Interaction-driven metal-insulator transitions in disordered fermion systems

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We study the effects of electron-electron interactions in disordered metals in and close to two dimensions (2D). We consider physical situations in which localization effects are suppressed. The field-theoretical renormalization-group (RG) calculation performed recently by Finkelstein is interpreted and rederived in terms of perturbative results. Surprisingly, except for the density of states, the scaling behavior is independent of the interaction range. We further extend the model to several new universality classes. In the presence of a strong magnetic field the metal is unstable in 2D and undergoes a metal-insulator transition in $d=2+\epsilon$. The conductivity exponent, defined by $\sigma\sim(n-n_c)^{\mu}$, is universal with $\mu=1+O(\epsilon)$ but $N(E_F)$ depends not only on the range of the interaction but also on its strength for short-ranged interactions. In 2D the conductivity has a universal temperature dependence $[\delta\sigma(T)=\sigma_N(2-2\ln2)\ln(T\tau),\ \sigma_N=e^2/2\pi^2\hbar]$ if the interaction is Coulombic. If magnetic impurities (or strong spin-orbit scattering with a weak magnetic field) are present instead, the noninteracting fixed point is stable for short-ranged interactions ($\mu=\frac{1}{2}$). For the Coulomb interaction the interaction is relevant and drives a metal-insulator transition in $d=2+\epsilon$ with universal critical properties ($\mu=1$). In 2D the conductivity also has a universal temperature dependence $[\delta\sigma(T)=\sigma_N\ln(T\tau)]$. We also discuss the behavior of the dielectric constant on the insulator side and the frequency (temperature) dependence of the conductivity at criticality. Remarks are made on the relationship of the above to experiments.

I. INTRODUCTION

In the past few years, a great deal of progress has been made towards understanding the behavior of electrons in a random potential. A scaling theory for the localization problem, i.e., the behavior of a single electron in a random potential, was formulated[1] and justified by a mapping to a field theory of coupled matrices.[2] On the other hand, perturbation theory in the interaction strength between electrons shows that in the presence of disorder, a strong deviation from Fermi-liquid theory occurs and logarithmic corrections to the conductivity and single-particle density of states in two dimensions are discovered.[3,4] It is clearly important to develop a scaling theory which effectively sums these logarithmic series and permits one to discuss the strong-coupling region of the metal-insulator transition. Attempts to construct a scaling theory by the brute-force calculation of the perturbation theory[5,6] have not been completely successful. Recently, an important advance was made by Finkelstein,[7] who produced a field-theory mapping of the interacting disordered problem. His theory explicitly suppressed the maximally crossed diagrams that give rise to logarithms in the localization problem and can be considered as a scaling theory for the "pure" interaction problem. He concluded that in two dimensions, the resistance scales to weak coupling, ending up in a perfect conductor at zero temperature.

The physics of the Finkelstein solutions was much clarified in a recent paper by Altshuler and Aronov,[8] who pointed out that it is very useful to decompose the density-density fluctuation into singlet and triplet spin channels. They further pointed out that in the presence of spin-orbit or spin-flip scattering, the Finkelstein solution for the resistance scales to strong coupling, so that a metal-insulator transition is possible in $d>2$.

In this work we generalize Finkelstein's solution in several directions. First, Finkelstein has considered the case of long-range Coulomb interactions. We extend his considerations to short-range interactions only, which turns out to provide an interesting contrast with the long-range case. Experimental systems where a short-range—interaction model may be realized include the ^{3}He Fermi liquid adsorbed on a disodered substrate and the (MOSFET) metal-oxide-semiconductor field-effect transistor system with a sufficiently thin oxide so that the image charge of the metal layer cuts off the Coulomb interaction.[9] Second, we write the results for the spin-flip—scattering case first proposed by Altshuler and Aronov.[8] Third, we consider the application of a large magnetic field which causes the spin splitting of the electron states. This last case is of special interest because the field also suppresses the maximally crossed diagrams, so that we have the solution of a model for a realistic physical problem. In particular, it may be relevant to experiments performed in a strong magnetic field.[10] However, before developing these extensions, we present in the next section a pedagogical discussion of Finkelstein's theory. Finkelstein actually proves the renormalizability of his theory, and his work is very complete and difficult. With the benefit of the insight gained from Finkelstein's analysis, it is possible to perform simple perturbation theory and extract the scaling equations from the leading logarithmic singularity. We also clarify some subtle points in Finkelstein's renormalization procedure.

II. A PEDAGOGICAL DISCUSSION OF FINKELSTEIN'S SOLUTION

A. Effective diffusion propagator

We consider an interacting fermion system in a random potential. From perturbation theory we learned that the logarithmic singularity originates from the fact that density fluctuations in a random medium are diffusive, i.e., the particle-hole propagator $L_0^{-+}(\vec{q},\Omega)$ represented by the infinite series shown in Fig. 1 has a diffusion pole when the electron and hole lines are on opposite sides of the Fermi surface, i.e., when $\epsilon_n(\epsilon_n+\Omega_m)<0$,

$$L_0^{-+}(\vec{q},\Omega)=\langle u^2\rangle_{\text{av}}[\tau(\,|\,\Omega_m\,|\,+D_0q^2)]^{-1}\,, \tag{2.1}$$

where $\langle u^2\rangle_{\text{av}}$ is the average impurity-scattering strength, D_0 is the diffusion constant, and the scattering rate is given by $\tau^{-1}=2\pi N_0\langle u^2\rangle_{\text{av}}$ where N_0 is the single-spin density of states. $\epsilon_n=\pi T(2n+1)$ and $\Omega_m=2\pi Tm$ are the Matsubara frequencies at temperature T. The zero-temperature expression of L_0^{-+} is obtained substituting $|\,\Omega_m\,|$ with $-i\Omega$ in Eq. (2.1). For convenience we will drop the factor $u^2\tau^{-1}$ from the definition of L_0^{-+} from now on. In Finkelstein's theory, the disorder scattering is assumed to be weak, so that the theory is evaluated to lowest order in

$$t=1/(2\pi)^2N_0D_0\,. \tag{2.2}$$

However, in the interaction between electrons,

$$H_I=\sum_{\vec{k},\vec{p},\vec{q}}V(q)c^\dagger_{\vec{p}-\vec{q}}c^\dagger_{\vec{k}+\vec{q}}c_{\vec{k}}c_{\vec{p}}\,, \tag{2.3}$$

no weak-coupling assumption is made. For simplicity, we shall present the theory for the short-range potential only. The extension to include the long-range Coulomb interaction is straightforward. In this section we perform a diagrammatic expansion for the polarization function $\pi(\vec{q},\Omega)$ valid to lowest order in $t\ln(\,|\,\Omega\,|\,\tau)$, and, following Finkelstein, to all order in V. Out of the perturbation theory, guided by the structure of the field theory, scaling equations in agreement with Finkelstein's can be constructed.

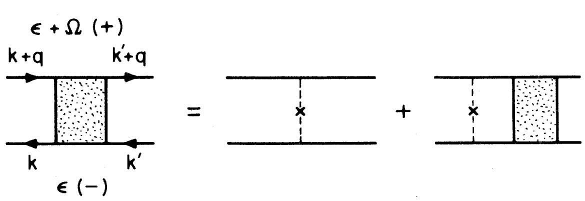


FIG. 1. Particle-hole propagator $L_0^{-+}(q,\Omega)$ under impurity scatterings using the "ladder approximation."

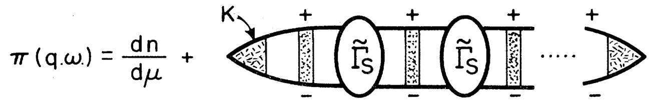


FIG. 2. General structure of the polarization bubble, explicitly showing the separation into static and dynamic parts.

The evaluation of $\pi(\vec{q},\Omega)$ is broken down into several stages. By general principles, its static limit is given by the thermodynamic density of states $dn/d\mu$,

$$\pi(\vec{q}\rightarrow\vec{0},\ \Omega=0)=\frac{dn}{d\mu}\,, \tag{2.4}$$

where n is the electron density and μ is the chemical potential. Furthermore, $dn/d\mu$ is not expected to have any logarithmic singularity because the physics of the present problem must be a smooth function of μ. Thus, we concentrate on the dynamic part of $\pi(\vec{q},\Omega)$. This can be written as an infinite series as shown in Fig. 2 involving vertex correction K, an effective interaction amplitude $\widetilde{\Gamma}_s$ to be discussed later, and a modified particle-hole propagator $L^{-+}(\vec{q},\Omega)$. This propagator includes modification by the interaction terms for which the initial energy of the electron or hole line is equal to the final one. In terms of these functions, we obtain

$$\pi(\vec{q},\Omega)=\frac{dn}{d\mu}-\frac{2N_0K^2\Omega L^{-+}}{1-2\widetilde{\Gamma}_s\Omega L^{-+}}\,. \tag{2.5}$$

Our first step is to analyze the propagator L^{-+}. To lowest order in V the diagrams are shown in Fig. 3 in addition to a set where the dressing of the particle and hole lines are interchanged. We also have the Hartree version of these diagrams, which are proportional to $2F$ where the factor 2 is for spin degeneracy and the constant F was

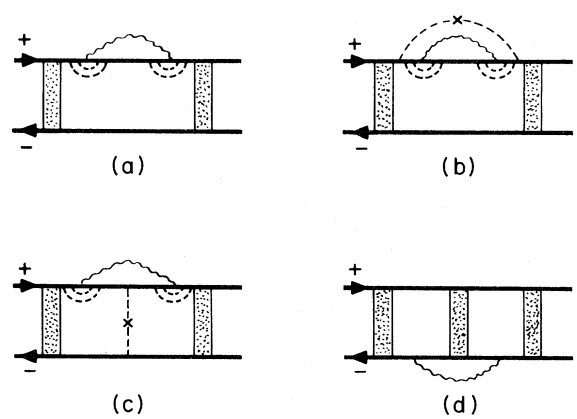


FIG. 3. Singular corrections to L^{-+} due to interactions. Four more graphs are obtained from interchanging electron and hole lines. Also present but not shown are the corresponding Hartree diagrams.

given in Ref. 3. Clearly, we can perform an infinite resummation and consider Fig. 3, as a self-energy correction to the propagator L^{-+}. Denoting the sum of Figs. 3(a)—3(c) and its Hartree counterpart by $\Sigma_{\rm abc}$, we note that the sum is easily done by recognizing that the integration over the electron Green's functions is just that given by Hikami,[11]

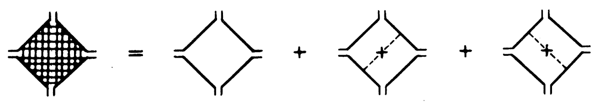


FIG. 4. Hikami four-point vertex, giving Eq. (2.6) for the total contributions of Figs. 3(a)—3(c).

$$\Sigma_{\rm abc}=-2T\sum_{\omega>\epsilon+\Omega}\sum_{\vec{k}}\frac{V(q=0)(1-2F)}{(D_0k^2+\omega_n)^2}[D_0q^2+\Omega_m+D_0k^2+\omega_n]\ , \tag{2.6}$$

where the [ ] is from the Hikami vertex shown in Fig. 4. From Fig. 3(d) we have

$$\Sigma_{\rm d}=2T\sum_{\omega>\epsilon}\sum_{\vec{k}}\frac{V(q=0)(1-2F)}{D_0(\vec{k}+\vec{q})^2+\Omega_m+\omega_n}\ . \tag{2.7}$$

The total "self-energy" correction is given by

$$\Sigma=\Sigma_{\rm abc}+\Sigma_{\rm d}\ . \tag{2.8}$$

We note that in the limit $\vec{q}=\vec{0}$, $\Omega=0$, Eqs. (2.6) and (2.7) exactly cancel, and thus Σ should be linear in Ω and q^2. To exhibit this cancellation, we anticipate that later we shall replace $V(q=0)$ by $V(\vec{q},\omega)$. Thus, we avoid any integration by parts and instead rewrite Eq. (2.7) as

$$\begin{aligned}\Sigma_{\rm d}=&\,2T\sum_{\epsilon+\Omega>\omega>\epsilon}\sum_{\vec{k}}\frac{V(q=0)(1-2F)}{D_0(\vec{k}+\vec{q})^2+\Omega_m+\omega_n}\\&+2T\sum_{\omega>\epsilon+\Omega}\sum_{\vec{k}}V(q=0)(1-2F)\left[\frac{1}{D_0k^2+\omega_n}+\frac{1}{D_0(\vec{k}+\vec{q})^2+\omega_n+\Omega}-\frac{1}{D_0k^2+\omega_n}\right]\ .\end{aligned} \tag{2.9}$$

