# One-dimensional Ising model in an incommensurate field

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For the one-dimensional Ising model with a special incommensurate field, the ground-state magnetization as function of the field exhibits a staircase with an infinite number of almost exponentially decreasing steps. With increasing temperature, the steps get smoother and disappear successively. The specific heat resembles that of the Ising model with a constant field.

## I. INTRODUCTION

In the last few years one-dimensional incommensurate systems have been studied intensively by several authors. For earlier references we refer to the review by Bak.[1]

One of the most striking features of the investigated systems seems to be the existence of a devil's staircase. The first explicit expression for a complete devil's staircase was given by Aubry[2] for the Frenkel-Kontorova (FK) model with piecewise parabola potential. Recently, Bak and Bruinsma[3] have proven rigorously that the ground-state magnetization as a function of the external field for a one-dimensional Ising model with long-range antiferromagnetic interactions exhibits a complete devil's staircase too. The equivalence of both models has been shown by Aubry.[4]

With this equivalence, the low-energy excitations of the FK model with parabola potential can be described, as shown by Aubry *et al.*,[5] by a one-dimensional Ising model with long-range antiferromagnetic interactions and an *incommensurate* field. The low-temperature specific heat which was deduced from that (Ref. 5) for an incommensurability ratio equal to $\phi=(\sqrt{5}-1)/2$, the inverse of the golden mean, exhibits as a function of the temperature an infinite number of peaks. These were interpreted as transition points (not real phase transitions) between commensurate phases related to the continued fraction expansion of $\phi$. This anomalous low-temperature behavior was already found before by Pietronero and Strässler.[6]

Two questions may arise now for one-dimensional incommensurate systems.

(i) How generic is the existence of a devil's staircase?

(ii) How generic is the anomalous behavior of the specific heat at low temperatures?

The following incommensurate Ising model leads to exact answers:

$$H=\sum_{i=1}^{N}(J\sigma_i\sigma_{i+1}-h_i\sigma_i),\quad \sigma_{N+1}=\sigma_1 \tag{1}$$

where $\sigma_i=\pm1$, $N$ represents the number of sites, and the magnetic field $h_i$ at the site $i$ is given by

$$h_i(\alpha)=h(i\phi+\alpha) \tag{2a}$$

with $h(t)$ being a periodic (period equal to 1) piecewise constant function:

$$h(t)=\begin{cases}a_0, & -\phi<t\le-\phi^3\\ a_1, & -\phi^3<t\le\phi^2\end{cases}$$
$$h(t+1)=h(t),\quad \phi=(\sqrt{5}-1)/2 \tag{2b}$$

with $a_0$ and $a_1$ representing constants. The phase $\alpha$ in Eq. (2a) $(0\le\alpha<1)$ can be chosen arbitrarily for the incommensurate case. We take $\alpha=0$. Studying the incommensurate Ising model may also be motivated by extending recent work by Derrida *et al.*,[7,8] Williams,[9] Doman and Williams,[10] Bruinsma and Aeppli,[11] and Aeppli and Bruinsma[12] for the Ising model with a random field.

## II. FINITE TEMPERATURE

We calculate the partition function for (1):

$$Z(\beta,N)=\mathrm{tr}\left[\prod_{i=1}^{N}\underline{T}(i\phi)\right],\quad \beta=(k_BT)^{-1} \tag{3}$$

with the use of the transfer matrix

$$\underline{T}(t)=\begin{pmatrix}\exp\{-[K-H(t)]\} & \exp[K+H(t)]\\ \exp[K-H(t)] & \exp\{[-K+H(t)]\}\end{pmatrix},$$
$$K=\beta J,\quad H(t)\equiv\beta h(t) \tag{4}$$

which is related to the one-dimensional Schrödinger equation with incommensurate potential, given by (2). This was recently studied independently by Kohmoto *et al.*[13] and Ostlund *et al.*[14] From Eq. (4), it follows that $\det\underline{T}(t)$ is independent of $t$. We define

$$\underline{M}^{(k)}=(\det\underline{T})^{-k/2}\prod_{i=1}^{k}\underline{T}(i\phi), \tag{5}$$

for which $\det\underline{M}^{(k)}=1$. From this, it follows that $\underline{M}^{(k)}$ can be obtained recursively:

$$\underline{M}^{(k_1+k_2)}(t)=\underline{M}^{(k_1)}(t+k_2\phi)\underline{M}^{(k_2)}(t), \tag{6}$$

with $\underline{M}^{(1)}(t)=\underline{M}(t)$ and where $\underline{M}(t)=(\det\underline{T})^{-1/2}\underline{T}(t)$. Then, Eq. (2) implies

$$\underline{M}(t)=\begin{cases}\underline{M}_0, & -\phi<t\le-\phi^3\\ \underline{M}_1, & -\phi^3<t\le\phi^2\end{cases}$$

where the matrix elements of $\underline{M}_0$ and $\underline{M}_1$ are constant.

