Fluctuations and limit of metastability in a periodically driven unstable system

F. de Pasquale, Z. Racz,* M. San Miguel,† and P. Tartaglia
Dipartimento di Fisica, Università degli Studi di Roma "La Sapienza," Piazzale Aldo Moro 2, I-00185 Roma, Italy
(Received 13 December 1983)

A mechanism for restoring broken symmetry in a system periodically driven above and below its instability point is discussed by studying the homogeneous, n-component, time-dependent Landau-Ginzburg model. We report numerical simulations for the $n=1$ case and an analytical solution for the spherical limit $n\rightarrow\infty$. In both cases the results show that fluctuations are enhanced by bringing the system near an unstable state periodically and as a consequence a well-defined shift in the instability point occurs. This shift can alternatively be characterized through the limit of the metastable behavior of the order parameter or by the large increase in the order-parameter fluctuations.

I. INTRODUCTION

The decay of unstable states is one of the fundamental problems of nonequilibrium statistical mechanics in which the understanding of the interplay of nonlinearities and fluctuations is crucial.[1,2] In most theoretical studies the unstable state is taken as a given initial condition, i.e., it is assumed that the initial condition is obtained by an instantaneous variation of a control parameter that brings the system from a stable to an unstable state. The effect of the finite velocity in the change of the control parameter has also been studied in particular cases.[3,4] A related situation is the one in which a system is periodically brought to an unstable situation by changing the control parameter back and forth through the instability point. This situation might be achieved in experiments, e.g., in the case of the decay of a laser unstable state[5] and in the spinodal decomposition of a polymer binary mixture.[6] By describing the above experiments in terms of Ginzburg-Landau models with the spatial inhomogeneities neglected, the processes can be related to the overdamped motion of a Brownian particle in a time-dependent potential.[7,8] In an interesting series of papers, Onuki[9] has considered the problem of spinodal decomposition[10] in a system periodically brought above and below the critical temperature. From an analysis of a time-dependent Ginzburg-Landau model, Onuki has concluded that the periodic variation of temperature leads to a strong enhancement of the fluctuations that causes nontrivial effects (for a summary, see Ref. 10). First, the critical temperature is shifted, and second, the transition becomes first order. These effects are also obtained when spatial inhomogeneities are neglected. Onuki states that his results cannot be conclusive due to the rather crude approximation scheme. In this paper we elucidate this problem.

Qualitatively, the combined effect of periodic modulation of the control parameter and fluctuations can be understood in the following way. In the absence of modulation, and beyond the instability point, the phase space of a finite-size system is divided into subspaces which correspond to the regions of symmetry breaking of the deterministic solutions. The system remains confined in one of these regions on physical time scales.[11] This corresponds to a metastable state that decays by the mixing of phase-space regions caused by fluctuations which restore the broken symmetry. This in general a very slow process. For example for an n-component vectorial Ginzburg-Landau model the mixing process is caused by Kramers escape for $n=1$ and by phase diffusion for $n\neq 1$.[12] In the presence of modulation a new mechanism of phase mixing appears. This mechanism leads to an enhancement of fluctuation effects. It can be explained in terms of the behavior of the stochastic trajectories. When the system is periodically brought close to an unstable state, a small fluctuation causes the escape from one phase region to another. This reduces drastically the lifetime of the otherwise metastable states. For this mechanism to be effective the system must spend a sufficiently long time around the unstable state. Essentially a time of the order of the mean first passage time is needed to leave the unstable state. The latter gives a lower bound of the period of modulation for which the new mechanism of phase-space mixing becomes important. It results that the transition from a metastablelike state to the disordered state occurs in a quite narrow range of the control parameters.

In this paper we show the above qualitative picture to be valid for a symmetric n-component model with a periodic modulation of the control parameter. We present the results of a numerical simulation for the $n=1$ case and we give a complete analytical solution of the problem in the spherical limit $n=\infty$. In this limit the two basic factors, nonlinearities and fluctuations, are taken into account in a nonperturbative way. In the spherical limit all the main features of the phenomenon previously described are maintained. The qualitative results are also in a reasonable agreement with the numerical simulation as far as the averaged field and intensity are concerned. Our calculations allow the qualification of Onuki's conclusions: the definition of the shifted instability point is related to a given observation time and it is shown to be meaningful only in an intermediate time scale which is much smaller than the decay rates of metastable states. The character of the transition cannot be properly established since there is no physical limit in which it becomes sharply defined.

The outline of the paper is as follows. In Sec. II we describe the symmetric n-vector model that we use in our calculations. We recall elementary facts about the model in the case in which the control parameter is constant in time. We also discuss the deterministic solution of the model when the control parameter becomes a periodic function of time. At the deterministic level the stability properties of the system remain unchanged. From the form of the deterministic solution we give an estimate of the value of the parameters and of the intensity of fluctuations for which they are expected to produce macroscopically large effects. In Sec. III we present the result of a numerical simulation of the $n=1$ model. It is shown that fluctuations modify the deterministic trajectories in a drastic way. As a consequence metastable states, which are long lived when the control parameter is constant, decay now after a few periods. The limit of metastablity (given an observation time) is quite well defined and coincides with the estimate in Sec. I. We obtain a fluctuation-modified phase diagram for the time-averaged order parameter which shows an effective shift of the instability point and a steplike growth of the averaged order parameter at the shifted instability point. In Sec. IV we discuss the analytical solution of the spherical limit of our model. We first describe the shift of the instability point in terms of an effective susceptibility and of a fluctuation-modified phase diagram. Second, we calculate the decay rate of metastable states. The analysis of the results for the decay rate gives an independent description of the enhancement of fluctuations. The shifted instability point is identified with the point at which the decay rate shows a pronounced change of magnitude. From this calculation we recover our first heuristic estimate. Details of the calculations are given in two appendixes.

II. THE MODEL

In this paper we consider the dynamics of a symmetric n-component model[12] defined by the following Langevin equation:

$$\dot{\varphi}_i(t)=-\frac{\partial\Phi(t)}{\partial\varphi_i(t)}+\sqrt{\epsilon}\xi_i(t),\quad i=1,\ldots,n \tag{2.1}$$

$$\Phi(t)=\frac{r_0(t)}{2}\sum_{i=1}^{n}\varphi_i^2(t)+\frac{1}{4n}\left[\sum_{i=1}^{n}\varphi_i^2\right]^2 . \tag{2.2}$$

The components φ_i of the vector $\vec{\varphi}\equiv\{\varphi_1,\ldots,\varphi_n\}$ are taken to be real numbers. The normalized modulus (intensity) of $\vec{\varphi}$ is

$$\psi(t)=\frac{1}{n}\sum_{i=1}^{n}\varphi_i^2 , \tag{2.3}$$

and $\psi(t)$ satisfies the following equation:

$$\dot{\psi}(t)=-2[r_0(t)+\psi(t)]\psi(t)+F(t) , \tag{2.4}$$

$$F(t)=\frac{2\sqrt{\epsilon}}{n}\sum_{i=1}^{n}\varphi_i(t)\xi_i(t) . \tag{2.5}$$

The stochastic driving forces $\xi_i(t)$ are assumed to be independent Gaussian white noise with

$$\langle\xi_i(t)\rangle=0 ,\qquad \langle\xi_i(t)\xi_j(t')\rangle=\delta_{ij}\delta(t-t') . \tag{2.6}$$

The parameter ϵ measures the noise strength. The coefficient $r_0(t)$ is assumed to be a periodic function of time. We are specifically interested in studying the role of fluctuations in the presence of such a modulation.