The first term in Eq. (2.9) yields the following contribution:

$$2T\sum_{\epsilon+\Omega>\omega>\epsilon}\sum_{\vec{k}}\frac{V(q=0)(1-2F)}{D_0(\vec{k}+\vec{q})^2+\omega_n+\Omega_m}\approx\frac{2\Omega}{2\pi}\sum_{\vec{k}}V(q=0)\frac{1-2F}{D_0k^2+\bar{\omega}}\ , \tag{2.10}$$

where $\bar{\omega}$ is of the order of Ω_m. The first term in the large square brackets of Eq. (2.9) cancels the last two terms of Eq. (2.6). The remaining terms in Eq. (2.9) can be expanded to leading powers of Ω and q^2. Together, we have

$$\Sigma=-(D_0q^2+\Omega_n)2I_1+D_0q^2I_2+\Omega_nI_3\ , \tag{2.11}$$

where

$$I_1=2T\sum_{\omega>\Omega/2}\sum_{\vec{k}}\frac{V(0)(1-2F)}{(D_0k^2+\omega_n)^2}\ , \tag{2.12}$$

$$I_2=\frac{8}{d}T\sum_{\omega>\Omega/2}\sum_{\vec{k}}\frac{V(0)(1-2F)Dk^2}{(D_0k^2+\omega_n)^3}\ , \tag{2.13}$$

$$I_3=\frac{2}{2\pi}\sum_{\vec{k}}\frac{V(0)(1-2F)}{D_0k^2+\Omega/2}\ . \tag{2.14}$$

Note that I_3 derives from Eq. (2.10) and involves no energy integration. In these equations we have used the fact that $0<\epsilon+\Omega<\Omega$ and replaced the limit of integration in Eqs. (2.12) and (2.13) by the typical value $\epsilon+\Omega\approx\Omega/2$, and in Eq. (2.14) we replaced $\bar{\omega}\approx\Omega/2$. In two dimensions, the integrals I_1, I_2, and I_3 are all logarithmically divergent and this replacement should be permissible. The modified particle-hole propagator $L^{-+}(\vec{q},\Omega)$ is now written as

$$\begin{aligned}L^{-+}(\vec{q},\Omega_n)&=[(L_0^{-+})^{-1}-\Sigma]^{-1}\\&=[(D_0q^2+|\Omega_n|)(1+2I_1)\\&\qquad-D_0q^2I_2-|\Omega_n|I_3]^{-1}\ .\end{aligned} \tag{2.15}$$

Within the accuracy of our calculation, this can be written in the following suggestive form:

$$L^{-+}(\vec{q},\Omega_n)=\zeta^2/(D'q^2+z|\Omega_n|)\ . \tag{2.16}$$

where

$$\zeta=1-I_1\ , \tag{2.17}$$

$$D'=D_0(1-I_2)\ , \tag{2.18}$$

$$z=1-I_3\ . \tag{2.19}$$

Such a separation is, of course, completely arbitrary, but can be motivated as follows. Firstly, we note that while I_1, I_2, and I_3 are all logarithmically divergent, they are really different integrals and it is natural that they should renormalize different quantities. Secondly, if we consider an electron system interacting via a Coulomb potential, the potential $V(q)$ must be replaced by the dynamically screened Coulomb potential

$$V_C(\vec{q},\omega)=V_c(q)/[1+V_c(q)\pi(\vec{q},\omega)]\ . \tag{2.20}$$

In two dimensions, this takes the form

$$V_C(\vec{q},\omega)=\frac{2\pi e^2}{|\vec{q}|}\Big/ 1+\frac{2\pi e^2}{|\vec{q}|}\frac{dn}{d\mu}\frac{Dq^2}{Dq^2+|\omega|} . \tag{2.21}$$

Then we note that while I_2 and I_3 are proportional (at $T=0$) to $t\ln(|\Omega|\tau)$, I_1 is proportional to $t\ln^2(|\Omega|\tau)$ and is, in fact, simply the correction to the single-particle density of states.[3] It is therefore reasonable to treat this $t\ln^2(|\Omega|\tau)$ singularity separately as a "wave-function" renormalization and maintain this procedure for the case of short-range forces also. Once this choice is made, D' is simply the perturbation-theory correction to the conductivity,[3] and z given in Eq. (2.19) is in agreement with the frequency-renormalization factor introduced by Finkelstein. We should point out that in Finkelstein's approach, the wave-function renormalization ζ does not appear explicitly. A detailed discussion of the two approaches is given in Appendix A.

B. General form of the renormalization parameters

Let us return to Eq. (2.16). Once we recognize that singular behavior comes from electrons and holes on opposite sides of the Fermi surface with small total q and ω, it is easy to generalize the above discussion formally to all orders in the coupling constant by introducing the static amplitudes $\tilde{\Gamma}_1^{(0)}$ and $\tilde{\Gamma}_2^{(0)}$. These amplitudes include interactions to all powers in $N_0V(q)$ which do not involve L^{-+}. $\tilde{\Gamma}_1^{(0)}$ and $\tilde{\Gamma}_2^{(0)}$ are defined in terms of their spin structures in Fig. 5, and we can simply replace $N_0V(0)$ by $\tilde{\Gamma}_1^{(0)}$ and $N_0FV(0)$ by $\tilde{\Gamma}_2^{(0)}$ in Eqs. (2.12)—(2.14). Furthermore, a ladder summation of $\tilde{\Gamma}^{(0)}$ does not introduce higher powers in $t\ln(|\Omega|\tau)$. This leads to a replacement of $\tilde{\Gamma}^{(0)}$ by the dynamic interaction $\tilde{U}(q,\omega_n)$ as discussed by Finkelstein and which we shall discuss later. For simplicity, we shall carry on our discussion in terms of "skeleton" graphs involving $\tilde{\Gamma}^{(0)}$, with the understanding that $\tilde{\Gamma}^{(0)}$ can be replaced by $\tilde{U}(q,\omega)$ afterwards, in the integrals for the evaluation of the physical quantities.

It is useful at this point to introduce the singlet (short-range) and the triplet amplitudes,

$$\tilde{\Gamma}_s^{(0)}=\tilde{\Gamma}_1^{(0)}-\tfrac{1}{2}\tilde{\Gamma}_2^{(0)} , \tag{2.22a}$$

$$\tilde{\Gamma}_t^{(0)}=-\tfrac{1}{2}\tilde{\Gamma}_2^{(0)} . \tag{2.22b}$$

The first refers to the spin singlet for the electron-hole pair, the second refers to the corresponding triplet channel. This separation is useful because in Fig. 2, which describes density fluctuations, the total spin of the particle-hole pair is conserved and equal to 0; therefore the singlet amplitude $\tilde{\Gamma}_s$ appears in Fig. 2 and in Eq. (2.5). In the case of the long-range (LR) Coulomb interaction, we further separate the singlet amplitude into

$$\tilde{\Gamma}_s^{(0)\mathrm{LR}}=\tilde{\Gamma}_0^{(0)}+\tilde{\Gamma}_s^{(0)} , \tag{2.22'}$$

where $\tilde{\Gamma}_s^{(0)}$ is defined as the part of the amplitude which cannot be separated by cutting a Coulomb line. Since the polarization function is irreducible for cutting a Coulomb line, it is $\tilde{\Gamma}_s^{(0)}$, which enters Fig. 2 and Eq. (2.5) in lowest order.

Returning to Eqs. (2.5) and (2.16); the polarization function now takes the form

$$\pi(q,\Omega_m)=dn/d\mu-\frac{2N_0K^2\zeta^2\Omega_m}{D'q^2+(z-2\tilde{\Gamma}_s\zeta^2)\Omega_m} . \tag{2.23}$$

While this equation satisfies the $\Omega=0$ limit by design, it must also satisfy the requirement that

$$\pi(\vec{q}=\vec{0},\ \Omega\to 0)=0 . \tag{2.24}$$

This requires that the singularities in $K^2\zeta^2$ and in $z-2\tilde{\Gamma}_s\zeta^2$ cancel each other, i.e.,

$$dn/d\mu(z-2\tilde{\Gamma}_s\zeta^2)=2N_0K^2\zeta^2 . \tag{2.25}$$

As we shall demonstrate in perturbation theory in $t\ln$, it turns out that each of the two terms are separately nonsingular, i.e., each term turns out to be an invariant.

From Eq. (2.25) it follows that π acquires the characteristic diffusion form

$$\pi(q,\Omega_m)=\frac{dn/d\mu D''q^2}{(\Omega_m+D''q^2)} , \tag{2.26a}$$

where

$$D''=D'/(z-2\tilde{\Gamma}_s\zeta^2) , \tag{2.26b}$$

with the conductivity σ given by

$$\sigma=\lim_{\Omega\to 0}\lim_{k\to 0}\frac{\Omega}{k^2}\pi(k,\Omega)=\frac{dn}{d\mu}D'' . \tag{2.27}$$

This last equation specifies the Einstein relation. For the present case, D' appearing in L^{-+} is proportional to the true diffusion constant provided $z-2\tilde{\Gamma}_s\zeta^2$ is an invariant as follows from perturbation theory.

C. Renormalized effective couplings

Returning to our perturbative analysis, the diagrams which satisfy the requirement (2.24) were already identified in Ref. 6 to lowest order in $Vt\ln$. Thus, it is straightforward to demonstrate the cancellation in $K^2\zeta^2$. The singular correction to K is shown in Fig. 6. We find that

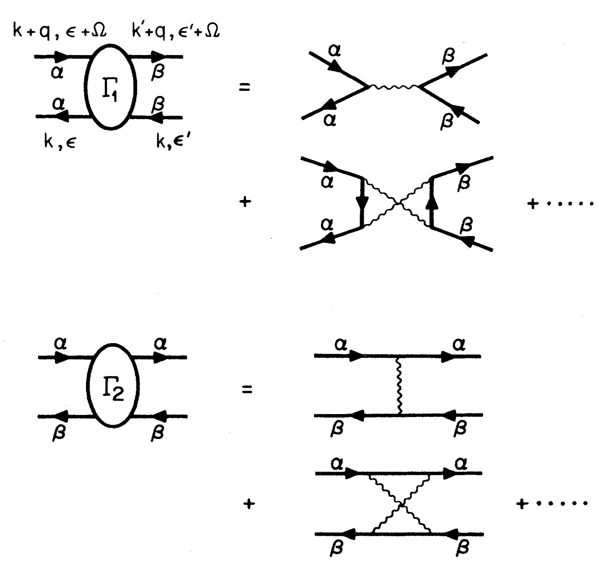


FIG. 5. Graphs contributing to the total static amplitudes $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ to zeroth order in t. Note the spin structures.

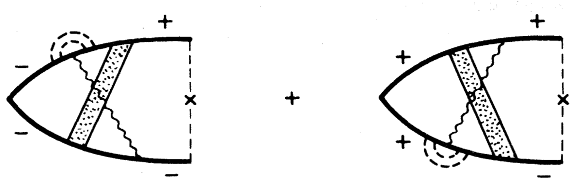


FIG. 6. Order t ln corrections to the vertex K.

$$K/K_0=1+I_1\ , \tag{2.28}$$

where $K_0=1-2\tilde{\Gamma}_1^{(0)}+\tilde{\Gamma}_2^{(0)}$ is the nonsingular Fermi-liquid value. Thus, we conclude that $K\zeta=K_0$ is nonsingular to lowest order in t ln.

Next we study the combination $z-2\tilde{\Gamma}_s\zeta^2$ appearing in Eq. (2.23). $\tilde{\Gamma}_s$ is defined as the effective singlet amplitude between electron and hole lines with positive and negative energy, respectively. To zeroth order in t ln, it is given by Eq. (2.22). Now we have to consider corrections to $\tilde{\Gamma}_s$ to first order in t ln. For this purpose it is easier to consider the more general problem of the renormalization of the amplitudes $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ separately. It is convenient to define

$$\Gamma_1=\tilde{\Gamma}_1\zeta^2 \tag{2.29}$$

and

$$\Gamma_2=\tilde{\Gamma}_2\zeta^2\ . \tag{2.30}$$

We shall see that Γ_1 and Γ_2 correspond to the same quantities introduced by Finkelstein.

Let us consider logarithmic corrections to the amplitude $\tilde{\Gamma}$. One class of diagrams which renormalizes $\tilde{\Gamma}_2$ is shown in Fig. 7 and similar diagrams renormalize $\tilde{\Gamma}_1$. Note that this diagram is first order in $\tilde{\Gamma}^{(0)}$ and the energy variables are fixed by the external legs. Thus, no energy integration is involved in the logarithmic integral. Denoting its correction to $\tilde{\Gamma}_i$ by $\delta\tilde{\Gamma}_i^{(7)}$, we find

$$\delta\tilde{\Gamma}_1^{(7)}=-t\tilde{\Gamma}_2\ln(T\tau) \tag{2.31}$$

and

$$\delta\tilde{\Gamma}_2^{(7)}=-t\tilde{\Gamma}_1\ln(T\tau)\ . \tag{2.32}$$

We identify the interaction line in Fig. 7 as $\tilde{\Gamma}_1$ because it is forced to carry a small momentum. As pointed out in Ref. 6, this class of diagrams were left out of the perturbation expansion in Ref. 5. Inclusion of these processes leads to many contributions of order $Vt^2\ln^2(T\tau)$ and $V^2t^2\ln^2(T\tau)$. These have recently been calculated.[12] Equations (2.31) and (2.32) are exactly what is needed to cancel the singularity in z [Eq. (2.19)] in the combination $z-2\Gamma_s$, again to first order in $t\ln(T\tau)$.

