**, 1939 (1984).

[13]G. D. Mahan, *Many Particle Physics* (Plenum, New York, 1981).

[14]L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).

[15]G. D. Mahan and W. Hänsch, J. Phys. F **13**, L47 (1983).

[16]G. D. Mahan, J. Phys. F **14**, 941 (1984).

[17]T. Holstein, Ann. Phys. (Leipzig) **29**, 410 (1964).

[18]D. E. Chimenti and B. W. Maxfield, Phys. Rev. B **7**, 3501 (1973).

[19]R. Fletcher, Can. J. Phys. **60**, 679 (1982).

[20]G. D. Mahan, J. Phys. F **13**, L257 (1983).

[21]N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders, Philadelphia, 1976).

[22]D. Shoenberg, Phys. Kondens. Mater. **9**, 1 (1969).

[23]Z. Altounian, C. Verge, and W. R. Datars, J. Phys. F **8**, 75 (1978).

[24]J. E. Huffman, M. L. Snodgrass, and F. J. Blatt, Phys. Rev. B **23**, 484 (1981).

[25]R. C. Shulka and R. Taylor, J. Phys. F **6**, 531 (1976).

[26]R. Taylor, C. R. Leavens, and R. C. Shukla, Solid State Commun. **19**, 809 (1976).

[27]D. Gugan, Proc. R. Soc. London, Ser. A **325**, 223 (1971).

[28]J. W. Ekin and B. W. Maxfield, Phys. Rev. B **4**, 4215 (1971).

[29]J. M. Luttinger, Phys. Rev. **135**, A1505 (1964).

[30]G. Grimvall, Phys. Scr. **14**, 63 (1976).

[31]J. L. Opsal, J. Phys. F **7**, 2349 (1977).

[32]A. M. Guenault, J. Phys. F **2**, 316 (1972).

[33]J. C. Swihart, D. J. Scalapino, and Y. Wada, Phys. Rev. Lett. **14**, 106 (1965).

[34]J. C. Swihart, D. J. Scalapino, and Y. Wada, Low Temp. Phys. **LT9**, 607 (1965).

[35]N. W. Ashcroft and J. W. Wilkins, Phys. Lett. **14**, 285 (1965).

[36]B. J. Thaler and J. Bass, J. Phys. F **5**, 1554 (1975).

[37]A. Vilenkin and P. L. Taylor, Phys. Rev. B **18**, 5280 (1978).

[38]Y. A. Ono and P. L. Taylor, Phys. Rev. B **22**, 1109 (1980).

[39]J. J. Ksempasky and A. Schmid, J. Low Temp. Phys. **34**, 107 (1979).

[40]A. Vilenkin, Phys. Rev. B **22**, 4085 (1980).

[41]R. J. Douglas and R. Fletcher, Philos. Mag. **32**, 73 (1975).