We first recall basic elementary facts of the model in the case in which $r_0(t)$ has a constant value r_0. In this case the system shows an instability at $r_0=0$: In the deterministic limit $\epsilon=0$, Eq. (2.4) has stable stationary solutions at $\psi_0=0$ for $r_0>0$ and $\psi_0=-r_0$ for $r_0<0$. The stationary solution $\psi_0=0$ becomes unstable at $r_0=0$ (Fig. 1). In the same limit it follows from (2.1) that the initial direction of the vector $\vec{\varphi}$ is kept fixed in time. Therefore, deterministically the model reduces to the $n=1$ case in which the stable stationary solutions of the variable φ are $\varphi_0=\pm\sqrt{\psi_0}$. In the presence of fluctuations the process is better described in terms of the probability distribution $P(\vec{\varphi},t)$. This distribution obeys a Fokker-Planck equation whose stationary distribution is

$$P_{st}(\vec{\varphi})=Ne^{-(2/\epsilon)\Phi} , \tag{2.7}$$

where N is a normalization constant. This distribution is peaked around the degenerate minimum of Φ and becomes sharply peaked as $\epsilon\to0$. For the mean value of ψ, small fluctuations only produce a rounding of the deterministic behavior (Fig. 1). Only for $\epsilon=0$ is the instability defined. The effect of fluctuations is more dramatic for the mean value of a component of the field $m(t)=\langle\varphi_i(t)\rangle$. Owing to the symmetry of the problem, $m(t)\to0$ as $t\to\infty$: fluctuations restore the symmetry which is broken in the deterministic limit by initial conditions. Nevertheless, for small fluctuations the restoring of the symmetry only occurs in a very long time scale. The phase diagram remains meaningful in the sense that the states with $m\neq0$ correspond to very long-lived metastable states. The decay of these states occurs via a different mechanism in the cases $n=1$ and $n\neq1$. For $n=1$ the different time scales of evolution of the systems are quite well understood.[13] The decay of metastable states occurs in the largest time scale given by Kramers escape time. For $n\neq1$ one can distinguish two well-separated time scales. In the first, ψ reaches its stationary value. In the

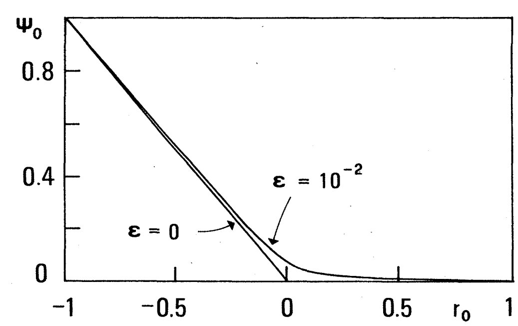

FIG. 1. Stationary value of the intensity ψ_0 in the absence of modulation in the deterministic limit $\epsilon=0$ and for $\epsilon=10^{-2}$.

second, a slow process of phase diffusion occurs. As a consequence the system will appear as "ordered" if the observation time is short in comparison to the typical decay time associated with phase diffusion. An explicit calculation of these time scale in the limit $n \to \infty$ is given in Ref. 12.

An alternative mechanism for symmetry restoring could appear in the case in which $r_0(t)$ is a periodic function of time such that the system is swept periodically above and below the instability point $r_0=0$. If the system comes sufficiently close to the unstable state, fluctuations are expected to be able to produce a transition to the disordered state restoring the symmetry in times which are much smaller than the diffusion time. For simplicity we consider instantaneous changes of $r_0(t)$ at times t_j as shown in Fig. 2. We have

$$r_0(t)=r_0+f(t)\ , \tag{2.8}$$

where $f(t)$ is a periodic function of period T given by

$$f(t)=\begin{cases} R, & t_{2j-1}<t<t_{2j}=t_{2j-1}+\dfrac{T}{2} \\ -R, & t_{2j}<t<t_{2j+1}\ . \end{cases} \tag{2.9}$$

R is the amplitude of the periodic modulation. We define three dimensionless parameters

$$\mu=RT,\quad \sigma=\frac{r_0}{R},\quad \nu=\frac{\epsilon}{R^2}\ . \tag{2.10}$$

μ and ν measure the amplitude of the modulation and the intrinsic fluctuations, respectively. For $|\sigma|<1$ the system is periodically driven through the instability. For $\sigma>0$ or $\sigma<0$ the system is, in the absence of modulation, below or above the instability point, respectively.

In order to see under which conditions the system is periodically brought close to the unstable state we first analyze the solutions of (2.1) and (2.4) with $r_0(t)$ given by (2.8), in the deterministic limit $\nu=0$. In this limit, Eq. (2.4) can be written in linear form, defining a new variable $x(t)$:

$$x(t)=\frac{\psi(0)}{\psi(t)}\ , \tag{2.11}$$

$$\dot{x}(t)=2r_0(t)x(t)+2\psi(0)\ . \tag{2.12}$$

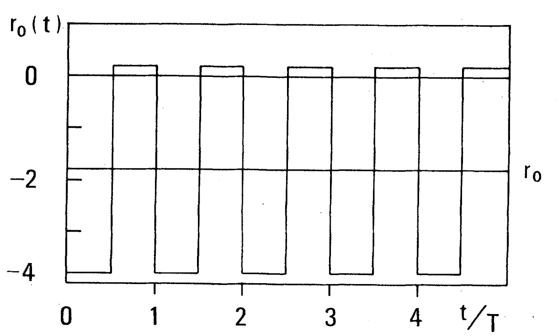

FIG. 2. Modulation $r_0(t)$ of Eq. (2.8) with $\gamma_0=-1.8$, $R=2$, $T=2$.

The equation for the field (2.1) is also brought into linear form defining

$$z(t)=\left[\frac{\varphi_i(0)}{\varphi_i(t)}\right]^2\ . \tag{2.13}$$

The variable $z(t)$ satisfies the same equation (2.12) for any i. The solution of (2.12) for $t\to\infty$ diverges for $r_0>0$ and goes to a periodic solution for $r_0<0$. This indicates that we still have an instability point at $r_0=0$. At this point $\psi(t)$ and $\varphi_i(t)$ change from a constant zero value to a time-dependent periodic function that we denote by $\psi^\infty(t)$ and $\varphi_i^\infty(t)$. The limit-cycle state for $r_0<0$ is[9]

$$x^\infty(t)=2\psi(0)\int_0^\infty ds\,\exp\left[2\sigma\mu\frac{s}{T}-2\int_t^{t-s}ds'f(s')\right]\ . \tag{2.14}$$

Taking into account that

$$\int_0^\infty ds\,\exp\left[2\sigma\mu\frac{s}{T}-2\int_t^{t-s}ds'f(s')\right]=\sum_{j=0}^{\infty}I_j(t) \tag{2.15}$$

and

$$\begin{aligned} I_j(t)&=\int_{t_{2j}}^{t_{2j-1}}ds\,\exp\left[2\mu\sigma\frac{s}{T}-2\int_t^{t-s}ds'f(s')\right] \\ &=e^{2\sigma\mu}I_{j-1}(t)\ , \end{aligned} \tag{2.16}$$

we have

$$x^\infty(t)=\frac{2\psi(0)}{1-e^{2\mu\sigma}}\int_0^T ds\,\exp\left[2\sigma\mu\frac{s}{T}-2\int_t^{t-s}ds'f(s')\right]\ . \tag{2.17}$$

Substituting the explicit form (2.9) of $f(s)$ we obtain from (2.13) two solutions of opposite sign for $\varphi^\infty(t)$ that we will denote as $\varphi_\pm^\infty(t)$. The behavior of these solutions is better understood by considering in limiting cases the values of $[\varphi^\infty(t)]^2$ at time $t=t_{2j},t_{2j-1}$. We find ($\sigma<0$)

(a) for $\mu \ll 1$,

$$[\varphi^\infty(t_{2j-1})]^2=-r_0\left[1+\frac{\mu}{2}\right]+O(\mu^2)\ , \tag{2.18}$$

$$[\varphi^\infty(t_{2j})]^2=-r_0\left[1-\frac{\mu}{2}\right]+O(\mu^2)\ , \tag{2.19}$$

(b) for $\mu\gg1$, $|\sigma|>1$,

$$[\varphi^\infty(t_{2j-1})]^2\approx-(r_0-R)\left[1-\frac{2}{\sigma+1}e^{\mu(\sigma-1)}\right]\ , \tag{2.20}$$

$$[\varphi^\infty(t_{2j})]^2\approx-(r_0+R)\left[1-\frac{2}{\sigma+1}e^{\mu(\sigma+1)}\right]\ , \tag{2.21}$$

(c) for $\mu\gg1$, $|\sigma|<1$,

$$[\varphi^\infty(t_{2j-1})]^2\approx-(r_0-R)\left[1-\frac{2}{\sigma+1}e^{2\mu\sigma}\right]\ , \tag{2.22}$$

$$[\varphi^\infty(t_{2j})]^2 \approx r_0 \frac{1-\sigma^2}{2\sigma} e^{\mu(\sigma+1)} \,. \tag{2.23}$$

These results indicate that for small values of μ, $\varphi^\infty_\mp(t)$ oscillates with a small amplitude around $-r_0$. For large values of μ, and $|\sigma|>1$, $\varphi^\infty_\mp(t)$ reaches a value close to $[-(r_0-R)]^{1/2}$ at the end of a first semiperiod and close to $[-(r_0+R)]^{1/2}$ at the end of the second semiperiod. For $|\sigma|<1$ the behavior is the same in the first semiperiod but at the end of the second semiperiod $\varphi^\infty_\pm \sim 0$ because $r_0(t)>0$ for $t_{2j-1}<t<t_{2j}$. We conclude that for large values of μ the variable $\varphi^\infty_\pm(t)$ follows the periodic modulation going at the end of each semiperiod to a value close to the corresponding equilibrium one. In Fig. 3 we show this behavior of the deterministic periodic trajectories as obtained from (2.17).