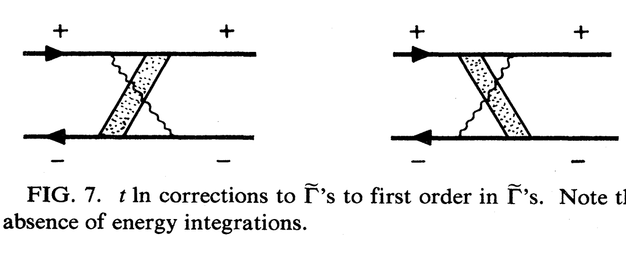


FIG. 7. t ln corrections to $\tilde{\Gamma}$'s to first order in $\tilde{\Gamma}$'s. Note the absence of energy integrations.

Additional diagrams which renormalize $\tilde{\Gamma}_2$ are shown in Fig. 8. These are second order in $\tilde{\Gamma}$ and given by

$$\delta\tilde{\Gamma}_2^{(8)}=-2t\tilde{\Gamma}_2\tilde{\Gamma}_1\ln(T\tau)\ . \tag{2.33}$$

The corresponding correction to $\tilde{\Gamma}_1$ is shown in Fig. 9(a),

$$\delta\tilde{\Gamma}_1^{(9a)}=-2t\tilde{\Gamma}_1^2\ln(T\tau)\ . \tag{2.34}$$

However, it is now possible to replace one of the $\tilde{\Gamma}_1$ in Fig. 9(a) by $\tilde{\Gamma}_2$ to produce Fig. 9(b),

$$\delta\tilde{\Gamma}_1^{(9b)}=4t\tilde{\Gamma}_1\tilde{\Gamma}_2\ln(T\tau)\ , \tag{2.35}$$

with the extra factor 2 coming from the spin sum. It is also possible to replace both $\tilde{\Gamma}_1$'s in Fig. 9(a) by $\tilde{\Gamma}_2$, producing Fig. 9(c),

$$\delta\tilde{\Gamma}_1^{(9c)}=-2t\tilde{\Gamma}_2^2\ln(T\tau)\ . \tag{2.36}$$

Finally, we have to consider corrections to $\tilde{\Gamma}$ coming from other second and higher orders in $\tilde{\Gamma}^{(0)}$. The number of skeleton graphs which is lowest order in $t\ln(T\tau)$ is limited and, as discussed by Finkelstein, involve ring-type diagrams. In Fig. 10 we show the diagrams correcting $\tilde{\Gamma}_2$ with internal $\tilde{\Gamma}_2$ vertices. By opening up the closed loop while joining two of the external legs in Fig. 10, we generate diagrams contributing to $\tilde{\Gamma}_1$ as shown in Fig. 11. These are simply Finkelstein's Figs. 9(a) and 9(c)–9(e). As explained by Finkelstein, if any internal $\tilde{\Gamma}_2$ is replaced by $\tilde{\Gamma}_1$, the spin indices are restricted in such a way that cancellations occur between Figs. 10(a) and 11(a), Figs. 10(b) and 11(b), etc. We obtain

$$\delta\tilde{\Gamma}_2^{(10)}=2t(\tilde{\Gamma}_2^2+\tilde{\Gamma}_2^3-\tilde{\Gamma}_2^3+\tfrac{1}{3}\tilde{\Gamma}_2^4)\ln(T\tau)\ . \tag{2.37}$$

where the four terms in parentheses correspond to Figs. 10(a), 10(b), 10(c), and 10(d), respectively, and

$$\delta\tilde{\Gamma}_1^{(11)}=\tfrac{1}{2}\delta\tilde{\Gamma}_2^{(10)}\ . \tag{2.38}$$

Now we are ready to demonstrate the cancellation of the singularities in $z-(2\tilde{\Gamma}_1-\tilde{\Gamma}_2)\zeta^2$. Recall from Eq. (2.17) that ζ is given by

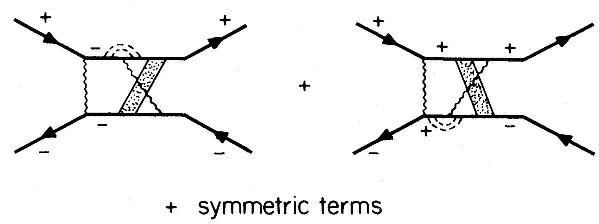
$$\zeta=1+t(\tilde{\Gamma}_1-2\tilde{\Gamma}_2)\ln(T\tau)\ . \tag{2.39}$$

It turns out that the singularity in ζ^2 is cancelled by

$$2(\delta\tilde{\Gamma}_1^{(9a)}+\delta\tilde{\Gamma}_1^{(9b)}+\delta\tilde{\Gamma}_1^{(9c)})-\delta\tilde{\Gamma}_2^{(8)}\ .$$

Furthermore, from Eqs. (2.37) and (2.38), it is clear that the higher-order contributions to $2\tilde{\Gamma}_1-\tilde{\Gamma}_2$ also cancel. Thus we see that the combination

$$z_1=z-2\Gamma_1+\Gamma_2=z-2\Gamma_s \tag{2.40}$$



+ symmetric terms

FIG. 8. Representative diagrams for t ln corrections to $\tilde{\Gamma}_2$ derivative from Fig. 7.

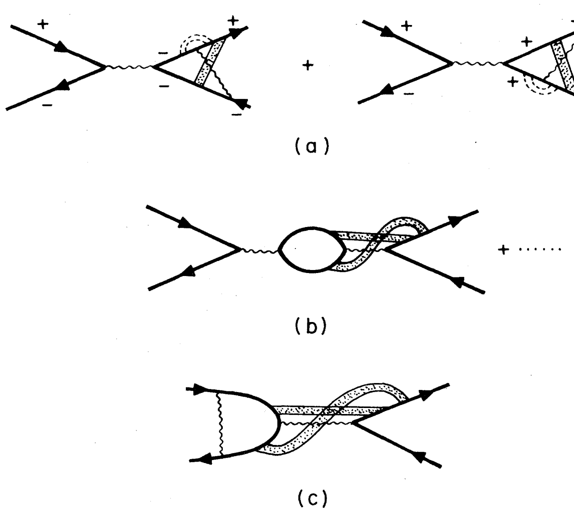


FIG. 9. The equivalent of Fig. 8 for $\tilde{\Gamma}_1$.

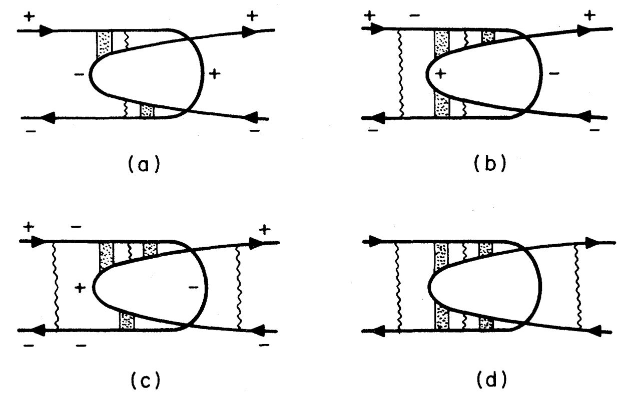


FIG. 11. The equivalent to Fig. 10 for $\tilde{\Gamma}_1$. The diagrams in Figs. 10 and 11 are arranged in pairs such that replacement of a $\tilde{\Gamma}_2$ by $\tilde{\Gamma}_1$ in any pair of diagrams results in cancellation; hence the absence of $\tilde{\Gamma}_1$ in both figures.

has no logarithmic singularity and will remain invariant under renormalization. As we have seen, this important invariance is closely connected with the physical requirement that $dn/d\mu$ is nonsingular together with the local conservation law expressed by Eq. (2.24). Equation (2.26) is now confirmed in perturbation theory and has the proper limit for the polarization function. It confirms the interpretation of D' as being proportional to the renormalized diffusion constant according to Eqs. (2.26) and (2.27).

It remains to calculate the correction to Γ and D' to all orders in $\Gamma^{(0)}$ (but first order in $t\,\ln$). As mentioned earlier, in D' this is accomplished by replacing $N_0V(1-2F)$ in Eq. (2.13) first by $\Gamma_1-2\Gamma_2$ where Γ_1 and Γ_2 are the renormalized static amplitudes, and then by replacing Γ_1 and Γ_2 by $U_1(q,\omega)$ and $U_2(q,\omega)$ through ladder summation. For this purpose it is easier to break the scattering amplitudes down into the singlet and triplet components, since the total electron-hole spin is conserved in the ladder sum. Then, using Eq. (2.16) for L^{-+}, one has

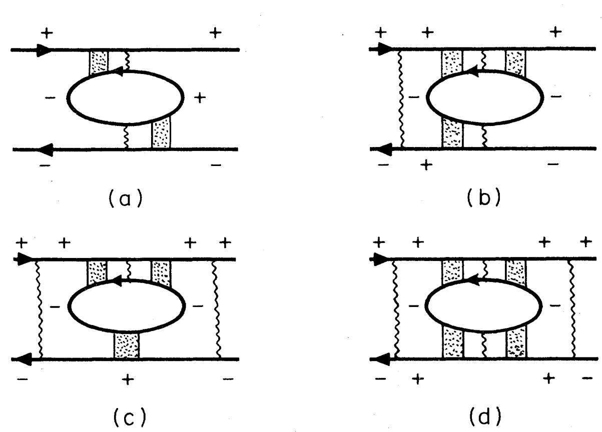


FIG. 10. $t\,\ln$ corrections to $\tilde{\Gamma}_2$ involving higher order in $\tilde{\Gamma}$'s. Note that only $\tilde{\Gamma}_2$'s are present in all the diagrams.

$$\begin{aligned} U_\alpha(q,\omega) &= \Gamma_\alpha + 2\tilde{\Gamma}_\alpha \omega L^{-+} U_\alpha(q,\omega) \\ &= \frac{\Gamma_\alpha(D'q^2+z\,|\,\omega\,|\,)}{|\,\omega\,|\,(z-2\Gamma_\alpha)+D'q^2}\,, \end{aligned} \tag{2.41}$$

where α stands for either triplet or singlet. The triplet component, from Eq. (2.22), is simply

$$U_2(q,\omega)=\Gamma_2\mathscr{D}_2(q,\omega)/\mathscr{D}_0(q,\omega)\,, \tag{2.42}$$

where

$$\mathscr{D}_0=(D'q^2+z\,|\,\omega\,|\,)^{-1}\,, \tag{2.43}$$

$$\mathscr{D}_2=[D'q^2+(z+\Gamma_2)\,|\,\omega\,|\,]^{-1}\,. \tag{2.44}$$

The propagator $\mathscr{D}_2$ describes spin fluctuation and is familiar in the theory of paramagnon, if the factor $z+\Gamma_2$ is replaced by the Stoner enhancement factor $(1-FV)^{-1}$. Associated with density fluctuation (the spin singlet channel), we have the propagator

$$U_s(q,\omega)=\Gamma_s\mathscr{D}_1/\mathscr{D}_0\,, \tag{2.45}$$

$$\mathscr{D}_1=[D'q^2+(z-2\Gamma_s)\,|\,\omega\,|\,]^{-1}\,. \tag{2.46}$$

The ladder-sum generalization for Γ_1 is given by

$$U_1(q,\omega)=\Gamma_1\mathscr{D}_1\mathscr{D}_2/\mathscr{D}_0^2\,, \tag{2.47}$$

which satisfies the relation

$$U_1(q,\omega)=U_s(q,\omega)+\tfrac{1}{2}U_2(q,\omega)\,.$$

Replacement of $V(1-2F)$ in Eq. (2.14) by $U_1(q,\omega)-2U_2(q,\omega)$ yields the leading logarithmic correction to the diffusion constant,

$$\frac{\delta D}{D_0}=\frac{4}{N_0}\int\frac{d\omega}{2\pi}\frac{d\vec{q}}{(2\pi)^2}(\Gamma_1\mathscr{D}_0\mathscr{D}_1\mathscr{D}_2-2\Gamma_2\mathscr{D}_0^2\mathscr{D}_2)Dq^2\,. \tag{2.48}$$

It is interesting that by combining one-fourth of the last term in Eq. (2.48) with the first term, we naturally separate out the singlet and triplet contribution as follows:

$$\frac{\delta D}{D_0}=\frac{4}{N_0}\int\frac{d\omega}{2\pi}\frac{d\vec{q}}{(2\pi)^2}\left[\Gamma_s\frac{\mathscr{D}_1}{\mathscr{D}_0}-\frac{3}{2}\Gamma_2\frac{\mathscr{D}_2}{\mathscr{D}_0}\right]\mathscr{D}_0^3Dq^2$$
$$=t\left[\frac{2\Gamma_s}{z}-\frac{z-2\Gamma_s}{2\Gamma_s}\ln\left[1+\frac{2\Gamma_s}{z-2\Gamma_s}\right]+3\left[1-\frac{z+\Gamma_2}{\Gamma_2}\ln\frac{z+\Gamma_2}{z}\right]\right]\ln(\tau T)\ . \qquad (2.49)$$