 

Thus, for arbitrary $t$, $\underline{M}(t)$ is equal either to $\underline{M}_0$ or to $\underline{M}_1$, which is a result quite similar to that of Ref. 13.

Therefore it follows from Ref. 13, that the recursion relation (6) for $k_1$ and $k_2$, restricted to the Fibonacci numbers $F_l$, becomes

$$\underline{M}_{l+1}=\underline{M}_{l-1}\underline{M}_l,\quad l\geq 1 \tag{7}$$

where

$$\underline{M}_l=\underline{M}^{(F_l)}\ (t=0)\ ,$$

$$\underline{M}_0=\underline{M}(-\phi^3),\quad \underline{M}_1=\underline{M}(0)$$

and $F_l$ are defined by

$$F_{l+1}=F_l+F_{l-1},\quad l\geq 1,\ \text{ with } F_0=1 \text{ and } F_1=1\ . \tag{8}$$

Using (7), it is easy to prove that the following identity (Ref. 13):

$$\underline{M}_{l+1}+\underline{M}_{l-2}^{-1}\equiv\underline{M}_{l-1}\underline{M}_l+\underline{M}_{l-1}\underline{M}_l^{-1} \tag{9}$$

is true.

Let $X_l=\mathrm{tr}\underline{M}_l$ (notice that we have omitted the factor $\frac{1}{2}$, compared to Ref. 13), then taking the trace of Eq. (9) we obtain

$$X_{l+1}=X_lX_{l-1}-X_{l-2}\ , \tag{10}$$

where we have used $\mathrm{tr}\underline{M}_l=\mathrm{tr}(\underline{M}_l^{-1})$ and $\underline{M}_l+\underline{M}_l^{-1}=(\mathrm{tr}\underline{M}_l)\mathbb{1}$ ($\mathbb{1}$ is the unit matrix) which holds for $2\times 2$ real matrices with a determinant equal to 1. Taking the trace of Eq. (5) and $k=F_l$ we obtain for the partition function $Z_l=Z(\beta, N=F_l)$:

$$\begin{aligned}Z_l(\beta,K,H_0,H_1)&\equiv\mathrm{tr}\left[\prod_{i=1}^{F_l}\underline{T}(i\phi)\right]\\&=(\det\underline{T})^{F_l/2}\mathrm{tr}\underline{M}_l\\&=(\det\underline{T})^{F_l/2}X_l\ .\end{aligned} \tag{11}$$

$\det T$ follows from Eq. (4). Then, substituting $X_l$ from Eq. (11) into Eq. (10) and using (8) we find

$$\begin{aligned}&Z_{l+1}=Z_lZ_{l-1}-[2\sinh(2K)]^{F_l-1}Z_{l-2},\quad l\geq 1\\&Z_{-1}=2\cosh(H_0-H_1)\ ,\\&Z_i=2\exp(-2K)\cosh H_i,\quad H_i=\beta a_i,\quad i=0,1\ .\end{aligned} \tag{12}$$

The asymptotic behavior of $Z_l$, for $l\to\infty$, is

$$Z_l\to\exp(-\beta f F_l)\ , \tag{13}$$

where $f(\beta,K,H_0,H_1)$ is the free energy per spin.

We have not succeeded in deriving an analytical expression for $f$ but Eq. (12) can easily be iterated on a computer. Performing only $l=30$ iterations, the partition function for $N=F_{30}\cong 1.3\times 10^6$ spins is obtained for given $K$, $H_0$, and $H_1$. To our knowledge there is no other example apart from the exactly solvable models where this can be done with such little effort.

Because of Eq. (13) we have iterated $\ln Z_l$. This was performed for $H_0=-H_1=H$. $F_l^{-1}\ln Z_l$ converges rapidly such that for $l=30$, $F_l^{-1}\ln Z_l\cong f$. From this, one easily obtains the specific heat per spin:

$$c(\beta,K,H)=-k_B\beta^2\frac{\partial^2}{\partial\beta^2}[\beta f(\beta,K,H)] \tag{14}$$

and the magnetization (per spin) with respect to the local field direction:

$$\begin{aligned}m'(\beta,K,H)&=\lim_{l\to\infty}F_l^{-1}\sum_{i=1}^{F_l}\langle\sigma_i\rangle\mathrm{sgn}h_i\\&=-\frac{\partial}{\partial H}[\beta f(\beta,K,H)]\ ,\end{aligned} \tag{15}$$

which are represented in Figs. 1 and 2. Whereas the specific heat resembles that for the Ising model in a constant field, the "magnetization" exhibits a peculiar behavior which can be understood by the exact results for the ground state, which will be discussed now.