In order to have a phase diagram for the model we define averaged quantities over a period T in the limit-cycle state:

$$\bar{\psi}^\infty = \frac{1}{T}\int_0^T dt\, \psi^\infty(t) \,, \tag{2.24}$$

$$\bar{\psi}^\infty_\pm = \frac{1}{T}\int_0^T dt\, \varphi^\infty_\pm(t) \,. \tag{2.25}$$

The explicit calculation of these quantities is carried out in Appendix A.

We find $(r_0<0)$

$$\bar{\varphi}^\infty_\pm = \pm(\bar{\psi}^\infty)^{1/2} = \pm(-r_0)^{1/2} \,. \tag{2.26}$$

That is, the averaged quantities have the same value as in the absence of periodic modulation. Therefore, the modulation does not change the phase diagram of the system if ψ_0, φ_0 are replaced by $\bar{\psi}^\infty, \bar{\varphi}^\infty$.

It is easy to understand that small fluctuation can produce important changes in the above deterministic picture. For $\mu \gg 1$ and $|\sigma|<1$ it follows from (2.23) that a small fluctuation at time $t=t_{2j}$ can drive a trajectory φ^∞_+ into φ^∞_- and vice versa. This change of the deterministic behavior cannot be described perturbatively. The value of the parameters for which fluctuations have a nontrivial effect can be identified with the point at which perturbation theory breaks down. The perturbation expansion around the deterministic trajectory is

$$\varphi(t) = \varphi^\infty(t) + \sqrt{\epsilon}\varphi_1(t) + \cdots \,. \tag{2.27}$$

Perturbation theory breaks down when the stochastic trajectory changes signs. This happens most probably when

$$[\varphi^\infty(t_{2j})]^2 \approx \epsilon\langle \varphi_1^2(t_{2j})\rangle \,. \tag{2.28}$$

The computation of φ_1 is straightforward and the result gives

$$[\varphi^\infty(t_{2j})]^2 \approx \frac{\epsilon}{R} \,, \tag{2.29}$$

which can be easily derived as a dimensional estimate. Taking into account (2.23),

$$e^{-\mu(\sigma+1)} \frac{1-\sigma^2}{2} = \nu \,. \tag{2.30}$$

Asymptotically for large μ and small ν

$$\sigma_c \approx -1 - \frac{\ln\nu}{\mu} \,. \tag{2.31}$$

σ_c is an estimate of the value of σ for which fluctuations are expected to modify the macroscopic behavior of the system. It is worth noting that in the limiting case $\sigma \to 0$. Equation (2.30) gives

$$T \approx -\frac{1}{R}\ln\nu = -\frac{1}{R}\ln\left[\frac{\epsilon}{R^2}\right] \,. \tag{2.32}$$

This result can be interpreted as a lower bound for T for the existence of phase-space mixing. The right-hand side of Eq. (2.32) is actually an asymptotic estimate (for very small ν) of the mean first passage time[14] that the system needs to leave the unstable state.

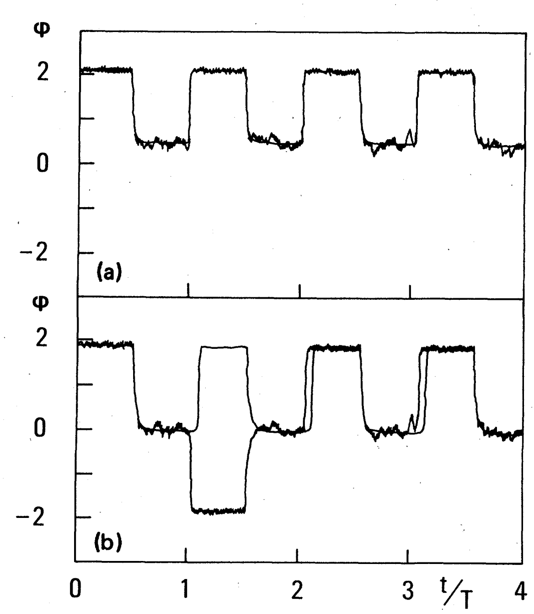

FIG. 3. Deterministic trajectory in the periodic stationary state (solid lines) and stochastic counterparts for $\nu=0.25\times10^{-2}$, $\mu=100$. (a), $\sigma=-1.1$; (b), $\sigma=-0.75$.

III. FLUCTUATIONS AND METASTABILITY IN THE $n=1$ CASE

In order to get a better understanding of the effect of fluctuations, it is convenient to look at the stochastic trajectories of the process (2.1). We made a numerical simulation of this process with $r_0(t)$ given by (2.8) and (2.9), following the method of Ref. 15. (See also Ref. 16.) Representative stochastic trajectories are shown in Figs. 3 and 4. For small values of μ, fluctuations only produce small deviations from the deterministic trajectory. For $\mu \gg 1$ the effect can be more dramatic depending on the values of σ. We distinguish four cases.

(i) $\sigma<-1$ [Fig. 3(a)]. The control parameter $r_0(t)$ oscillates but always being negative, that is, beyond the deterministic instability point. Starting from a deterministic periodic and positive trajectory, fluctuations produce small deviations around it. We assign to this

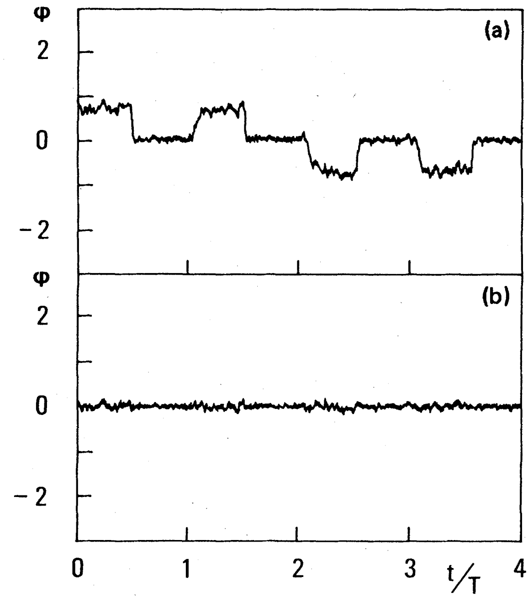

FIG. 4. Same as Fig. 3 ($\nu=0.25\times10^{-2}$; $\mu=100$), but (a) $\sigma=0.75$, (b) $\sigma=2$.

behavior an "ordered" state characterized by positive trajectories which are, on the average, periodic. Of course, this is not a stable state: After a long time there will always occur a sufficiently large fluctuation that will drive the trajectory to negative values. For small fluctuations this only happens for very long times so that very long-lived metastable states exist. The main effect of fluctuations is to change the stable periodic "ordered" state to a long-lived metastable state. This is completely analogous to what we discussed in Sec. II for the case with constant control parameter $r_0<0$.