Note that the triplet term, or the paramagnon contribution, tends to enhance conductivity as temperature decreases.[13] We might add that in the case of Coulomb interaction the static amplitude is $\tilde{\Gamma}_0=K^2N_0V_C(\vec{q}\to\vec{0},\omega=0)$ where V_C is given by Eq. (2.21). From Eq. (2.25) it follows that

$$\Gamma_0=\zeta^2\tilde{\Gamma}_0=\tfrac{1}{2}(z-2\Gamma_s)=z_1/2\ , \qquad (2.50)$$

i.e.,

$$2\Gamma_s^{\rm LR}=z\ . \qquad (2.51)$$

It then follows from Eq. (2.41) that the dynamic singlet amplitude in this case reads

$$U_s^{\rm LR}(q,\omega)=\frac{\Gamma_s^{\rm LR}}{Dq^2}\frac{1}{\mathscr{D}_0}=\frac{z}{2Dq^2}\frac{1}{\mathscr{D}_0}\ . \qquad (2.52)$$

The correction to D is now

$$\frac{\delta D}{D_0}=\frac{4}{N_0}\int\frac{d\omega}{2\pi}\frac{d\vec{q}}{(2\pi)^2}\left[\frac{z}{2}\left[\frac{Dq^2+z\omega}{Dq^2}\right]-\frac{3}{2}\Gamma_2\frac{\mathscr{D}_2}{\mathscr{D}_0}\right]\mathscr{D}_0^3Dq^2 \qquad (2.53a)$$

$$=t\left[1+3\left[1-\frac{z+\Gamma_2}{\Gamma_2}\ln\frac{z+\Gamma_2}{z}\right]\right]\ln(T\tau)\ . \qquad (2.53b)$$

The first term in Eq. (2.53a) is the singlet contribution. Upon a change of variable, $\omega\to z\omega$, we see that this term is a universal number multiplied by $\ln(\Omega\tau)$, thus confirming the observation first made by Althuler and Aronov.[8]

From Eqs. (2.31) to (2.38) we have calculated the leading logarithmic corrections to $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$, from which it should be possible to construct scaling equations. However, it is ambiguous whether the scaling variable should be $\tilde{\Gamma}$ or Γ. We argue that it is more convenient to use Γ. To see this it is more transparent to consider the long-range case, where the $\tilde{\Gamma}_1$ terms in Eq. (2.39) for ζ and (2.33)–(2.35) for $\delta\tilde{\Gamma}_1$ and $\delta\tilde{\Gamma}_2$ become the $\ln^2(T\tau)$ singularity. The term (2.33) of $\delta\tilde{\Gamma}_2$, which contains $\tilde{\Gamma}_1$, is cancelled by the analogous term in $2(\delta\zeta)\tilde{\Gamma}_2$ so that the combination $\Gamma_2=\zeta^2\tilde{\Gamma}_2$ does not contain $\ln^2(T\tau)$. A similar cancellation occurs for $\Gamma_1=\zeta^2\tilde{\Gamma}_1$. Thus, Γ_1 and Γ_2, even in the long-range-force case, are free of $\ln^2(T\tau)$ singularities, which would otherwise cause trouble in writing the group equations. From Eqs. (2.31) to (2.38) we construct the leading logarithmic correction to Γ_1 and Γ_2. After the cancellation with ζ^2 only the terms (2.31), (2.36), and (2.38) for Γ_1 are left:

$$\delta\Gamma_1=-t\ln(T\tau)(\Gamma_2+\Gamma_2^2-\tfrac{1}{3}\Gamma_2^4)\ . \qquad (2.54)$$

For Γ_2, to the terms (2.32) and (2.37), one has to add the term $-4\tilde{\Gamma}_2^2\ln(T\tau)$, which is left from the cancellation with ζ^2:

$$\delta\Gamma_2=-t\ln(T\tau)(\Gamma_1+2\Gamma_2^2-\tfrac{2}{3}\Gamma_2^4)\ . \qquad (2.55)$$

where on the right-hand side (rhs) we have replaced $\tilde{\Gamma}$ by Γ, which is consistent with the accuracy we are working with.

Finally, we extend the renormalization of Γ to all orders in Γ by the replacement $\Gamma\to U(q,\omega)$ in the "skeleton" diagrams given in Figs. 7–11. Since only U_2 is involved, the replacement is straightforward. Here we just quote the result, as given by Finkelstein,

$$\delta\Gamma_1=-t\ln(T\tau)[\Gamma_2-\Phi(z,z_2)]\ , \qquad (2.56)$$

$$\delta\Gamma_2=-t\ln(T\tau)[\Gamma_1-2\Phi(z,z_2)]\ , \qquad (2.57)$$

where $z_2=z+\Gamma_2$ and

$$\Phi=-2\Gamma_2^2f_1(z,z_2)+\frac{\Gamma_2^2}{z_2}+\frac{\Gamma_2^3}{2z_2}f_2(z,z,z_2)-\frac{\Gamma_2^3}{2z}f_2(z_2,z_2,z)+\Gamma_2^4\frac{z^{-1}+z_2^{-1}-2f_1(z,z_2)}{(z-z_2)^2}\ . \qquad (2.58)$$

The first term in Φ corresponds to Eqs. (2.33) and (2.36) and the remaining terms come from Figs. 10 and 11. The functions

$$f_1(a,b)=\ln(a/b)/(a-b)$$

and

$$f_2(a,b,c)=[2b\,f_1(a,b)-2c\,f_1(a,c)]/(b-c)\ .$$

Equation (2.58) can be simplified considerably,

$$\Phi=\Gamma_2^2/z_2-2\Gamma_2\ln(z_2/z)\ . \qquad (2.59)$$

III. SCALING EQUATIONS FOR THE SHORT-RANGE–INTERACTION PROBLEM

Now that we know the leading logarithmic singularity of the quantities z, Γ_1, Γ_2, and t, we can write the scaling equation, under the assumption that a scaling theory exists. In the usual field-theory problem, this is accomplished by integrating out the degrees of freedom in the range $\lambda\tau^{-1}<\omega<\tau^{-1}$, so that, for example, $\ln(T\tau)$ in Eq. (2.56) is replaced by $\ln\lambda\equiv-\xi$. An equation for $\partial\Gamma/\partial\xi$ can then be derived. However, the present problem is more subtle because the singularity comes from q and ω integration. In principle, renormalization is achieved by integrating out all the fast modes at each stage. A possible renormalization scheme would integrate over the regions

$$0<|\omega|\tau<\lambda,\ \ \lambda'<Dq^2\tau<\lambda$$

and (3.1a)

$$\lambda'<|\omega|\tau<\lambda,\ \ 0<Dq^2\tau<\lambda' .$$

This region is illustrated in Fig. 12 by the shaded strips and the dark square. This is apparently the renormalization procedure adopted by Finkelstein. However, when he writes his integrals, the region of integration seems to be further restricted to

$$\lambda'<|\omega|\tau<\lambda,\ \ \lambda'<Dq^2\tau<\lambda , \tag{3.1b}$$

i.e., the dark square region in Fig. 12. Since the integrals are dominated by the poles in the integrands, this scheme is sufficient provided the poles are all situated at $Dq^2\approx\omega$. In fact, the poles are at $Dq^2\approx z_i\omega$. While this scheme is correct initially, when z, z_1, and z_2 are of order unity, under renormalization z and z_2 diverge while z_1 remains finite and the scheme according to Eq. (3.1b) breaks down.

We find that it is much more natural to adopt an alternate renormalization scheme in which ω in Eq. (3.1a) is replaced by $z\omega$,

$$0<z|\omega|\tau<\lambda,\ \ \lambda'<Dq^2\tau<\lambda$$

and (3.2a)

$$\lambda'<z|\omega|\tau<\lambda,\ \ 0<Dq^2\tau<\lambda' .$$

This has the advantage that for the renormalization of t, z_i, and Γ_i (but *not* for the density of states, as will be discussed in Appendix B), it is possible to simplify the region of integration to

$$\lambda'<z|\omega|\tau<\lambda,\ \ \lambda'<Dq^2\tau<\lambda . \tag{3.2b}$$

The above discussion is most readily illustrated by considering the triplet contribution to the renormalization of the conductance [second term in Eqs. (2.49) or (2.53)]. According to Eq. (3.1b), we have

$$\frac{\delta D_t}{D_0}=-\frac{4}{N_0}\int_{\lambda'}^{\lambda}\frac{d\omega}{2\pi}\int_{\lambda'}^{\lambda}\frac{d(Dq^2)}{4\pi D}\times\frac{3}{2}\frac{\Gamma_2 Dq^2}{[Dq^2+(z+\Gamma_2)\omega](Dq^2+z\omega)^2} . \tag{3.3}$$

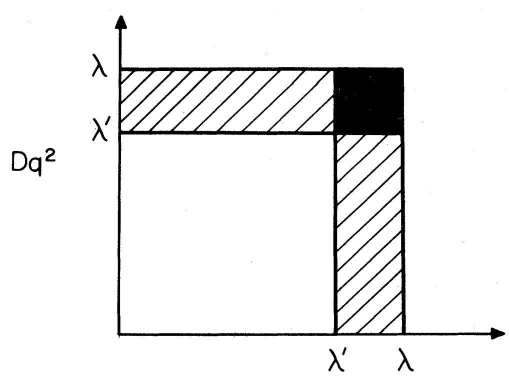


FIG. 12. The proper renormalization procedure is to integrate out the shaded and dark regions at each stage.

If z and Γ_2 are constants of order unity, by standard arguments, we find the desired result that the rhs is proportional to $\ln\lambda$. However, in Finkelstein's solution, both z and Γ_2 scale to infinity, with Γ_2/z finite, in which case the $z\omega$ factors in the denominator dominates Dq^2 over the entire range of the q integration and the integral is no longer $\ln\lambda$. This difficulty is avoided in the renormalization scheme given by Eq. (3.2b), by a simple change of variable from ω to $z\omega$. A similar situation obtains in the renormalization of Γ, in that the nonlinear terms given by Φ in Eqs. (2.56) and (2.57) all involve z and z_2. The linear term and the renormalization of z will also work, because that logarithm involves a spatial integration only [see Eq. (2.14)],

$$I_3\propto\int_{\lambda'}^{\lambda}\frac{d(Dq^2)}{4\pi D}\frac{1}{Dq^2+z_i\lambda'} , \tag{3.4}$$

where z_i may be z, z_2, or z_1. (Note that unlike z and z_2, z_1 remains invariant upon scaling.) Again, the renormalization scheme of (3.1b) runs into difficulty when z_i exceeds Dq^2 in the range of integration, whereas according to (3.2b), $z_i\lambda'$ should be replaced by $z_i/z\lambda'$ and can always be ignored compared with Dq^2; hence Eq. (3.4) is proportional to $\ln\lambda$, as desired. We conclude that, using Eq. (3.2), the renormalization procedure is consistent and Finkelstein's scaling is recovered.

According to Finkelstein, z diverges only logarithmically with λ, and the distinction between (3.1) and (3.2) is in most cases small, leading to no more than a logarithmic correction to the temperature scale in two dimensions. However, the singlet contribution δD_s to the conductivity [first term in Eqs. (2.49) and (2.53)] requires special discussion. The long-range case of Eq. (2.53) is satisfactorily treated by the procedure of Eq. (3.2b), but in the short-range case, the integral involves both z_1 and z, and special care is required. We have, from Eq. (2.49),

$$\frac{\delta D_s}{D_0}=\frac{4}{N_0}\int_{\lambda'}^{\lambda}\frac{d\omega'}{2\pi}\int_{\lambda'}^{\lambda}\frac{d(Dq^2)}{4\pi D}\times\frac{(\Gamma_s/z)Dq^2}{[Dq^2+(z_1/z)\omega'](Dq^2+\omega')^2} , \tag{3.5}$$

where we have renormalized according to Eq. (3.2b) and made the change of variable $\omega'=z\omega$. Initially, z and z_1 are both of order unity and both poles are inside the integration limit. However, after some steps in the renormalization, z_1/z becomes much less than unity, since z_1 is invariant while z diverges, and the factor $(z_1/z)\omega'$ in the denominator may be ignored compared to Dq^2. This, together with the fact that $\Gamma_s/z\rightarrow\frac{1}{2}$ under renormalization (see below) immediately removes the distinction between Eqs. (2.49) and (2.53). If instead, the integral is performed in the entire region given by (3.2a), we find an extra contribution proportional to z_1/z which becomes negligible as the renormalization proceeds. We shall see that, with the exception of the density of states, the long- and short-range problems become identical under renormalization.