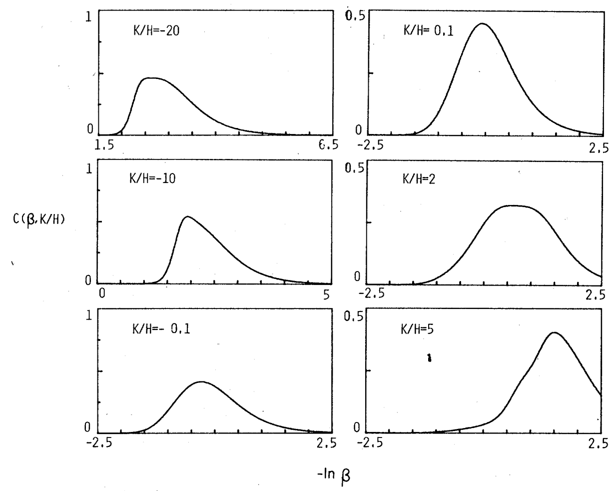


FIG. 1. Specific heat (in units of $k_B$) as a function of $\ln\beta$ ($\beta$ in units of $J^{-1}$) for different $K/H$ and $l=30$.

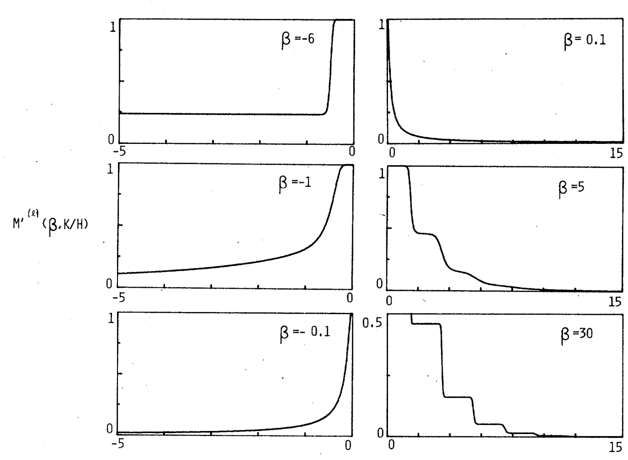


FIG. 2. Magnetization as function of $K/H$ for different temperatures and $l=30$.

## III. GROUND STATE

For this purpose, some properties of the sequence

$$\{\epsilon_i(0)\}_l=\{\text{sgn}h_i(0)\mid 1\le i\le F_l\}$$

will be crucial. Because of the periodic boundary conditions [we have chosen in (1)], two sequences $\{h_i\}$ and $\{h_i'=h_{i+n}\}$ $(h_{N+1}=h_1$, $n$ is an arbitrary integer) are physically equivalent. It is easy to prove that in that sense $\{\epsilon_i(\alpha)\}_l$ and $\{\epsilon_i(0)\}_l$ are equivalent for arbitrary phases $\alpha$.

For convenience, we choose $\alpha=2\phi-1=\phi^3$. The sequence $s_l=\{\epsilon_i(\phi^3)\}_l$ has the following property.

(P1) $s_l$ is the union of $s_{l-1}$ and $s_{l-2}$:

$$s_l=s_{l-1}\cup s_{l-2},\quad l\ge 5 \tag{16}$$

with

$$s_3=-+-\ ,$$

$$s_4=-+-+-\ .$$

For simplicity we omit the union symbol in the explicit representation of $s_l$ by plus and minus signs, as for $s_3$ and $s_4$ given above. We only sketch the proof of (P1), which is analogous to that of Eq. (7). Using Eq. (2) with $\alpha=\phi^3$ and Eq. (8), we obtain

$$s_l=\{\epsilon_i(\phi^3)\}_{1\le i\le F_l}=\{\epsilon_i(\phi^3)\}_{1\le i\le F_{l-1}}\cup\{\epsilon_{F_{l-1}+i}(\phi^3)\}_{1\le i\le F_{l-2}}=s_{l-1}\cup\{\text{sgn}h(F_{l-1}\phi+i\phi+\phi^3)\}_{1\le i\le F_{l-2}}\ . \tag{17}$$

Using a further property of $\phi$ (Ref. 15),

$$F_l\phi-F_{l-1}=(-1)^l\phi^{l+1}\ , \tag{18}$$

one can easily prove that

$$\text{sgn}h(F_{l-1}\phi+i\phi+\phi^3)=\text{sgn}h(i\phi+\phi^3) \tag{19}$$

for $1\le i\le F_{l-2}$ and $l\ge 5$. Then, we obtain, from (17),

$$s_l=s_{l-1}\cup\{\text{sgn}h(i\phi+\phi^3)\}_{1\le i\le F_{l-2}}$$

$$=s_{l-1}\cup s_{l-2}\quad\text{for } l\ge 5\ ,$$

which was what we set out to prove.