(ii) $-1<\sigma<0$ [Fig. 3(b)]. The average value of the control parameter r_0 is still negative but now the system is swept periodically through the deterministic instability point $r_0(t)=0$. The difference with the previous case is that the deterministic trajectory $\varphi^\infty(t)$ now takes a very small value for $t=t_{2j}$. At each of these points a very small fluctuation can destabilize the trajectory driving it from positive to negative values or vice versa. This does not happen in (i) because there $\varphi^\infty(t_{2j})$ is a finite quantity. The effect of small fluctuations is to destroy the "ordered" periodic positive trajectory, producing a trajectory that goes randomly from positive to negative values. We assign to this behavior a "disordered" state characterized by a mixture of the two possible deterministic ordered states. The mechanism of mixing of the positive and negative regions of phase space is here different, and much more efficient, than Kramers escape over a barrier. It leads to the decay of a metastable state after a few periods. The time average over the trajectory of $\varphi(t)$ is zero in this situation. The existence of this disordered state for $r_0<0$ will produce an effective shift of the instability point due to fluctuations. We remark that the difference between the cases of $\sigma<-1$ and $-1<\sigma<0$ is only a matter of time scales of observation. For $t\to\infty$ both situations coincide, but there is an important physical difference for a large but finite observation time.

(iii) $0<\sigma<1$ [Fig. 4(a)]. The deterministic trajectory $\varphi^\infty(t)=0$. The system is periodically swept through the instability point $r_0(t)=0$ but now with $r_0>0$. Again the fluctuations change the qualitative form of the deterministic trajectory. Small fluctuations allow large departures of the trajectory to values close to $\pm[-(r_0-R)]^{1/2}$. These excursions are forbidden in the deterministic limit. The time average over the trajectory is zero. The main difference with the deterministic case is that the time average of the intensity ψ is nonzero and much larger than ν.

(iv) $\sigma>1$ [Fig. 4(b)]. For this value of σ, $r_0(t)>0$ for any t and the deterministic trajectory is $\varphi^\infty(t)=0$. Fluctuations produce small deviations around zero.

A quantitative characterization of the above description can be given in terms of the following averaged quantities:

$$\overline{m}_\pm(t)=\frac{1}{T}\int_t^{t+T}dt'\langle\varphi_\pm(t')\rangle\ , \tag{3.1}$$

$$\Delta\overline{m}_\pm(t)=\frac{1}{T}\int_t^{t+T}dt'[\langle\varphi_\pm^2(t')\rangle-\langle\varphi_\pm(t')\rangle^2]\ , \tag{3.2}$$

$$\langle\overline{\psi}(t)\rangle=\frac{1}{T}\int_t^{t+T}dt'\langle\psi(t')\rangle\ . \tag{3.3}$$

$\varphi_\pm(t)$ indicates, respectively, a stochastic trajectory with a finite positive or negative initial condition. The quantity $\langle\varphi_\pm(t)\rangle$ is the ensemble average over these trajectories, and $\overline{m}_\pm(t)$ is the time average over a period of this ensemble average. The quantity defined in (3.2) is the variance of $\overline{m}_+(t)$ and $\langle\overline{\psi}(t)\rangle$ is the average over a period of the ensemble average of the intensity $\psi(t)$. Owing to the symmetry of the model we know that for very long times the fluctuations destroy the memory of the initial condition, so that

$$\lim_{t\to\infty}\overline{m}_\pm(t)=0\ . \tag{3.4}$$

The difference between cases (i) and (ii) above is given by the decay time of $\overline{m}_\pm$ to zero. In case (ii) we expect that after an initial transient of a few periods $\overline{m}_\pm(t)\simeq0$. In case (i) we expect that after an initial transient $\overline{m}_\pm(t)$ reaches a plateau value and then decays very slowly to zero. Given a finite observation time $\tau\gg T$ we define

$$\overline{m}\,^\tau_\pm=\overline{m}_\pm(t=\tau)\ . \tag{3.5}$$

This quantity is the stochastic counterpart of the deterministic $\overline{\varphi}\,^\infty_\pm$ [Eq. (2.26)]. Its value is expected to depend weakly on τ. The two cases that we are discussing are separated by a fuzzy region in which $\overline{m}\,^\tau_\pm$ becomes nonzero. The explicit behavior of $\overline{m}_\pm(t)$ is shown in Figs. 5 and 6 for two representative cases together with $\Delta\overline{m}_+(t)$. In these figures $\overline{m}_\pm(t)$ is calculated as the time average over a single stochastic trajectory

$$\overline{m}_\pm(t)=\frac{1}{t}\int_0^t dt'\varphi_\pm(t')\ . \tag{3.6}$$

We note that it is only for $\tau\gg T$ that the time average over a period is a physical meaningful quantity. For $\mu\gg1$, $\sigma<0$, and τ of the order of a few periods, we are just observing repeatedly the decay of an unstable state.

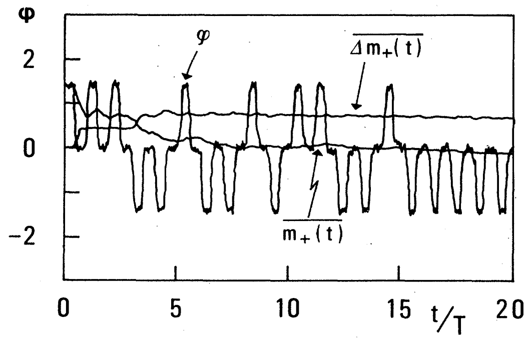

FIG. 5. A typical trajectory $\varphi(t)$ for $\mu=20$, $\sigma=0$, and $\nu=0.25\times10^{-2}$ and the corresponding time average $\overline{m}_+(t)$ and variance $\Delta\overline{m}_+(t)$.

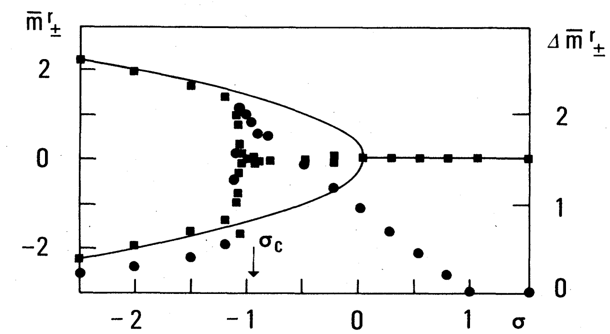

FIG. 7. "Phase diagram" for the time average over a period $\overline{m}^{\tau}_{\pm}$ (■) for $\mu=100$, $\nu=0.25\times10^{-2}$, and the corresponding variance $\Delta\overline{m}^{\tau}_{\pm}$(●). The solid line is the deterministic result.

We have calculated $\overline{m}^{\tau}_{\pm}$ for different values of the parameters μ, ν, σ. The resulting "phase diagram" is shown in Fig. 7. This diagram cannot be taken in a strict sense: There is no well-defined instability point and the values of $\overline{m}^{\tau}_{\pm}$ depend on the observation time and on the value of fluctuations.[17] Nevertheless, it is clear that fluctuations change the "phase diagram" in a nontrivial way. There exists a narrow region of values of around a "critical" value σ_c ($-1<\sigma_c<0$) in which $\overline{m}^{\tau}_{\pm}$ goes from zero to a finite value close to the corresponding deterministic one (2.26). This phenomenon can be interpreted as an effective macroscopic shift of the instability point, from $\sigma=0$ to σ_c, induced by fluctuations. The value of σ_c cannot be defined unambiguously since for $\nu=0$ the whole effect disappears. But it is not a small effect, since it happens in the absence of periodic modulation. The quantity σ_c plays a role here analogous to the "cloud point" in the nucleation problem.[10] It is a stability limit which depends on the observation time. As shown in Fig. 7 this stability limit also manifests itself in a large increase of the variance $\Delta\overline{m}(t=\tau)$ for $\sigma\simeq\sigma_c$. From the results shown in Fig. 7 it is seen that the numerical value of σ_c is in agreement with our estimate (2.31). The width of the transition region around σ_c diminishes when increasing μ or decreasing ν. Also $|\sigma_c|$ becomes larger in the same circumstances, in agreement with (2.31). Another important feature that distinguishes the behavior of $\overline{m}^{\tau}_{+}$ from that of $\overline{\varphi}^{\infty}$ is that while $\overline{\varphi}^{\infty}$ grows continuously from zero at $\sigma=0$, $\overline{m}^{\tau}_{+}$ has a steplike growth at $\sigma\approx\sigma_c$. Since $\overline{\varphi}^{\infty}\approx\overline{m}^{\tau}_{\pm}$ for $\sigma<\sigma_c$, the effect of fluctuations is pictorially summarized as a cut of the phase diagram at $\sigma\approx\sigma_c$. They are not important for $\sigma<\sigma_c$ and they reduce the order parameter to zero for $\sigma>\sigma_c$.