We now proceed to adopt the renormalization scheme, Eq. (3.2b), and generate the scaling equations for the effective couplings from the leading logarithmic singularities. From Eqs. (2.19) and (2.14) we see that the replacement $V(1-2F)$ by $U_1(q,\omega)-2U_2(q,\omega)$ does not change the results, so that only terms to lowest order in Γ contribute to the z renormalization,

$$dz/d\xi=-t(\Gamma_1-2\Gamma_2)\ . \tag{3.6}$$

From Eqs. (2.56)–(2.59) we have

$$\frac{d\Gamma_1}{d\xi}=t\left[\Gamma_2-\frac{\Gamma_2^2}{z+\Gamma_2}+2\Gamma_2\ln\left[\frac{z+\Gamma_2}{z}\right]\right] \tag{3.7}$$

and

$$\frac{d\Gamma_2}{dz}=t\left[\Gamma_1-2\frac{\Gamma_2^2}{z+\Gamma_2}+4\Gamma_2\ln\left[\frac{z+\Gamma_2}{z}\right]\right]\ . \tag{3.8}$$

From Eq. (2.49) we have

$$\frac{dt}{d\xi}=t^2\left[\frac{2\Gamma_s}{z}+3\left[1-\frac{z+\Gamma_2}{\Gamma_2}\ln\frac{z+\Gamma_2}{z}\right]\right]\ , \tag{3.9}$$

where we have left out the transient term in z_1/z. As discussed earlier, Eq. (3.9) is applicable at a late stage in the renormalization process, when z becomes larger. As already mentioned, Eqs. (3.6)–(3.8) imply that the singlet combination is invariant,

$$\frac{d(z-2\Gamma_1+\Gamma_2)}{d\xi}=0\ . \tag{3.10}$$

To simplify Eqs. (3.7)–(3.9) further, it is useful to recognize that the natural scaling variables are

$$\gamma_1=\Gamma_1/z \tag{3.11}$$

and

$$\gamma_2=\Gamma_2/z\ , \tag{3.12}$$

in that the scaling (3.9) for t can be written entirely in terms of γ_1,γ_2. This is evident from Eq. (3.9) and can be traced back to the fact that in performing the integrals over ω in Eqs. (2.49) or (2.53) the variable z can be eliminated by a charge of variable from ω to $z\omega$ and replacing Γ/z by γ. Similarly, equations for γ_1 and γ_2 can be written as follows:

$$\frac{d\gamma_1}{d\xi}=t\left[\gamma_1^2-2\gamma_1\gamma_2+\gamma_2-\frac{\gamma_2^2}{1+\gamma_2}+2\gamma_2\ln(1+\gamma_2)\right]\ , \tag{3.13}$$

$$\frac{d\gamma_2}{d\xi}=t\left[\gamma_1(1+\gamma_2)-2\gamma_2^2\left[\frac{2+\gamma_2}{1+\gamma_2}\right]+4\gamma_2\ln(1+\gamma_2)\right]\ , \tag{3.14}$$

and

$$\frac{dt}{d\xi}=t^2\left[2\gamma_1-\gamma_2+3\left[1-\frac{1+\gamma_2}{\gamma_2}\ln(1+\gamma_2)\right]\right]\ . \tag{3.15}$$

We should solve Eqs. (3.13) and (3.14) for the fixed points γ_1^* and γ_2^*. This task is simplified further by using the invariant condition, Eq. (3.10). Suppose

$$z-2\Gamma_1+\Gamma_2=c\ . \tag{3.16}$$

Then,

$$1-2\gamma_1+\gamma_2=c/z\ . \tag{3.17}$$

It can be shown that the rhs of Eqs. (3.7) and (3.8) is always positive, for positive Γ_1 and Γ_2, and the fixed point does not exist for finite Γ_1 and Γ_2. It turns out that the fixed point of interest corresponds to $z^*\to\infty$. Thus, Eq. (3.17) becomes

$$1-2\gamma_1^*+\gamma_2^*=0\ . \tag{3.18}$$

We can combine this condition with Eq. (3.13) to obtain a single equation for γ_2^* and solve numerically to obtain $\gamma_2^*\approx 4$. Upon substitution of γ_1^* and γ_2^* into the rhs of Eq. (3.15), we find that

$$dt/d\xi=t^2\left[4-3\frac{1+\gamma_2^*}{\gamma_2^*}\ln(1+\gamma_2^*)\right]\ . \tag{3.19}$$

The rhs of Eq. (3.19) is always negative, so that t scales towards weak coupling. The ground state in two dimensions is a perfect conductor. It is useful to compare the solution of the short-range problem with Finkelstein's solution for the long-range problem. After some initial transient, the scaling equation for t becomes identical to the long-range problem. The equations for z, Γ_1, and Γ_2 are the same except that Γ_1 must include the screened Coulomb interaction Γ_0. The invariance condition becomes, e.g., (2.51), which can be written as

$$1-2\gamma_1'+\gamma_2=0\ , \tag{3.20}$$

where $\gamma_1'=(\Gamma_1+\Gamma_0)/z$. The situation is therefore even simpler, and the fixed-point value for γ_2^* is the same in the long- and short-range cases because Eqs. (3.20) and (3.18) are identical.

Next we discuss the single-particle density of states. This involves the renormalization parameter ζ and, performing the usual substitution of V by $U(q,\omega)$ in Eq. (2.12), we obtain

$$\frac{\delta N}{N}=t\left[-\frac{1}{2}\ln\frac{z_1}{z}-\frac{3}{2}\ln\frac{z_2}{z}\right]\ln(T\tau)\ . \tag{3.21}$$

To obtain its scaling behavior we note from Eqs. (A10) and (A13) that as the scale is changed from λ to λ', the $N(\lambda)$ satisfy the relation

$$N(\lambda')=\zeta(\lambda'/\lambda)N(\lambda)\ , \tag{3.22}$$

where ζ is given by Eqs. (2.17) and (2.12) in the perturbation theory. Performing the usual substitution of V by $U(q,\omega)$ in Eq. (2.12) and performing the integral in the entire region denoted by (3.2a), we find (see Appendix B)

$$\zeta(\lambda/\lambda')=1+\tfrac{1}{2}t(\ln z)\ln(\lambda/\lambda')\ . \tag{3.23}$$

Thus the equation for the density of states is

$$\frac{d\ln N}{d\xi}=-(t/2)\ln z\ . \tag{3.24}$$

Close to the fixed point, $t \approx 1/2\xi$ and $\ln z \approx \frac{11}{4}\ln\xi$, so (3.24) implies

$$\ln N \approx -\tfrac{11}{32}\ln^2\xi \ . \tag{3.25}$$

Up to logarithmic accuracy we interpret λ as the temperature T or frequency ω, and so

$$N(T) \sim e^{-(11/32)(\ln\ln T)^2} , \tag{3.26}$$

indicating that $N(T)$ remains essentially constant until very low T and then drops abruptly to zero.

In Appendix B we also discuss the long-range problem, where perturbation theory yields a $\ln^2 T$ correction. Special care is needed to handle this singularity, and instead of Eq. (3.24) we find

$$d\ln N/d\xi = -\tfrac{1}{2}t\xi \ . \tag{3.27}$$

The solution of the equation yields

$$N(T) \sim T^{1/4} \ . \tag{3.28}$$

Equations (3.27) and (3.28) were first written by Finkelstein. We reiterate that the density of states is the only quantity that distinguishes the long- and short-range problems in the scaling region.

IV. THE SINGLET-ONLY PROBLEM

The reason that Finkelstein's solution scales to the perfect conductor is that the triplet amplitude Γ_2 scales to infinity, and the resulting spin-fluctuation contribution to the conductivity enhancement overcomes the singlet contribution. Thus, this theory cannot describe a metal-insulator transition in $2+\epsilon$ dimensions, because one always scales to a metal. As pointed out by Altshuler and Aronov, the situation changes if the triplet fluctuations are suppressed. Physically, the triplet diffusion channel will be suppressed in the presence of spin-orbit scattering or spin-flip scattering. We then have a model which exhibits a metal-insulator transition in at least two experimentally realizable situations: (i) strong-spin-flip scattering due to magnetic impurities, or (ii) strong spin-orbit scattering in addition to a small magnetic field to suppress weak antilocalization. Thus, it is interesting to work out the scaling properties of this case.

A. Long-range case

This case was treated by Altshuler and Aronov. The scaling equation for t simplifies in that only the first term in Eq. (2.53) survives, and gives a universal contribution to the scaling equation,

$$dt/d\xi = t^2 \ . \tag{4.1}$$

In $2+\epsilon$ dimensions, we introduce a dimensionless resistance $t = \lambda^{\epsilon/2+\epsilon}(4\pi^2 N_0 D)^{-1}$ as a generalization of Eq. (2.2). Equation (4.1) becomes

$$\frac{dt}{d\xi} = \frac{-\epsilon}{2+\epsilon}t + t^2 \ . \tag{4.2}$$

The factor $2+\epsilon$ on the rhs of Eq. (4.2) comes about because λ is a frequency scale, and the renormalized diffusion constant D enters in relating it to a momentum scale [see Eq. (3.1) or (3.2)]. However, to the accuracy in the ϵ expansion that we are working in, $2+\epsilon$ can be replaced by 2 in Eq. (4.2) and we shall do so from now on. Equation (4.2) has the same form as the pure-localization problem. The standard argument will produce a conductivity exponent near the critical concentration n_c,

$$\sigma = (n-n_c)^{\mu} \ , \tag{4.3}$$

such that

$$\mu = 1 + O(\epsilon) \ . \tag{4.4}$$

While the scaling equation for t is independent of Γ, it is still of interest to obtain the scaling equation for z and the singlet amplitude Γ_s^{LR} which are related to one another by the invariance relation (2.54), $z = 2\Gamma_s^{\mathrm{LR}}$. The z equation can be read off of Eq. (3.6) by breaking the rhs down into singlet and triplet amplitudes, and setting the triplet equal to zero,

$$dz/d\xi = -t\Gamma_s^{\mathrm{LR}} \ , \tag{4.5}$$

$$\frac{d\Gamma_s^{\mathrm{LR}}}{d\xi} = -\frac{t}{2}\Gamma_s^{\mathrm{LR}} \ . \tag{4.6}$$

Thus, both Γ_s and z scales to zero. The ratio $\gamma_s = \Gamma_s^{\mathrm{LR}}/z = \frac{1}{2}$ is universal, and produces the universal correction to t shown in Eq. (4.1). In $2+\epsilon$ dimensions, $t^* = \epsilon/2$ and we solve Eq. (4.5) to obtain

$$z = e^{-\epsilon\xi/4} = \lambda^{\epsilon/4} \ . \tag{4.7}$$

As first pointed out by Finkelstein,[14] the fact that $z \to 0$ affects the critical exponent describing the T dependence of σ at criticality. At the fixed point the diffusion constant is given by

$$D = t^* L^{-\epsilon} \ . \tag{4.8}$$

To convert this to a temperature dependence, we have to relate the length scale L to ω. According to Eq. (3.2) we have

$$DL^{-2} = z\omega = \lambda \ . \tag{4.9a}$$

Combined with Eq. (4.7), we see that

$$DL^{-2} = \omega^{1/(1-\epsilon/4)} \ . \tag{4.9b}$$

Combining Eqs. (4.9) and (4.8), we obtain

$$D = \omega^{\epsilon/(2+\epsilon/2)} \ . \tag{4.10}$$

This is in contrast with the usual scaling argument[8] which ignores the critical behavior of z and gives $D = \omega^{\epsilon/d}$.

The density of states obeys the same equation as in the weak-magnetic-field case, except that t is replaced by t^*,

$$d\ln N/d\xi = -\frac{t^*\xi}{2} \ , \tag{4.11}$$

which gives

$$N \sim e^{-t^*(\ln\lambda)^2/4} \ . \tag{4.12}$$

Substituting $t^* = \epsilon/2$ and $\lambda = zT = \lambda^{\epsilon/4}T$, we obtain

$$N \sim \exp\frac{-\epsilon}{8}\left[\frac{1}{1-\epsilon/4}\right]^2 (\ln T)^2$$

$$\sim e^{-(\epsilon/8)(\ln T)^2} \tag{4.13}$$

to lowest order in ϵ.

B. Short-range case

In this case we need to derive a scaling equation for the singlet amplitude Γ_s. Once again from the equation for z and the invariance (2.40), we immediately obtain

$$\frac{d\Gamma_s}{d\xi} = -\frac{t}{2}\Gamma_s \; . \tag{4.14}$$

We note that in contrast to the long-range case, z scales to a finite constant. Equation (4.14) states that the singlet amplitude scales to zero. The interaction becomes irrelevant! This is the only case we know of in which the pure-localization fixed point is stable with the introduction of interaction. The scaling theory of the pure-localization problem therefore applies. The consequences are that, in $2+\epsilon$ dimensions, in a system with spin-flip scattering (time-reversal symmetry breaking), the scaling equation takes the form

$$dt/d\xi = -\tfrac{1}{2}\epsilon t + \gamma t^3 \; , \tag{4.15}$$

where γ is a constant given by the noninteracting scaling theory. Standard arguments give the conductivity exponent $\mu = \frac{1}{2} + O(\epsilon)$. The single-particle density of states stays finite, as in the noninteracting case.