(P1) implies the following properties.

(P2) In $s_l$ the plus signs are always isolated, the minus signs are either isolated or appear in pair and the largest subsequence of $s_l (l\ge 4)$ with alternating signs is $s_4$.

(P3) The number of minus and plus signs in $s_l$ is equal to $F_{l-1}$ and $F_{l-2}$, respectively.

(P3) follows from (P1) and Eq. (8) by induction.

Using Eq. (16) $(n-1)$ times, $s_l$ can be composed as the union of only $s_{l-n+1}$ and $s_{l-n}$:

$$s_l=s_{l-n+1}\cup s_{l-n}\cup\cdots\cup s_{l-n+1}\ , \tag{20}$$

with $2\le n\le l-3$. Representing $s_{l-n+1}$ by a minus sign and $s_{l-n}$ by a plus sign in Eq. (20) the obtained sequence $s_l^{(n)}$ has the following property.

(P4) $s_l^{(n)}$ is equivalent to $s_n$.

This again is easily proven by induction. These properties are related to the "irrational decimation" discussed by Feigenbaum and Hasslacher.[16]

Now let us investigate the ground-state problem of (1) for $N=F_l$. We take the phase $\alpha=\phi^3$ without restricting generality.

For $K/H=0$, the ground-state spin configuration is

$$\sigma_i^{(0)}=\text{sgn}h_i(\phi^3)\ . \tag{21}$$

From (P1) it follows that this has the form

$$\cdots(-+-)(-+-+-)(-+-+-)(-+-)(-+-+-)\cdots \tag{22}$$

which we represent as

$$\cdots F_3F_4F_4F_3F_4\cdots\ , \tag{23}$$

because there are only blocks of $F_3$ and $F_4$ alternating spins. For $J<0$ the modulated ground state (22) becomes unstable if

$$2J+a=0$$

(we have chosen $a_0=-a_1=a$) because then the energy to flip the isolated up spins [because of (P2)] vanishes. Thus for the ferromagnetic case there is a transition at

$$K/H=-0.5$$

from the modulated state (22) with $m_0'^{(l)}=1$ to the ferromagnetic state with all spins down and $m'^{(l)}=(F_{l-1}-F_{l-2})/F_l$ [where (P3) is used]. $m'^{(l)}\to\phi^3$ for $l\to\infty$.

The antiferromagnetic case is much more interesting. Because the $F_3$ and $F_4$ blocks in (22) (displaced by parentheses) are antiferromagnetic, they already have minimum energy. But the bonds between two adjacent blocks are frustrated. Decreasing the magnetic field, the modulated ground state becomes unstable at a critical value $(K/H)_1$ where flipping the $F_3$ blocks does not cost energy. Representing in (22) the $F_3$ block by plus signs and the $F_4$ blocks by minus signs, one obtains a sequence which is equivalent to $s_{l-3}$, because of (P4). Therefore the sequence (23) consists of only two types of "alternating" blocks [(P2)]:

$$(F_4F_3F_4)$$

and

$$(F_4F_3F_4F_3F_4)\ .$$

Therefore, when the $F_3$ blocks flip a new ground state with only two types of antiferromagnetic blocks is obtained. These blocks consist of

$$2F_4+F_3=F_6$$

and

$$3F_4+2F_3=F_7$$

spins and are called the $F_6$ and $F_7$ blocks, respectively. Thus, the new ground state is represented as

$$\cdots F_6F_7F_6F_7F_7F_6\cdots \; . \tag{24}$$

At $(K/H)_2$, the $F_6$ blocks become unstable, forming larger antiferromagnetic blocks. This procedure continues until the ground state is built up of antiferromagnetic $F_{3n}$ and $F_{3n+1}$ blocks:

$$\cdots F_{3n}F_{3n+1}F_{3n}F_{3n+1}F_{3n+1}\cdots \; . \tag{25}$$

Representing in (25) $F_{3n}$ and $F_{3n+1}$ by plus and minus signs, respectively, the obtained sequence is equivalent to $s_{l-3n}$ [because of (P4)]. Therefore (25) consists of only two types of "alternating" blocks [because of (P2)]:

$$(F_{3n+1}F_{3n}F_{3n+1})$$

and

$$(F_{3n+1}F_{3n}F_{3n+1}F_{3n}F_{3n+1}) \; .$$

Thus at $(K/H)_n$ the $F_{3n}$ blocks flip forming two types of antiferromagnetic blocks with

$$2F_{3n+1}+F_{3n}=F_{3(n+1)}$$

and

$$3F_{3n+1}+2F_{3n}=F_{3(n+1)+1}$$

spins and are called the $F_{3(n+1)}$ and $F_{3(n+1)+1}$ blocks. $(K/H)_n$ is determined by the condition that the energy to flip an $F_{3n}$ block just vanishes. This happens if

$$-2J+(x_na)=0 \; , \tag{26}$$

where $(x_na)$ is the field energy of an $F_{3n}$ block. (We would like to remind the reader that we have chosen $a_0=-a_1=a$.) With $(y_na)$, the field energy of the $F_{3n+1}$ block, we obtain the recursion relations:

$$x_{n+1}=2y_n-x_n, \quad n\geq 1$$

$$y_{n+1}=3y_n-2x_n, \quad n\geq 1$$

$$x_1=3, \quad y_1=5$$

from which one finds easily

$$x_n=4n-1, \quad n\geq 1 \; . \tag{27}$$

Equations (26) and (27) yield finally,

$$(K/H)_n=2n-\tfrac{1}{2}, \quad n\geq 1 \; . \tag{28}$$

The ground-state magnetization $m'^{(l)}_n$ for $(K/H)_n < K/H < (K/H)_{n+1}$ is just $m'^{(l)}_{n-1}$ reduced by twice the magnetization of an $F_{3n}$ block, which equals $2x_n$, times $F_{l-2-3n}/F_l$, the numbers of $F_{3n}$ blocks per spin:

$$m'^{(l)}_n=m'^{(l)}_{n-1}-2(4n-1)F_{l-2-3n}/F_l \tag{29}$$

for $1\leq n\leq[(l-2)/3]$ ($[x]$ is the integer part of $x$). If we use (see Ref. 15)

$$F_{l-2-3n}=(-1)^nF_{3n}F_l+(-1)^{n+1}F_{3n+1}F_{l-1}$$

and

$$\lim_{l\to\infty} F_{l-1}/F_l=\phi$$

we obtain for the infinite system from (29),

$$m'^{(\infty)}_n=(1+4n\phi^3)\phi^{3n}, \quad n\geq 1 \; . \tag{30}$$

## IV. RESULTS AND DISCUSSION

For a special incommensurate external field the partition function can be determined recursively. Performing only $l=30$ iteration steps on a computer, we obtained the specific heat and the magnetization for a system with $N\cong 1.3\times 10^6$ spins.

The specific heat as a function of temperature does not exhibit a number of peaks but resembles the behavior of the Ising model in a constant field. The flat region occurring for $K/H\approx 2$ and $K/H\approx -20$ may be related to the existence of two typical energy gaps in the excitation spectrum leading each to a Schottky anomaly such that superimposing both may result in such a flat behavior.

Whereas the ground-state magnetization as a function of $K/H$ has only one transition point (no real phase transition) for $J<0$ it exhibits for $J>0$ and for the infinite system an ordinary staircase, i.e., first-order transitions with an infinite number of decreasing steps, according to (28) and (30). Thus, there is *no* devil's staircase. This result is quite similar to that obtained by Williams[9] for the random-bond Ising model, but there $m'^{(\infty)}_n\sim 1/n$ for $n\to\infty$ which is different from (30).

For the commensurate case, replacing $\phi$ by $F_{l-1}/F_l$ only $[(l-2)/3]$ steps appear and the ground state is periodic with period $F_l$. This is quite similar to the Frenkel-Kontorova model.[17] We would like to mention that a similar behavior has been found by Nadal *et al.*[18] for $h(t)=h\cos(2\pi\phi t)$.

For increasing temperature the steps become smoother and disappear one after the other, first the smaller and finally the largest ones (Fig. 2). The "transition" points are still given by (28) for $i=1,2,\ldots,n_0$, where $n_0$ depends on the temperature and the "magnetization" is well approximated by (30) for not too high temperatures. The same smoothening of the steps was found by Doman and Williams[10] for the random-bond Ising model.

Thus the magnetization for the incommensurate Ising model with nearest-neighbor interactions differs qualitatively from the Ising model with long-range interactions studied in Ref. 3 or equivalently the Frenkel-Kontorova model in Refs. 2 and 5, but resembles that for the random-bond Ising model studied in Refs. 9 and 10.

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