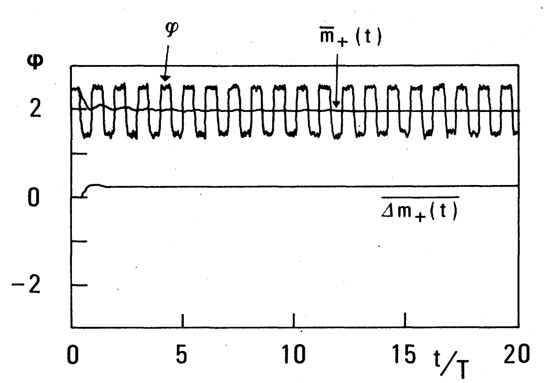

FIG. 6. Same as Fig. 5 for $\sigma=-2$; $\mu=20$, $\nu=0.25\times10^{-2}$.

We finally discuss the results of the numerical simulation for $\langle\overline{\psi}(t)\rangle$. After an initial transient this quantity reaches a steady-state value and it remains constant for $t\to\infty$. This value practically coincides with the deterministic one $\overline{\psi}^{\infty}$ for $|\sigma|>1$. As we discussed before, the steady-state value for $0<\sigma<1$ is macroscopically different from the zero value of the deterministic case. In this sense, the instability point for the intensity is effectively shifted by fluctuations in the opposite direction as compared to $\overline{\varphi}^{\infty}$. It is worth noting that this shifted point is not accompanied by an increase of the variance of the intensity fluctuations. For $-1<\sigma<0$, $\langle\overline{\psi}(t)\rangle$ also has a larger value than in the deterministic case. The behavior of $\langle\overline{\psi}(t)\rangle$ can be interpreted as an enhancement of fluctuations caused by the periodic modulation.

IV. SPHERICAL LIMIT

A. Equations

We have already mentioned in Sec. II that the effect of fluctuations in the presence of a periodic modulation of the control parameter cannot be studied by a perturbative treatment. Perturbation theory also fails to describe the relaxation of an unstable system in the case without periodic modulation.[1] The more elaborate quasideterministic theory (QDT) that describes the decay of an unstable state[1] also fails in the case with periodic modulation. The reason is that in QDT, fluctuations saturate after the initial transient of interest. For $t\to\infty$, QDT becomes deterministic. In our case fluctuations are important at the end of every period of the deterministic periodic state. It is then desirable to have a model including the two basic ingredients of the phenomenon, described in Sec. III, i.e., nonlinearity and fluctuations, for which a nonperturbative analytical solution can be given. Such a model is given by the spherical limit $n\to\infty$ of (2.1). The solution of this model will be useful to check the validity of the estimate (2.31) and it also gives the behavior of the system for $\sigma > \sigma_c$ when perturbation theory cannot be used. One may only expect a qualitative agreement between the model and the $n=1$ simulation because of the problem of phase diffusion discussed in Sec. II.

In order to take the spherical limit of (2.4) we consider the properties of the stochastic force $F(t)$ (Ref. 18):

$$\langle F(t) \rangle = \epsilon \; , \tag{4.1}$$

$$\langle F(t)F(t') \rangle = \frac{4\epsilon}{n}\psi(t)\delta(t-t') \; . \tag{4.2}$$

We can write

$$F(t) = \epsilon + 2\left[\frac{\epsilon}{n}\right]^{1/2}\eta(t) \tag{4.3}$$

with $\eta(t)$ having zero mean value and $(4\epsilon/\eta) \times \langle \eta(t)\eta(t') \rangle = \langle F(t)F(t') \rangle$. In the $n \to \infty$ limit, $F(t) \to \epsilon$ and Eq. (2.4) for ψ becomes deterministic:

$$\dot{\psi}(t) = -2(r_0(t)+\psi(t)]\psi(t)+\epsilon \; . \tag{4.4}$$

It is important to note that this equation is not the same as the deterministic limit of (2.4) but it includes the effect of fluctuations. Equation (2.1) can now be written as

$$\dot{\varphi}_i(t) = -[r_0(t)+\psi(t)]\varphi_i(t)+\sqrt{\epsilon}\xi_i(t) \; . \tag{4.5}$$

Equations (4.4) and (4.5) give a complete description of the problem in the limit $n \to \infty$. In this limit an exact solution is possible since (4.4) is deterministic and, given a solution of (4.4), (4.5) is a linear stochastic differential equation for $\varphi_i(t)$. The mean value $m(t)$ of $\varphi_i(t)$ can also be directly obtained from a closed linear integro-differential equation that follows from (4.4) and (4.5). We have

$$\dot{m}(t) = -r(t)m(t) \; , \tag{4.6}$$

$$r(t) \equiv r_0(t)+\psi(t) \; . \tag{4.7}$$

We define a new variable $y(t)$,

$$y(t) \equiv \left[\frac{m(0)}{m(t)}\right]^2 . \tag{4.8}$$

From (4.5) we obtain an equation for $y(t)$ which, with the use of the solution of (4.4) becomes

$$\tfrac{1}{2}\dot{y}(t) = r_0(t)y(t)+\psi(0)+\epsilon\int_0^t dt' y(t') \; . \tag{4.9}$$

This equation gives an alternative description of the problem to the one provided by (4.4) and (4.5).

In Fig. 8 we compare a simulated trajectory for $n=1$ and a trajectory corresponding to (4.5) with the same sequence of random numbers. Two main differences are shown. An obvious difference is that for the spherical model φ_i reaches quite different values in the semiperiods $t_{2j} < t < t_{2j+1}$. A second difference is that the spherical trajectory changes sign at different times than the corresponding $n=1$ trajectory. This means that the spherical model does not give a good approximation of the $n=1$ trajectories in a one-to-one correspondence. Nevertheless it will be shown that it reproduces qualitatively the results

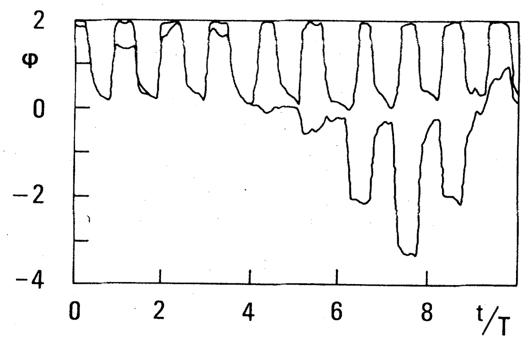

FIG. 8. Stochastic trajectories for $n=1$ (positive values) and $n=\infty$ (positive and negative values).

of the $n=1$ case. It also gives a good quantitative agreement for the effective susceptibility defined below Eq. (4.10).

In the remainder of this section we characterize the phenomena described in Sec. III in terms of the two alternative descriptions mentioned above. We first (Sec. IV B) give a static description in terms of the behavior of the effective susceptibility (Fig. 9)

$$\bar{r}^{\infty} = \lim_{t\to\infty}\frac{1}{T}\int_t^{t+T} ds\, r(s) \; . \tag{4.10}$$

Secondly (Sec. IV C), we give a dynamic description in terms of the eigenvalues of (4.9). These eigenvalues give the decay rate of the metastable states characterized by $\overline{m}\,^T_{\pm} \neq 0$. The values of the parameters for which the decay rate is no longer very small characterize the shifted instability point σ_c. In the asymptotic limit of large μ and small fluctuations this point becomes well defined and we recover our first estimate (2.31).