V. "HIGH"-MAGNETIC-FIELD PROBLEM

In the preceding section the singular contributions from the maximally crossed graphs are suppressed by the presence of spin-flip scattering. In Finkelstein's paper it is the external magnetic field that provides the cutoff for these graphs. However, the magnetic field also introduces Zeeman splitting and this must be taken into account if the temperature is sufficiently low or the magnetic field sufficiently large. Thus, it is particularly interesting to study high-field limits. The conditions are $g\mu_B H \gg T$ and $g\mu_B H \gg \tau_{\text{s.o.}}^{-1}$. In this case, the $S_z = \pm 1$ channels for the particle-hole ladder are cut off by the spin-splitting frequency, so that

$$L^{-+}_{S_z=\pm 1}(q,\Omega) = (Dq^2 + |\Omega| - ig\mu_B H)^{-1} \; , \tag{5.1}$$

so that these channels may be ignored at sufficiently low frequency or temperature. The only singular channels that remain are those for $S_z = 0$ (see Fig. 13). This is also the situation for ferromagnetic metals below T_c, even if the field is infinitesimally small since the spin splitting would be large compared to $k_B T$ (unless one is very close to T_c).

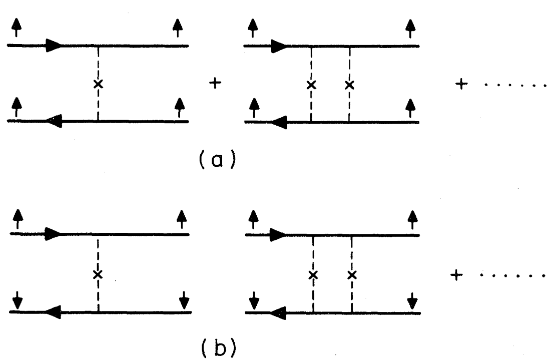


FIG. 13. Owing to Zeeman splitting, (b) is no longer diffusive.

A. Short-ranged interaction

The physics is most transparent if we first discuss the short-range case. It is convenient to rewrite the interaction part of Eq. (A2) as

$$\Gamma_H(Q^{\uparrow\uparrow}Q^{\uparrow\uparrow} + Q^{\downarrow\downarrow}Q^{\downarrow\downarrow}) + \Gamma_1(Q^{\uparrow\uparrow}Q^{\downarrow\downarrow} + Q^{\downarrow\downarrow}Q^{\uparrow\uparrow}) - \Gamma_2(Q^{\uparrow\downarrow}Q^{\downarrow\uparrow} + Q^{\downarrow\uparrow}Q^{\uparrow\downarrow}) \; , \tag{5.2}$$

where only the spin indices are displayed. Thus, Γ_H, which turns out to be the relevant coupling in the present case, is the interaction amplitude for processes where $S_z = 0$ in both the direct and exchange channels. Comparison with (A2) means the "bare" value of Γ_H is $\tilde{\Gamma}_1^{(0)} - \tilde{\Gamma}_2^{(0)}$.

As remarked earlier, $|S_z| = 1$ channel is not diffusive and so the Hartree diagrams should no longer carry a spin sum (the exchange diagrams already have no spin sum). Thus, in a perturbation calculation, $1-2F$ in Eqs. (2.12)–(2.14) must be replaced by $1-F$. As discussed in Sec. II, one then obtains the corrections to lowest order in $t\ln$ by replacing $N_0 V(0)(1-F)$ by the appropriate dynamic amplitude—in this case $U_H(q,\omega)$, which we now derive.

$U_H(q,\omega)$ is given by the graphs shown in Fig. 14. A general graph has an arbitrary number of Γ_H's and zero or an even number of Γ_1's. For $\omega = 0$ only the first graph remains, identifying Γ_H as the total static amplitude. Evaluating the graphs and summing, one obtains

$$U_H(q,\omega) = U_1(q,\omega) - U_2(q,\omega) = U_s(q,\omega) - \tfrac{1}{2}U_2(q,\omega) \; , \tag{5.3}$$

where U_1, U_2, and U_s are given by Eqs. (2.47), (2.42), and (2.45), respectively. z_1 and z_2 now read

$$z_1 = z - \Gamma_H - \Gamma_1 \tag{5.4}$$

$$z_2 = z - \Gamma_H + \Gamma_1 \; . \tag{5.5}$$

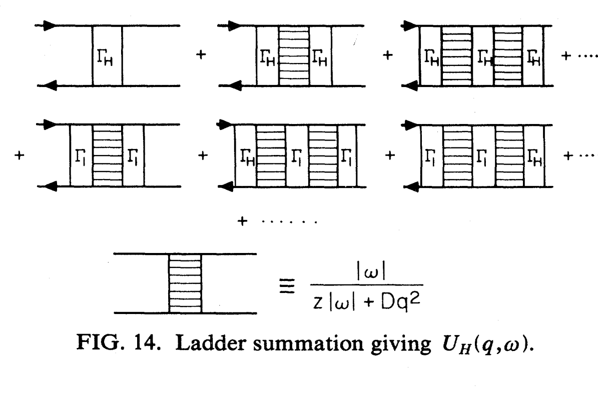


FIG. 14. Ladder summation giving $U_H(q,\omega)$.

Inserting $U_H(q,\omega)$ into Eq. (2.13) we obtain the conductivity scaling equation as

$$dt/d\xi = t^2 \left[2 - \frac{z-\Gamma_H-\Gamma_1}{\Gamma_1+\Gamma_H} \ln \frac{z}{z-\Gamma_H-\Gamma_1} - \frac{z+\Gamma_1-\Gamma_H}{\Gamma_1-\Gamma_H} \ln \frac{z+\Gamma_1-\Gamma_H}{z} \right] . \tag{5.6}$$

Note that this is the same as Eq. (2.49) [or Eq. (3.9) plus the transient term] when only one-third of the triplet contribution is taken; this follows from (5.2). To interpret this equation one also needs the scaling equations for z, Γ_1, and Γ_H. The corrections to Γ_H and Γ_1 are given to first order in Γ's by the graphs shown in Figs. 15(a) and 15(b), respectively. Note that Fig. 15(b) involves the $|S_z|=1$ channel and is nonsingular and contributes only a trivial additive correction to Γ_1. Figure 15(a), however, is singular. Higher-order corrections are given by the graphs in Figs. 9 and 10 in Finkelstein's paper. Both of them are, in the present case, corrections to Γ_H. Although each graph in Fig. 10 has a relative minus sign to a corresponding graph in Fig. 11, they do not cancel in Finkelstein's problem since there is a factor of 2 associated with the spin sum in Fig. 11. However, it is precisely this spin sum that the Zeeman splitting removes, and so for the present case all of the higher-order graphs cancel. The scaling equations for Γ_H and Γ_1 are then

$$\frac{d\Gamma_1}{d\xi} = 0 \tag{5.7}$$

and

$$\frac{d\Gamma_H}{d\xi} = -t\Gamma_H . \tag{5.8}$$

The scaling equation for z can be obtained by substituting $N_0V(0)(1-2F)$ in Eq. (2.14) by $U_H(q,\omega)$. Alternatively, we can use the invariance of $z_1 = z - 2\Gamma_s = z - \Gamma_H - \Gamma_1$, together with the invariance of Γ_1 for the present case, to give

$$dz/d\xi = -t\Gamma_H . \tag{5.9}$$

The fixed-point values for these parameters are then

$$\Gamma_1^* = \tilde{\Gamma}_1^{(0)} , \tag{5.10}$$

$$\Gamma_H^* = 0 , \tag{5.11}$$

$$z^* = 1 - \tilde{\Gamma}_H^{(0)} = 1 - \tilde{\Gamma}_1^{(0)} + \tilde{\Gamma}_2^{(0)} . \tag{5.12}$$

Substituting these values into the conductivity equation (5.6) gives

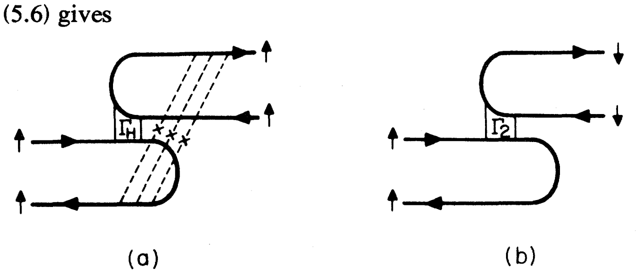


FIG. 15. (a) Correction to $\tilde{\Gamma}_H$ to first order in $\tilde{\Gamma}$ is singular. (b) The corresponding correction to $\tilde{\Gamma}_1$ is not.

$$\frac{dt}{d\xi} = t^2 f(x) ,$$

where

$$f(x) = 2 + x \ln \left[\frac{x-1}{x+1} \right] + \ln \left[\frac{x^2}{x^2-1} \right], \quad x = \frac{z^*}{\tilde{\Gamma}_1^{(0)}} . \tag{5.13}$$

Note that the stability of the electron gas requires $z_1 > 0$ so $x \geq 1$. The function $f(x)$ is monotonic in the interval $[1,\infty]$ with $f(1) = 2 - 2\ln 2$ and $f(\infty) = 0$; hence t always increases as T decreases and the system is driven into the insulating regime. For fixed $\tilde{\Gamma}_2^{(0)}$, as $\tilde{\Gamma}_1^{(0)}$ increases from 0, x decreases from ∞, and $f(x)$ increases, and so the drive to an insulator is faster, the "stronger" the interaction. However, it is important to remember that the real picture is more complicated since x increases with $\tilde{\Gamma}_2^{(0)}$, which is also a measure of the strength of the interaction.

For $d = 2+\epsilon$ dimensions, the scaling equation to first order in ϵ is then

$$dt/d\xi = -\frac{\epsilon}{2} t + f(x) t^2 , \tag{5.14}$$

and so a metal-insulator transition occurs when $t > t^* = \epsilon/2f(x)$. The conductivity exponent is again

$$\mu = 1 + O(\epsilon) . \tag{5.15}$$

The density-of-states correction is given by Eq. (2.12) with $N_0V(0)(1-2F)$ replaced by $U_H(q,\omega)$. Performing the integration we obtain the singlet term plus one-third of the triplet term of Eq. (3.21). Inserting the fixed-point values, we obtain

$$dN/d\xi = t/2N \ln \frac{(z^*)^2 - \Gamma_1^2}{(z^*)^2} = \theta t N . \tag{5.16}$$

Since $\theta < 0$, the density of states must go to zero as $T \to 0$. In fact, the behavior is, in $d = 2+\epsilon$, $N(T) \sim T^\delta$, where

$$\delta = t^* |\theta| = [\epsilon/2f(x)] |\theta| \tag{5.17}$$

is nonuniversal.

B. Coulomb interaction

In the long-range case the amplitude Γ_0 is, according to Eq. (2.50), given by

$$z - \Gamma_H - \Gamma_1 = 2\Gamma_0 . \tag{5.18}$$

The various scaling equations are modified as follows. The conductivity correction is now obtained by taking the singlet term plus one-third of the triplet term of Eq. (2.53) to give

$$dt/d\xi = t^2 \left[2 - \frac{z+\Gamma_1-\Gamma_H}{\Gamma_1-\Gamma_H} \ln \frac{z+\Gamma_1-\Gamma_H}{z} \right] . \tag{5.19}$$

The first-order correction to Γ_H now involves an additional term coming from the replacement of Γ_H by Γ_0 in Fig. 12(a). Hence,

$$\frac{d\Gamma_H}{d\xi}=-t(\Gamma_H+\Gamma_0)\ . \tag{5.20}$$

The $|S_z|=1$ channels are still suppressed, and so Γ_1 remains an invariant. From (5.18) we have

$$\frac{dz}{d\xi}=\frac{d\Gamma_H}{d\xi}\ . \tag{5.21}$$

The fixed-point values are then

$$\Gamma_H^*=-\widetilde{\Gamma}_0^{(0)} \text{ and } z^*=\widetilde{\Gamma}_0^{(0)}+\widetilde{\Gamma}_1^{(0)}\ . \tag{5.22}$$

The conductivity equation evaluated at these values is

$$\frac{dt}{d\xi}=(2-2\ln 2)t^2\ , \tag{5.23}$$

and the metal is unstable in 2D. Note that the coefficient of t^2 is the same as that for the short-ranged interaction when $z^*=\widetilde{\Gamma}_1^{(0)}$. This is due to the fact that this coefficient is independent of Γ_0. Setting $\Gamma_0=0$ immediately gives $z^*=\widetilde{\Gamma}_1^{(0)}$.

In $2+\epsilon$ dimensions the metal-insulator transition occurs for $t>\epsilon/[2(2-2\ln 2)]$, with an exponent $\mu=1+O(\epsilon)$.