B. Susceptibility

In order to describe the behavior of $\bar{r}^{\infty}$, we first calculate $\psi(t)$. (For details see Appendix A.) Equation (4.4) is easily solved in each semiperiod $t_j < t < t_{j+1}$ in which $r_0(t)$ is constant. Matching these solutions, we obtain recursion relations that give the value of ψ at the end of a period as a function of its value at the end of the previous period

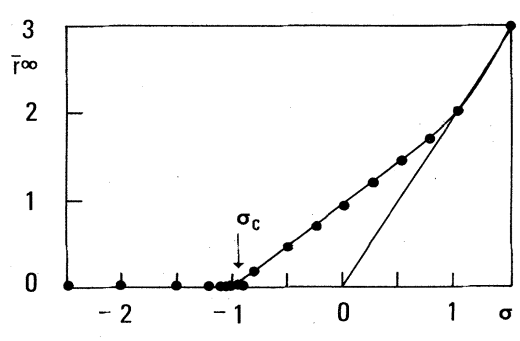

FIG. 9. Effective susceptibility $\bar{r}^{\infty}$ as a function of σ for $\epsilon=0$ and $\epsilon=10^{-2}$ for $n=1$ (dots) and $n=\infty$ (solid line).

$$\psi(t_{2j})=F_+(\psi(t_{2j-2})) , \tag{4.11}$$

$$\psi(t_{2j+1})=F_1(\psi(t_{2j-1})) . \tag{4.12}$$

The recursion relations have fixed point solutions $\psi^*_\pm$

$$\psi^*_+=F_+(\psi^*_+) , \tag{4.13}$$

$$\psi^*_-=F_-(\psi^*_-) . \tag{4.14}$$

For $t\rightarrow\infty$, $\psi(t)$ tends to a periodic function $\psi^\infty_\epsilon(t)$

$$\psi^\infty_\epsilon(t)=\lim_{t\rightarrow\infty}\psi(t) . \tag{4.15}$$

Discretizing the time in units of $T/2$, $\psi^\infty_\epsilon(t)$ evolves in a limit cycle of period 2 taking values ψ^*_+ and ψ^*_-. For large values of μ and $\sigma<1$, $\psi^*_\epsilon(t)$ oscillates with a large amplitude going close to zero at times t_{2j}. The periodic solution $\psi^\infty_\epsilon(t)$ can be expressed, for any t, in terms of the fixed points $\psi^*_\pm$ and other auxiliary quantities. The average over a period gives (see Appendix A)

$$\begin{aligned}\bar{\psi}^\infty_\epsilon&=\frac{1}{T}\int_t^{t+T}dt'\psi(t')\\&=\frac{R}{2}\left\{(a_++a_-)+\frac{1}{\mu}\ln\left[1+\left[\frac{\psi^*_+}{R}-a_-\right]\frac{1-e^{-\mu\Gamma_-}}{\Gamma_-}\right]\right.\\&\qquad\left.+\frac{1}{\mu}\ln\left[1+\left[\frac{\psi^*_-}{R}-a_+\right]\frac{1-e^{-\mu\Gamma_+}}{\Gamma_+}\right]\right\} ,\end{aligned} \tag{4.16}$$

where

$$a_\pm=\tfrac{1}{2}[\Gamma_\pm-(\sigma\pm1)] , \tag{4.17}$$

$$\Gamma_\pm=[(\sigma\pm1)^2+2\nu]^{1/2} . \tag{4.18}$$

$\bar{\psi}^\infty_\epsilon$ and $\psi^\infty_\epsilon(t)$ differ from the corresponding deterministic quantities $\bar{\psi}^\infty$ and $\psi^\infty(t)$ due to the ϵ term in (4.4).

The effective susceptibility $\bar{r}^\infty$ (4.10) is given by

$$\bar{r}^\infty=r_0+\bar{\psi}^\infty_\epsilon . \tag{4.19}$$

We recall that in the case without modulation ($R=0$) (Ref. 12),

$$\bar{r}^\infty=\tfrac{1}{2}[r_0+(r_0^2+2\epsilon)^{1/2}] . \tag{4.20}$$

In the deterministic limit $\epsilon=0$ (4.20) gives

$$\bar{r}^\infty=\begin{cases}0, & r_0<0\\ r_0, & r_0>0 .\end{cases} \tag{4.21}$$

which indicates that the system undergoes an instability at $r_0=0$. For $\epsilon\neq0$ the instability is not sharply defined but $\bar{r}^\infty$ deviates from (4.21) only in small quantities of the order $\epsilon^{1/2}$. In the case with periodic modulation of $r_0(t)$ and in the deterministic limit, we obtain from (4.19) and (2.26) the same result (4.21). That is, a well-defined instability at $r_0=0$ independently of the value of R. The combined effect of a periodic modulation of $r_0(T)$ and fluctuations of the system is described by $\bar{r}^\infty$ obtained from (4.19) and (4.16) (Fig. 10). The small effect of fluctuations for $R=0$ is now enhanced, producing a macroscopic shift, from $\sigma=0$, of the value of σ at which $\bar{r}^\infty$ becomes significantly different from zero. This value of σ is identified with σ_c. It shows the same qualitative behavior discussed in Sec. III. It is also seen that the value obtained from (4.16) and (4.19) agrees very well with the one obtained from the numerical simulation for $n=1$. We note that $\bar{r}^\infty$ is a stationary quantity independent of any observation time: The effective shift of the instability point is described here without reference to metastable states and intermediate time scales.

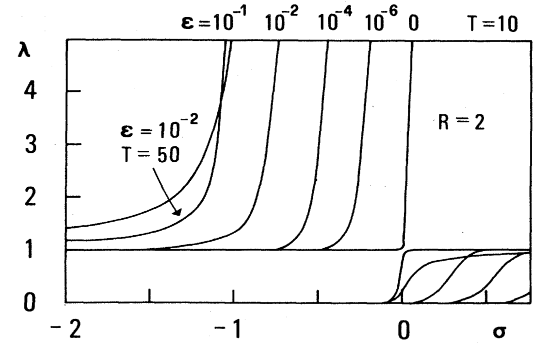

FIG. 10. Eigenvalues λ of the equation for y vs σ for different values of ϵ for $R=2$, $T=10$ and 50.

Finally, it is worth remarking that the averaged intensity $\bar{\psi}^\infty_\epsilon$ itself is macroscopically modified by fluctuations. This is in fact what causes the behavior of $\bar{r}^\infty$ that we have discussed.

C. Metastability and decay rates

Taking the time derivative of (4.9) we have

$$\tfrac{1}{2}\ddot{y}(t)=r_0(t)\dot{y}(t)+\epsilon y(t) . \tag{4.22}$$

In a given semiperiod, $r_0(t)$ is constant and we construct a matrix that, when applied to the vector $\vec{V}(t)=(y(t),\dot{y}(t))$ at the beginning of the semiperiod, gives the values of $\vec{V}$ at the end of the semiperiod:

$$\vec{V}(t_{2j+1})=\underline{A}_-\vec{V}(t_{2j}) , \tag{4.23}$$

$$\vec{V}(t_{2j+2})=\underline{A}_+\vec{V}(t_{2j+1}) . \tag{4.24}$$

We diagonalize the matrix $\underline{A}_+\underline{A}_-$ by a transformation $\underline{U}$

$$\underline{B}=\underline{U}\underline{A}_+\underline{A}_-\underline{U}^{-1}=\begin{bmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{bmatrix} , \tag{4.25}$$

so that

$$\vec{W}(t_{2j})=\begin{bmatrix}\lambda_+^j & 0\\ 0 & \lambda_-^j\end{bmatrix}\vec{W}(t=0) , \tag{4.26}$$

where

$$\vec{W}=\underline{U}\vec{V} . \tag{4.27}$$

Note that by definition $\vec{V}(t=0)\neq\vec{0}$. The two eigenvalues λ_+ and λ_- of B govern the dynamics of the system. At a time t_{2j}, y is a linear combination of λ_+^j and λ_-^j,

$$y(t_{2j})=\alpha\lambda_+^j+\beta\lambda_-^j . \tag{4.28}$$

The explicit calculation of these eigenvalues gives (see Appendix B)

$$\lambda_{\pm}=e^{\mu\sigma}[\Omega\pm(\Omega^2-1)^{1/2}]\ , \tag{4.29}$$

where

$$\Omega=\frac{(1-\delta_-^2)\cosh(\mu\delta_-)-(1-\delta_+^2)\cosh(\mu\delta_+)}{\delta_+^2-\delta_-^2}\ , \tag{4.30}$$

$$\delta_{\pm}=\frac{\Gamma_+\pm\Gamma_-}{2}\ . \tag{4.31}$$

$\Gamma_{\pm}$ are defined through (4.18).

The eigenvalues $\lambda_{\pm}$ are plotted in Fig. 10 for several values of the parameters. Since $\lambda_-<1$, for large j

$$y(t_{2j})\approx\alpha\lambda_+^j\ . \tag{4.32}$$

For large μ, λ_+ changes quite drastically from a value near 1 to a large value. This change, which occurs in a narrow region of values of σ for $\mu\gg1$ and small ν, explains the two main effects discussed in Sec. III: the effective shift of the instability point and the fact that $\overline{m}_+^{\tau}$ has a steplike growth from 0 to a value close to the deterministic one. First, for λ_+ near 1, y grows very slowly with j, which means that $m(t)$ decays very slowly to zero. This corresponds to the metastable states in which $\overline{m}_+^{\tau}\neq0$. When λ_+ becomes very large, $m(t)$ decays very fast. Therefore the region of values of σ for which λ_+ starts to grow abruptly must be interpreted as the instability region $\sigma\approx\sigma_c$ in which for fixed and large observation time τ, $\overline{m}_{\pm}^{\tau}$ becomes nonzero. This region coincides with the one of Figs. 7 and 9. It shows in a different way the same macroscopic shift of the instability point from $\sigma=0$ to σ_c. The steplike growth of $\overline{m}_{\pm}^{\tau}$ at the instability point σ_c is also clearly explained.

For $\sigma<\sigma_c$, λ_+ is close to one, which is its deterministic value (see below), so that $m_{\pm}^{\tau}$ is close to its deterministic large value. For $\sigma>\sigma_c$, λ_+ becomes very large and $\overline{m}_{\pm}^{\tau}$ close to zero. A better understanding of the problem is obtained considering some limiting situations of (4.29).

(i) We first consider the case in which r_0 has a constant value (absence of modulation, $R=0$). In this case (4.29) reduces to

$$\lambda_{\pm}=e^{\mu[\sigma\pm(\sigma^2+2\nu)^{1/2}]}\ . \tag{4.33}$$

In particular $\lambda_-<1$. Repeating the discussion above for λ_+, we conclude the existence of a smoothed instability at $\sigma=0$. In the limit $\nu=0$

$$\lambda_+=\begin{cases}1 & \text{for } \sigma<0\\ e^{2\sigma\mu} & \text{for } \sigma>0\ .\end{cases} \tag{4.34}$$

This shows how the instability becomes well defined for $R=0$ and $\nu=0$ as we already discussed.

(ii) In the limit of vanishing fluctuations $\nu=0$ (but $R\neq0$) we recover again (4.34) in agreement with the deterministic discussion of Sec. II and the behavior of $\overline{r}^{\infty}$. We note that in this limit λ_+ is independent of R.

(iii) The interplay of periodic modulation of r_0 and fluctuations is made explicit considering the value of λ_+ at $\sigma=0$. For small fluctuations ν we obtain the following from (4.29):

$$\lambda_+=\begin{cases}1+\nu e^{\mu} & \text{for } \mu\gg1\\ 1+\sqrt{2\nu}\mu & \text{for } \mu\ll1\ .\end{cases} \tag{4.35}$$

For $\mu\ll1$, λ_+ only has a small increase from the deterministic value, but for large μ the effect of fluctuations is amplified by an exponential factor in μ. This amplification is another way of understanding the failure of any perturbative treatment around the deterministic trajectory. For $\nu\ll1$ and $\mu\approx1$, we obtain

$$\lambda_+=1+2\sqrt{\nu}(\cosh\mu-1)^{1/2}\ . \tag{4.36}$$

These results for λ_+ and the general ones shown in Fig. 10 make it clear that only for large μ and small fluctuations is the effect of fluctuations amplified in a nontrivial sense. In fact, the lower bound (2.32) for T is here recovered from (4.35). Therefore we now study the value of λ_+ in the limit $\mu\gg1$. Since in this limit $\lambda_+(\sigma=0)$ can already deviate significantly from 1, we only consider the case $\sigma<0$.

(iv) For $\mu\gg1$, $\nu/(\sigma+1)^2\ll1$, and $\sigma<0$, (4.26) reduces to

$$\lambda_+=e^{\mu[\nu\sigma/(1-\sigma^2)]}+\frac{2\nu}{(1-\sigma^2)^2}e^{\mu(\sigma+1)}\ . \tag{4.37}$$

This formula is not valid near $\sigma\approx0$, a situation which we have already studied, nor in a small region of the order of ν near $\sigma=-1$. For realistic values of μ and ν we can assume that $\mu\nu\sigma/|1-\sigma^2|\ll1$. In this case,

$$\lambda_+=\begin{cases}1+\dfrac{2\nu}{(1-\sigma^2)^2}e^{\mu(\sigma+1)} & \text{for } |\sigma|<1\\ 1+\dfrac{\nu\mu\sigma}{1-\sigma^2} & \text{for } |\sigma|>1\ .\end{cases} \tag{4.38}$$

Equation (4.38) implies a large increase of λ_+ for $\sigma>-1$. For $\sigma<-1$ there is only a small correction to the deterministic value $\lambda_+=1$. For $\sigma>-1$ the effect of a small fluctuation is again amplified by an exponential factor in $\mu(\sigma+1)$. From this expression we obtain an estimate of the value σ_c for which λ_+ becomes significantly different from one. This is fixed by

$$\frac{2\nu}{(1-\sigma_c^2)^2}e^{\mu(\sigma_c+1)}=a\ ,$$

a being a finite quantity larger than ν. In the asymptotic limit in which (4.38) is correct, the value of σ_c is independent of a and we again obtain (2.31). This value of σ_c first obtained as a limit of stability of deterministic trajectories against small fluctuations is again obtained as the point at which metastable states become short lived.

V. CONCLUSIONS AND OUTLOOK

In this paper we have shown that in a periodically driven unstable system, fluctuations produce an important macroscopic effect. There is a change in the limit of metastability of the system that can be interpreted as an effective shift of the instability point. The change in the limit of metastability is a consequence of the existence of a mechanism which restores the broken symmetry in time scales much shorter than those of the Kramers escape or phase-diffusion processes. Although results in that case seem to be quite general, we have given an estimate of σ_c only for a particular form of the periodic modulation and with the neglect of inhomogeneous fluctuations.

The phenomenon we have studied can be of importance in discussing the nature of asymptotic metastable states. In general, however, is a further analysis of the role of inhomogeneous fluctuations needed. In particular, the role of the dimensionality of the system should be investigated. In the case where the system is not driven periodically, symmetry is not restored for dimensionality $d>2$ and metastability disappears. In the case we consider here, the mechanism of symmetry restoring for $-1<\sigma<0$ seems to be independent of dimensionality. We finally point out that the phenomena we have discussed may be relevant to provide evidence for the role of metastability in disordered systems.

APPENDIX A

The recursion relations for ψ can be obtained by solving Eq. (4.4) in each semiperiod. The result is

$$\psi(t_{2n+1})=\frac{(a_- - b_- e^{-\mu\Gamma_-})\psi(t_{2n})-a_-b_-(1-e^{-\mu\Gamma_-})}{(1-e^{-\mu\Gamma_-})\psi(t_{2n})-(b_- - a_- e^{-\mu\Gamma_-})} ,$$

$$\psi(t_{2n})=\frac{(a_+ - b_+ e^{-\mu\Gamma_+})\psi(t_{2n-1})-a_+b_+(1-e^{-\mu\Gamma_+})}{(1-e^{-\mu\Gamma_+})\psi(t_{2n})-(b_+ - a_+ e^{-\mu\Gamma_+})} ,$$

with

$$a_\pm=-\tfrac{1}{2}(\sigma\pm1-\Gamma_\pm) ,$$

$$b_\pm=-\tfrac{1}{2}(\sigma\pm1+\Gamma_\pm) ,$$

and

$$\Gamma_\pm=[(\sigma\pm1)^2+2\nu]^{1/2} ,$$

from which one obtains Eqs. (4.11) and (4.12). The fixed-point solutions result then from the conditions

$$\psi(t_{2n})=\psi(t_{2n-1})=\psi_+^* ,$$

$$\psi(t_{2n+1})=\psi(t_{2n-1})=\psi_-^* .$$

Explicitly, the first equation is

$$[(1-e^{-\mu\Gamma_+})(a_- - b_- e^{-\mu\Gamma_-})-(1-e^{-\mu\Gamma_-})(b_+ - a_+ e^{-\mu\Gamma_+})](\psi_+^*)^2$$
$$+[(b_+ - a_+ e^{-\mu\Gamma_+})(b_- - a_- e^{-\mu\Gamma_-})-(a_+ - b_+ e^{-\mu\Gamma_+})(a_- - b_- e^{-\mu\Gamma_-})]\psi_-^*$$
$$-\frac{\epsilon}{2}(a_+ - b_+ e^{-\mu\Gamma_+})(1-e^{-\mu\Gamma_-})+\frac{\epsilon}{2}(b_- - a_- e^{-\mu\Gamma_-})(1-e^{-\mu\Gamma_+})=0 ,$$