The presence of long-range interaction results perturbatively in a logarithmic square correction to the density of states, which dominates the short-ranged contributions, as discussed in Appendix B. As in the "singlet-only" case, the density of states in $2+\epsilon$ dimensions does not vanish like some power of temperature, behaving instead as

$$N(T)\sim \exp\left[-\frac{\epsilon}{8(2-2\ln 2)}\ln^2 T\right]\ . \tag{5.24}$$

Thus, in the presence of "high" magnetic field, both short-ranged and Coulomb interactions give rise to a metal-insulator transition in $2+\epsilon$ dimensions. The conductivity exponent is universal and is equal to 1 in both cases. The density of states, however, behaves very differently for the two situations. For short-ranged interactions, it vanishes as a power, albeit a nonuniversal one, of the temperature. For the Coulomb interaction, it vanishes as exponential of a logarithmic square. In 2D, the metal is unstable in both cases, the conductivity decreasing logarithmically with temperature. The coefficient is nonuniversal for the short-ranged interaction but is universal ($=2-2\ln 2$) for the Coulomb interaction. This predicts that in a high magnetic field ($g\mu_B H \gg kT$ and $g\mu_B H \gg \tau_{\text{s.o.}}^{-1}$, the spin-orbit scattering rate), the temperature dependence of the conductivity in 2D is given by the universal dependence,

$$\sigma(T)=\sigma_0-\frac{1}{2\pi^2}\frac{e^2}{\hbar}(2-2\ln 2)\ln(T\tau)\ . \tag{5.25}$$

It will be very interesting to test this prediction experimentally.

VI. CONCLUSIONS

In this paper we provided a diagrammatic perturbation treatment of the disordered interacting fermion problem. Throughout this work, the maximally crossed diagrams important for pure localization are suppressed. This suppression may be due to spin-flip scattering or an external field and we explore how these effects modify Finkelstein's original solution. We also study the case of short-range interaction only. Not surprisingly, we find that each situation corresponds to a different universality class. We summarize as follows.

(1) The density-density correlation function is assumed to be diffusive for all spin channels. The long-range case is the problem originally treated by Finkelstein. Since these conditions contradict the suppression of a maximally crossed diagram, this case may exist in reality only as a transient region in temperature. Finkelstein found that the conductance in 2D scales to infinity and the single-particle density of states vanishes like a power law. The perfect conductivity is driven by a divergence of the triplet amplitude Γ_2. Surprisingly, we found that the short-range case basically scales onto the long-range problem after some initial transient, and the scaling behavior becomes identical. The density of states, on the other hand, scales to zero differently, depending on the interaction range [Eqs. (3.26) and (3.28)]. The predicted conductivity rise, even though restricted to a limited temperature range, has not yet been observed experimentally.

(2) We next introduce spin-flip scattering or a strong magnetic field. These cases are interesting because a metal-insulator transition is predicted in $2+\epsilon$ dimensions and its critical properties can be calculated. In both cases the dimensionless conductance remains a scaling variable, and the scaling equation takes the form

$$\frac{dt}{d\ln\lambda}=\frac{\epsilon t}{2+\epsilon}-At^2\ , \tag{6.1}$$

where A is a constant at the fixed point. [See the discussion following Eq. (4.2).] This determines a fixed point $t^*=\epsilon/[(2+\epsilon)A]$. Near the fixed point Eq. (6.1) can be integrated out to a cutoff scale $\widetilde{\lambda}$ which separates the critical region from either the metallic or insulating region

$$\widetilde{\lambda}=\left[\frac{t-t^*}{t^*}\right]^p \tag{6.2}$$

and

$$p=\frac{2+\epsilon}{\epsilon}[1+O(\epsilon)]\ , \tag{6.3}$$

The conductivity σ is given by

$$\sigma\approx t^*/\xi^\epsilon\ , \tag{6.4}$$

where ξ is the length-scale change corresponding to $\widetilde{\lambda}$. According to the renormalization scheme given in Eq. (3.2), we have

$$D\xi^{-2}=\widetilde{\lambda}\ , \tag{6.5}$$

where $D=(dn/d\mu)^{-1}\sigma$. Combining Eqs. (6.5) and (6.4) we find

$$\xi=\widetilde{\lambda}^{-1/(2+\epsilon)}=\left[\frac{t-t^*}{t^*}\right]^{-\nu}\ . \tag{6.6}$$

The critical exponent ν is found from Eq. (6.3) to be

$$\nu=p/(2+\epsilon)=(1/\epsilon)[1+O(\epsilon)]\ . \tag{6.7}$$

Setting Eq. (6.6) into Eq. (6.4) we find

$$\sigma \approx \left[\frac{t-t^*}{t^*} \right]^{\mu} , \tag{6.8}$$

where

$$\mu = \frac{p\epsilon}{2+\epsilon} = 1 + O(\epsilon) . \tag{6.9}$$

The above line of arguments produces exponents that are identical to $O(\epsilon)$ to those in the noninteracting problem, because the basic equation, (6.1), has the same structure as the noninteracting case. However, because of the appearances of the parameter z in the renormalization scheme given by Eq. (3.2), other scaling properties may differ. Here we summarize the results for the two cases separately.

(a) A magnetic field high enough to cause spin splitting. The conditions are $g\mu_B H \gg k_B T$ and $g\mu_B H \gg \tau_{\mathrm{s.o.}}^{-1}$. Alternatively, the spin splitting might be the result of a ferromagnetic transition. The two-dimensional long-range case is particularly interesting in that a universal logarithmic temperature dependence is predicted for the conductivity [Eq. (5.25)], which should be accessible to experimental testing. The ideal geometry would be to apply a large field parallel to the plane, with a small normal component to suppress weak localization. This avoids the problem of having to deal with quantized Landau orbits.

In $2+\epsilon$ dimensions, a metal-insulator transition exists. The scaling behavior is particularly simple because z scales to a constant, so that the energy scale Δ can be identified with the cutoff scale $\tilde{\lambda}$ given in Eq. (6.2). The scaling behavior for the conductivity on the metallic side and the dielectric constant ε on the insulating side is identical to the noninteracting case, and we can follow the arguments of McMillan[15] and their elaborations.[5,16] Let us denote by ω the frequency or temperature, whichever is greater. For $\omega > \Delta$, we are in the critical regime, the conductivity is given by

$$\sigma(\omega) = \omega^{\epsilon/(2+\epsilon)} , \quad \omega > \Delta . \tag{6.10}$$

For $\omega < \Delta$, on the metallic side, we scale to a length scale $\tilde{\lambda}$, at which point we can do perturbation using renormalized parameters. This gives

$$\sigma(\omega) = \sigma(0)[1 + (\omega/\Delta)^{\epsilon/2}] , \tag{6.11}$$

Combining this with Eq. (6.8) shows that in the coefficient of the $T^{\epsilon/2}$ correction to the conductivity should diverge near the critical point as $[\sigma(0)]^{(2-d)/2}$.

On the insulator side, scaling is towards strong coupling and the perturbative methods break down. However, we can still estimate the dielectric constant by scaling up to the cutoff scale $\tilde{\lambda}$ and argue that the polarizability of the insulator is dominated by the metal-like screening which operates up to that cutoff scale. We write the dielectric constant ε in three dimensions as

$$\varepsilon = 1 + \frac{4\pi e^2}{q^2} \pi(q,\omega) \approx 4\pi e^2 \frac{dn}{d\mu} \frac{D'}{D' q^2 - i z_1 \omega} . \tag{6.12}$$

Recall that in the high-magnetic-field case, z scales to a constant, so that according to the renormalization scheme, the two terms $D'q^2$ and $i\omega z_1$ in Eq. (6.12) remain similar in magnitude upon scaling. At a cutoff scale $\tilde{\lambda}$, $q \approx \xi^{-1}$ and we have the result

$$\varepsilon = 4\pi e^2 \frac{dn}{d\mu} \xi^2 . \tag{6.13}$$

Thus the dielectric constant is predicted to diverge near the metal-insulator transition as $|t-t^*|^{-2\nu}$ with ν given by Eq. (6.7). Furthermore, combined with Eq. (6.4) we conclude that in three dimensions (3D), the product

$$\varepsilon \sigma^2 = \text{invariant} , \tag{6.14}$$

where ε and σ are measured equidistant from the transition. This result is the same as in the noninteracting problem.

The single-particle density of states vanish at the metal-insulator transition as a power law with a nonuniversal exponent in the short-range case [Eq. (5.17)]. In the long-range case the density of states vanishes in a peculiar way given by Eq. (5.24).

(b) Large spin-flip scattering, $\tau_{\mathrm{SF}}^{-1} \gg k_B T$, or large spin-orbit scattering with a weak magnetic field to suppress weak antilocalization. For a short-range interaction this case scales to weak coupling, so that we recover the noninteracting localization problem. For example, in tunnelling experiments, or in MOSFET's with a thin oxide, charges on the metal plate may cut off the long-range Coulomb interaction in the sample and make this case relevant. On the other hand, if the interaction remains long ranged, the interaction is relevant. In two dimensions, the conductivity behaves in a universal way,

$$\sigma(T) = \sigma_0 + \frac{1}{2\pi^2} \frac{e^2}{\hbar} \ln(T\tau) , \tag{6.15}$$

as first suggested by Altshuler and Aronov. Existing data seem consistent with this prediction.[17]

In $2+\epsilon$, a metal-insulator transition is predicted. However, the critical behavior is different from the noninteracting case or from case (2a) because in this case the parameter z scales to zero. Thus the energy scale Δ is no longer the same as $\tilde{\lambda}$, but

$$\begin{aligned} \Delta &= \tilde{\lambda}/z \\ &= \tilde{\lambda}^{1-\epsilon/4} \\ &= \left| \frac{t-t^*}{t^*} \right|^{p(1-\epsilon/4)} , \end{aligned} \tag{6.16}$$

where we have used Eq. (4.7). As discussed in Sec. IV, the scaling behavior of the conductivity is given by

$$\sigma(\omega) = \omega^{\epsilon/(2+\epsilon/2)}, \quad \omega > \Delta \tag{6.17}$$

$$= \sigma(0)[1 + (\omega/\Delta)^{\epsilon/2}], \quad \omega < \Delta \tag{6.18}$$

where

$$\sigma(0) = \left| \frac{t-t^*}{t^*} \right|^{\mu} , \tag{6.19}$$

with $\mu = 1 + O(\epsilon)$ as before.

The dielectric constant in 3D can be estimated as before but now z scales to zero while z_1 remains invariant. Upon scaling, according to Eq. (3.2), the term $z_1\omega$ in the denominator of Eq. (6.12) dominates $D'q^2$ and we have

$$\varepsilon \approx 4\pi e^2 \frac{dn}{d\mu}\frac{D'z}{\widetilde{\lambda}} \approx 4\pi e^2 \frac{dn}{d\mu}\xi^2 z \ . \tag{6.20}$$

Near the critical point we find that the dielectric constant diverges as

$$\varepsilon \approx \left| \frac{t-t^*}{t^*} \right|^{-2\nu+px} , \tag{6.21}$$

where x is the exponent in $z \sim \lambda^x$ and was calculated in $2+\epsilon$ dimensions to be $\epsilon/4$ [Eq. (4.7)]. Thus we conclude that in three dimensions,

$$\varepsilon\sigma^2 = \left| \frac{t-t^*}{t^*} \right|^{\delta} , \tag{6.22}$$

where the exponent $\delta = px$ and is given in a $2+\epsilon$ expansion as $\delta = p\epsilon/4$. Equation (6.21) is to be contrasted with Eq. (6.14) for the high-magnetic-field or the noninteracting case, and should be subject to experimental testing.

Finally, the single-particle density of states in the long range vanishes at the transition according to Eq. (4.13).

The experimental situation remains unsettled, with some experiments reporting $\mu = 1$ (Refs. 10 and 18) and the experiment on Si:P reporting $\mu = \frac{1}{2}$ (Ref. 19). As emphasized in Ref. 5, it is possible that these experiments represent different universality classes. In particular, the experiment of Ref. 10 should be described by our high-magnetic-field model which predicts $\mu = 1 + O(\epsilon)$. It will be interesting to study the temperature dependence of the conductivity and to make dielectric-constant measurements on the insulating side to compare with the theory. At the same time, it will be very interesting to study the metal-insulator transition in Si:P in the presence of a strong magnetic field to see if μ changes from 0.5 to 1 as our theory would suggest.

Note added in proof. Recently, an error has been discovered in Finkel'shtein's expression for Φ so that Eq. (2.59) should read

$$\Phi = -\frac{\Gamma_2^2}{z} \ . \tag{2.59$'$}$$

Details and the physical consequences of this can be found in Castellani *et al.* [Phys. Rev. B (to be published)] and A. M. Finkel'shtein [Z. Phys. (to be published)]. The new result that is relevant to this paper is that the system no longer scales to a perfect conductor. Instead, the conductivity remains finite in Finkel'shtein's original problem. However, the major conclusion reached in Sec. III, that the short-range and long-range problems become identical under scaling, remain unchanged. The singlet-only and the high-magnetic-field problems (Secs. IV and V are independent of Φ and are obviously unaffected.

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APPENDIX A

In Sec. II we showed that in order to renormalize L^{+-} it is necessary to introduce a "wave-function" renormalization ζ. This renormalization is related to the singularity $(t \ln^2)$ in the density of states. We demonstrated that it is possible to define $\Gamma_i = \widetilde{\Gamma}_i \zeta^2$ so that the coupling constants' renormalizations become free of the $t \ln^2$ singularity.