and analogously for ψ_-^*. Then within the semiperiod $t_{2n}\le t\le t_{2n+1}$ for large n

$$\psi_\epsilon^\infty(t)=\lim_{t\to\infty}\psi(t)=\frac{[a_- - b_- e^{-2R\Gamma_-(t-t_{2n})}]\psi_+^* + \frac{\epsilon}{2}[1-e^{-2R\Gamma_-(t-t_{2n})}]}{[1-e^{-2R\Gamma_-(t-t_{2n})}]\psi_+^* - [b_- - a_- e^{-2R\Gamma_-(t-t_{2n})}]} ,$$

and similarly for $t_{2n+1}\le t\le t_{2n+2}$. Thus it is possible to calculate the average over a period for large n

$$\bar{\psi}_\epsilon^\infty=\frac{1}{T}\int_{t_{2n}}^{t_{2n+1}}dt'\psi_\epsilon^\infty(t')$$

resulting in Eq. (4.16).

APPENDIX B

Equation (4.9) can be solved in a semiperiod $t_{2n}\le t\le t_{2n+1}$, where $r_0(t)$ is a constant given by r_0-R, by taking the time derivative [Eq. (4.22)] with the result

$$y(t)=-\frac{2b_-y(t_{2n})+\dot{y}(t_{2n})}{2\Gamma_-}e^{-2a_-R(t-t_{2n})}+\frac{2a_-y(t_{2n})+\dot{y}(t_{2n})}{2\Gamma_-}e^{-2b_-R(t-t_{2n})}$$

in terms of the values of y and its derivative at the beginning of the semiperiod. In particular for $t=t_{2n+1}$

$$y(t_{2n+1})=-\frac{2b_-y(t_{2n})+\dot{y}(t_{2n})}{2\Gamma_-}e^{-\mu a_-}+\frac{2a_-y(t_{2n})+\dot{y}(t_{2n})}{2\Gamma_-}e^{-\mu b_-} ,$$

and from Eq. (4.9)

$$\dot{y}(t_{2n})=2(r_0-R)y(t_{2n})+2\psi(0)+2\epsilon\int_0^{t_{2n}}dt'y(t') .$$

With the use of an analogous formula for the semiperiod $t_{2n+1}\le t\le t_{2n+2}$ and after some algebra, we get Eqs. (4.23) and (4.24) with the matrices $\underline{A}_\pm$ given by

$$\underline{A}_{\pm}=\begin{bmatrix} \dfrac{a_{\pm}e^{-\mu b_{\pm}}-b_{\pm}e^{-\mu a_{\pm}}}{\Gamma_{\pm}} & \dfrac{e^{-\mu b_{\pm}}-e^{-\mu a_{\pm}}}{2R\Gamma_{\pm}} \\ \dfrac{2a_{\pm}}{\Gamma_{\pm}}\left[\sigma\mp 1-\dfrac{\nu}{2b_{\pm}}\right]e^{-\mu b_{\pm}} & \dfrac{1}{\Gamma_{\pm}}\left[\sigma\mp 1-\dfrac{\nu}{2b_{\pm}}\right]e^{-\mu b_{\pm}} \\ -\dfrac{2b_{\pm}}{\Gamma_{\pm}}\left[\sigma\mp 1-\dfrac{\nu}{2a_{\pm}}\right]e^{-\mu a_{\pm}} & -\dfrac{1}{\Gamma_{\pm}}\left[\sigma\mp 1-\dfrac{\nu}{2a_{\pm}}\right]e^{-\mu a_{\pm}} \end{bmatrix}.$$

The product $\underline{A}_{+}\underline{A}_{-}$ is then given by

$$\underline{A}_{+}\underline{A}_{-}=\begin{bmatrix} k[b_{-}\beta_1-a_{-}\beta_2-b_{-}\beta_3+a_{-}\beta_4] & \dfrac{k}{2R}[\beta_1-\beta_2-\beta_3+\beta_4] \\ 2Rk[-b_{-}(a_{+}+2)\beta_1+a_{-}(a_{+}+2)\beta_2 & k[-(a_{+}+2)\beta_1+(a_{+}+2)\beta_2 \\ +b_{-}(b_{+}+2)\beta_3-a_{-}(b_{+}+2)\beta_4] & +(b_{+}+2)\beta_3-(b_{+}+2)\beta_4] \end{bmatrix}$$

with

$$k=\frac{e^{\sigma\mu}}{\Gamma_{+}\Gamma_{-}},$$

$$\beta_1=(1-\delta_{+})e^{-\mu\delta_{+}},$$

$$\beta_2=(1-\delta_{-})e^{-\mu\delta_{-}},$$

$$\beta_3=(1-\delta_{-})e^{\mu\delta_{-}},$$

$$\beta_4=(1+\delta_{+})e^{\mu\delta_{+}},$$

$$\det(\underline{A}_{+}\underline{A}_{-})=e^{2\sigma\mu}.$$

The calculation of the two eigenvalues $\lambda_{\pm}$ is then straightforward with the result given by Eqs. (4.30)—(4.32).

*Permanent address: Institute for Theoretical Physics, Roland Eötvös University, H-1088 Budapest, Hungary.
†Permanent address: Departamento de Fisica Teorica, Universidad de Barcelona, Diagonal 647, Barcelona 28, Spain.

[1]F. de Pasquale, P. Tartaglia, and P. Tombesi, Phys. Rev. A **25**, 466 (1982); Z. Phys. B **43**, 353 (1981).
[2]M. Suzuki, Adv. Chem. Phys. **46**, 195 (1980).
[3]B. Caroli, C. Caroli, B. Roulet, and D. Saint James, Physica (Utrecht) **108A**, 233 (1981).
[4]G. Ahlers, M. C. Cross, P. C. Hohenberg, and S. Safran, J. Fluid Mech. **110**, 297 (1981).
[5]F. T. Arecchi, V. Degiorgio, and B. Querzola, Phys. Rev. Lett. **19**, 1168 (1967); Phys. Rev. A **3**, 1108 (1971).
[6]T. Nishi, T. T. Wang and T. K. Kwei, Macromolecules **8**, 227 (1975).
[7]Weidlich and G. Haag, Z. Phys. B **39**, 81 (1980).
[8]A. Bunde and J. F. Gouyot, Z. Phys. B **42**, 169 (1981).
[9]A. Onuki, Prog. Theor. Phys. **66**, 1230 (1981); **67**, 768 (1982); **67**, 787 (1982); Phys. Rev. Lett. **48**, 753 (1982).
[10]J. D. Gunton, M. San Miguel, and P. Sahni, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. Lebowitz (Academic, New York, in press).
[11]R. G. Palmer, Adv. Phys. **31**, 669 (1982).
[12]F. de Pasquale, Z. Ràcz, and P. Tartaglia, Phys. Rev. B **28**, 2582 (1983).
[13]N. G. Van Kampen, J. Stat. Phys. **17**, 71 (1977).
[14]F. T. Arecchi and A. Politi, Phys. Lett. **45**, 1219 (1980). F. Haken, J. W. Haus, and R. Glauber, Phys. Rev. A **23**, 3255 (1981).
[15]N. J. Rao, J. D. Bouvankar, and D. Ramkrishna, SIAM J. Control **12**, 124 (1974).
[16]J. M. Sancho, M. San Miguel, S. Katz, and J. D. Gunton, Phys. Rev. A **25**, 1589 (1982).
[17]In the approximation of Onuki (Ref. 9) the effect of fluctuations that leads to (3.4) is not preserved. The ordered states are described by Onuki as having constant nonvanishing value of the average order parameter. This average value becomes multivalued in a domain of bistability.
[18]We use throughout the Stratonovich interpretation of stochastic differential equations [see L. Arnol'd, *Stochastic Differential Equations* (Wiley, New York, 1974)].