In Finkelstein's field-theoretic calculation the Γ_i's appear as the natural coupling constants and the $t \ln^2$ divergences are cancelled everywhere. In fact no "wave-function" renormalization is necessary in his theory for reasons we now discuss. We recall that the field theory of Finkelstein takes the form

$$Z = \int \mathscr{D}\{Q\} \exp - \mathscr{F}(\{Q\}) \ , \tag{A1}$$

where

$$\mathscr{F}(\{Q\}) = T\frac{\pi N_0}{4} \int [D\,\mathrm{Tr}(\vec{\nabla} Q)^2 - 4z\,\mathrm{Tr}(\hat{\epsilon} Q) - \Gamma(Q\gamma_1 Q) + \Gamma_2(Q\gamma_2 Q)] d\vec{r} \ . \tag{A2}$$

The matrix Q is $Q^{ij,\alpha\beta}_{n_1 n_2}$, where n_1, n_2 labels the energy $\epsilon_n, \epsilon_{n'}$; i,j are the replica indices and α,β are spin indices. The matrix $\hat{\epsilon}$ is $\hat{\epsilon} = \epsilon_n \delta_{nm} \delta_{ij} \delta_{\alpha\beta}$, $\epsilon_n = (2n+1)\pi T$, and Γ and Γ_2 are interaction amplitudes distinguished by their spin structure,

$$\begin{aligned}(Q\gamma_1 Q) &= 2\pi T \sum_{n,i} Q^{ii;\alpha\alpha}_{n_1 n_2} Q^{ii;\beta\beta}_{n_3 n_4} \delta(n_1+n_3, n_2+n_4) \ , \\ (Q\gamma_2 Q) &= 2\pi T \sum_{n,i} Q^{ii;\alpha\beta}_{n_1 n_2} Q^{ii;\beta\alpha}_{n_3 n_4} \delta(n_1+n_3, n_2+n_4) \ ,\end{aligned} \tag{A3}$$

The matrix Q satisfies the constraint

$$Q^2 = I \ , \tag{A4}$$

$$\mathrm{Tr} Q = 0 \ , \tag{A5}$$

and

$$Q = Q^\dagger \ . \tag{A6}$$

These matrices can be written as

$$Q=U^{-1}\Lambda U\ , \tag{A7}$$

where U is unitary and

$$\Lambda_{nm}^{ij;\alpha\beta}=\mathrm{sgn}(\epsilon_n)\delta_{nm}\delta_{ij}\delta_{\alpha\beta}\ .$$

Finkelstein chose a renormalization group such that the constraint, Eq. (A4), is satisfied without rescaling at each stage of the renormalization process. This is accomplished by a parametrization

$$U=\exp(W/2),\quad W=-W^{+} \tag{A8}$$

and the matrix elements of W are considered as independent variables. Note that this is different from the parametrization used by Wegner[2] and Hikami,[11] where the elements Q^{+-}, i.e., $\epsilon_n>0$ and $\epsilon_{n'}<0$ were chosen as independent. This latter parametrization is more analogous to the usual O(n) model, and a wave-function renormalization will occur.

In the renormalization process, Finkelstein showed that upon a length-scale change from λ to λ',

$$Z(\lambda,D,z,\Gamma,\Gamma_2)=Z(\lambda',D',z',\Gamma',\Gamma_2')\ , \tag{A9}$$

and the renormalization of D, z, Γ, and Γ_2 are free of $\ln^2$ divergences. However, in order to calculate the single-particle density of states,

$$N(\epsilon_n)=-\frac{1}{\pi}\,\mathrm{Im}G(\epsilon+i\eta,\vec{r})=N_0\langle\Lambda Q_{nn}\rangle\ , \tag{A10}$$

or $\langle Q(\vec{r})Q(\vec{0})\rangle$ correlation functions, it is necessary to add source terms,

$$\mathscr{F}_h=\int d\vec{r}\sum h_{nn'}^{ij;\alpha\beta}(\vec{r})Q_{nn'}^{ij;\alpha\beta}(\vec{r})\ , \tag{A11}$$

to the action in Eq. (A2). The calculation of the renormalization of $h_{nn'}$ (we suppress the remaining indices) is very similar to Finkelstein's z renormalization except that only the term arising from $\langle\mathscr{F}_h\mathscr{F}_{\rm int}\rangle$ contributes [Eq. (3.34) in Finkelstein]. We find that

$$h_{nn'}'=h_{nn'}(1-I_1)=\zeta h_{nn'}\ , \tag{A12}$$

where I_1 is Eq. (2.12) cut off at λ'. Thus the renormalization of $h_{nn'}$ contains the $\ln^2\lambda$ singularity associated with the density of states. Using $\langle Q\rangle=\delta(\ln Z)/\delta h$, we see that

$$\langle Q\rangle_{\mathscr{F}}=\zeta\langle Q\rangle_{\mathscr{F}'}\ , \tag{A13}$$

and

$$\langle Q(\vec{r})Q(\vec{0})\rangle_{\mathscr{F}}=\zeta^2\langle Q(\vec{r}/\lambda)Q(\vec{0})\rangle_{\mathscr{F}'}\ , \tag{A14}$$

where $\mathscr{F}'$ is the renormalized action parametrized by D', z', Γ', and Γ_2'. Since the particle-hole propagator L^{-+} is related to $\langle Q^{+-}(\vec{r})Q^{-+}(\vec{0})\rangle$, Eq. (A14) is consistent with the perturbation-theory result given by Eq. (2.16).

It is worth emphasizing that in the present case, the "wave-function renormalization" ζ is very different from the renormalization constant z, whereas in the nonlinear σ model representation of the noninteracting localization problem, or in the usual O(n) model, there is only a single spin length renormalization. The distinction is that in Eq. (A2) the O($2n$)/[O(n)$\times$O(n)] symmetry has already been broken by the interaction term Γ. These terms, upon renormalization, generate terms which renormalize the combination Tr($\hat{\epsilon}Q$), leading to a renormalization of z different from that of $h_{nn'}$.

APPENDIX B

In this appendix we discuss the problems associated with renormalization of the density of states. These are twofold: (i) the perturbative calculation for the long-range interaction shows a "logarithmic square" singularity which must be renormalized properly; (ii) it is no longer sufficient to include in the integrals just the region given by (3.2b), and instead one must integrate out the full region of (3.2a).

First, we review how the perturbative calculation gives the "logarithmic square" singularity. As shown in Sec. II, the correction to the density of states is given by the integral I_1 [Eq. (2.12)]. The contribution to this from the long-range Coulomb interaction is

$$I_1^{\rm LR}=2\int_0^{\lambda}\frac{d\omega}{2\pi}\int_0^{\lambda_q}\frac{d\tilde{q}}{(2\pi)^2}\frac{1}{(\omega+Dq^2)^2}\frac{V_C(q)}{1+V_C(q)\pi(q,\omega)}\ , \tag{B1}$$

where $V_C(q)=2\pi e^2/|q|$ is the bare Coulomb interaction in 2D, and λ_q, the momentum cutoff, is given by $\sqrt{\lambda/D}$, and ϵ is the energy (temperature) in question. The polarization bubble π is at this level given by

$$\pi(q,\omega)=\frac{dn}{d\mu}\frac{Dq^2}{|\omega|+Dq^2}\ . \tag{B2}$$

The integration is straightforward and we obtain

$$I_1\sim\left[\ln\frac{\epsilon}{D\kappa^2}\right]^2\ , \tag{B3a}$$

for $\lambda>D\kappa^2$, and

$$I_1\sim\left[\ln\left[\frac{\lambda}{D\kappa^2}\right]+\ln\left[\frac{\epsilon}{D\kappa^2}\right]\right]\ln\frac{\epsilon}{\lambda}\ , \tag{B3b}$$

for $\lambda<D\kappa^2$, where the inverse screening length $\kappa=2\pi e^2(\partial n/\partial\mu)$.

According to Finkelstein, this leads to the following scaling equation for the density of states:

$$\frac{d\ln N}{d\xi}=-\frac{t}{2}\xi\ . \tag{B4}$$

While we agree with this equation we show that it is impossible to obtain this using the renormalization scheme given in Eq. (3.1b) or (3.2b). In this region the momentum is always finite and it is possible to neglect 1 compared with $V_C(q)\pi(q,\omega)$ in (A1). Replacing the bare perturbative expression by the correct dynamic one, we have

$$\delta\zeta \sim t\Gamma_0 \int_{\lambda'}^{\lambda} d\omega \int_{\lambda'}^{\lambda} dx \frac{1}{x}\frac{1}{x+z_1\omega} , \tag{B5}$$

where $x \equiv Dq^2$. Upon performing the integration we find

$$\delta\zeta \sim t\Gamma_0 \tfrac{1}{2}(\ln^2\lambda - 2\ln\lambda \ln\lambda' + \ln^2\lambda') = (t/2)\Gamma_0(\delta\xi)^2 , \tag{B6}$$

where $\delta\xi \equiv \ln(\lambda/\lambda')$, which implies

$$\frac{d\zeta}{d\xi} = 0! \tag{B7}$$

It is therefore necessary to integrate over the entire region given by Eq. (3.1a) or (3.2a). While the integration over the strip corresponding to the first line in Eq. (3.1a) or (3.2a) gives zero in the same manner as (B5), that for the second line does not. In fact, the integral is similar to (B1) with λ' replacing ϵ, and $z_1\omega$ replacing ω in the integral. Also, as we scale, λ becomes smaller than $D\kappa^2$ and so, from (B3b),

$$\delta\zeta = -\tfrac{1}{4}t(\ln\lambda + \ln\lambda')\ln(\lambda'/\lambda) , \tag{B8}$$

which leads to (B4).

From (3.19), we have $t = 1/2\xi$ for large ξ and so integrating (B4) one obtains $N \sim (T)^{1/4}$, the answer originally exhibited by Finkelstein.

Finally, we consider the contribution to the density of states from short-range interactions. The triplet part gives an ordinary logarithmic term and requires no discussion. The singlet part is

$$\delta\zeta \sim \int dx\, d\omega \frac{\Gamma_1 - \frac{1}{2}\Gamma_2}{(x+z_1\omega)(x+z\omega)} . \tag{B9}$$

As the poles are at $x = z_1\omega$ and $x = z\omega$, and $z_1/z \to 0$, this integral is not represented adequately by either (3.1b) or (3.2b). Again, we must include the entire (3.2a) region and this gives

$$\delta\zeta = \frac{\Gamma_1 - \frac{1}{2}\Gamma_2}{z} t \ln(z_1/z) d\xi . \tag{B10}$$

Ignoring $\ln z_1$ compared to $\ln z$ and noting that $\gamma_1^* - \frac{1}{2}\gamma_2^* = \frac{1}{2}$, this leads to

$$\frac{d\ln N}{d\xi} = -\frac{t}{2}\ln z . \tag{B11}$$

[1]E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).

[2]F. J. Wegner, Z. Phys. B **35**, 207 (1979); L. Schaffer and F. J. Wegner, *ibid.* **38**, 113 (1980).

[3]B. L. Altshuler, A. G. Aronov, and P. A. Lee, Phys. Rev. Lett. **44**, 1288 (1980); B. L. Altshuler, D. Khmelnitskii, A. I. Larkin, and P. A. Lee, Phys. Rev. B **22**, 5142 (1980).

[4]H. Fukuyama, J. Phys. Soc. Jpn. **48**, 2169 (1980).

[5]G. S. Grest and P. A. Lee, Phys. Rev. Lett. **50**, 693 (1983).

[6]C. Castellani, C. diCastro, G. Forgacs, and E. Tabet, Nucl. Phys. B **225** [FS3], 441 (1983).

[7]A. M. Finkelstein, Zh. Eksp. Teor. Fiz. **84**, 168 (1983) [Sov. Phys.—JETP **57**, 97 (1983)].

[8]B. L. Altshuler and A. G. Aronov, Solid State Commun. **46**, 429 (1983).

[9]B. L. Altshuler, A. G. Aronov, and A. Yu. Zuzin, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 15 (1982) [JETP Lett. **35**, 16 (1982)].

[10]S. von Molnar, A. Briggs, J. Flouquet, and G. Remenyi, Phys. Rev. Lett. **51**, 706 (1983).

[11]S. Hikami, Phys. Rev. B **24**, 2671 (1981).

[12]C. Castellani, C di Castro, and G. Forgacs, Phys. Rev. B (to be published).

[13]R. Oppermann, Sol. State Commun. **44**, 1297 (1982).

[14]A. M. Finkelstein, Pis'ma Zh. Eksp. Teor. Fiz. **37**, 436 (1983).

[15]W. L. McMillan, Phys. Rev. B **24**, 2739 (1981).

[16]Y. Imry, Y. Gefen, and D. J. Bergmann, in *Anderson Localization,* edited by Y. Nagaokar and H. Fukuyama (Springer, New York, 1983), and unpublished.

[17]R. S. Markiewicz and C. J. Rollins, Phys. Rev. B **29**, 735 (1984).

[18]G. Hertl, D. I. Bishop, E. G. Spencer, J. M. Rowell, and R. C. Dynes, Phys. Rev. Lett. **50**, 743 (1983).

[19]M. A. Paalenen, T. F. Rosenbaum, G. A. Thomas, and R. N. Bhatt, Phys. Rev. Lett. **48**, 1284 (1982).