Theory of the spin-1 bosonic liquid metal: Quasiparticle interactions and electrical resistivity in liquid metallic deuterium

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We examine the low-temperature electrical resistivity of a hypothetical quantum plasma comprised of light fermions and heavy *nonzero-spin* bosons. The system corresponds to a possible quantum liquid metallic phase of highly compressed deuterium (electrons and deuterons). Bose condensation is assumed and in this state a new quadratic quasiparticle branch is present even with nonmagnetic bare interactions. The consequences of these impurity-like excitations for electric current degradation are explored using the variational treatment of the Boltzmann equation. Quasiparticle collision rates among all possible types are analyzed. We find that the low-temperature electrical resistivity will exhibit an unusual $\sim T^{7/2}$ behavior and is attributable to electron scattering from the impurity-like excitations.

I. INTRODUCTION

The possible existence of low-temperature *liquid* metallic phases of highly compressed hydrogen and deuterium has recently been discussed.[1-3] Liquid metallic phases of hydrogen and deuterium will represent, respectively, examples of a two-component light-fermion and heavy-fermion liquid (electrons and protons) and a boson-fermion liquid (deuterons and electrons). Low-temperature liquidity in these systems will be associated with high ionic zero-point motion stemming, in turn, from a very low ionic mass. Quantum effects in the ionic degrees of freedom are of special interest in these liquids, which may thus be regarded in this context as quantum liquid metals.

The equilibrium and transport properties of an assumed normal liquid metallic phase of hydrogen were analyzed earlier using a two-component Landau—Fermi-liquid theoretic approach.[4,5] A further study[6] (referred to here as paper I) considered equilibrium properties of a hypothetical Bose-condensed, but otherwise normal, liquid metallic phase of deuterium. The present paper extends this theory and presents results both for the electrical resistivity and the relevant quasiparticle interactions for a similar model of liquid metallic deuterium (LMD).

As noted in I, a novel feature of LMD is the presence of *nonzero* boson spin (the deuteron has spin 1). We also noted there that nonzero boson spin leads to a new Goldstone quasiparticle excitation branch in the non-nuclear-spin-polarized, Bose-condensed phase, even though there are no explicit magnetic terms in the Hamiltonian.[6-9] These modes, which may be referred to as "impurity-like modes," have quadratic dispersion and actually dominate the specific-heat and thermal-expansion coefficients for all but the lowest temperatures.[6] The presence of impurity-like modes qualitatively distinguishes LMD from ^{3}He-^{4}He mixtures, another boson-fermion liquid, but one in which the boson has zero spin. A major goal of this work is to elucidate the interesting role the impurity-like modes play in electrical transport of a possible liquid phase of metallic deuterium.

The paper is organized as follows: In Sec. II we describe the model and provide background for analyzing the transport properties of liquid metallic deuterium. A variational calculation of the impurity-like excitation and phonon contributions to the electrical resistivity is presented in Sec. III. An examination of the many possible types of quasiparticle interactions is given in the Appendix, the purpose of which, in part, is to justify some of the assumptions made in Sec. III. Discussion and conclusions appear in Sec. IV.

II. LIQUID METALLIC DEUTERIUM—BACKGROUND

The system under consideration is a neutral ensemble of N electrons of mass m_e and N deuterons of mass m_d. We assume all mutual interactions to be Coulombic; we ignore spin-dependent interactions. We also assume that there is no nuclear-spin polarization and that Bose condensation has actually occurred (case "A" of I). The Bose-condensation temperature $\tilde{T}_B$ of an ideal Bose gas[10] corresponding to LMD is 43 K at $r_s = 1.6$ (the choice[11] $r_s \approx 1.6$ is thought to be relevant for a possible liquid phase[2]). The Bose-condensation temperature for the interacting system T_B is very probably of the same order of magnitude as this; we restrict our attention to temperatures much lower than T_B, say, $T \lesssim 5$ K. Aside from Bose condensation, we also assume for this analysis that the system is electronically "normal," i.e., there is no electron Cooper pairing, no electron ferromagnetism, no charge-density-wave distortion, etc.[6]

The low-temperature equilibrium and near-equilibrium states of LMD may be characterized in terms of the dis-

tribution functions for the three types of quasiparticles present.[6] There are fermion quasiparticles corresponding to dressed electrons (e), phonons (ph), and impurity-like excitations (i). Each type of quasiparticle is considered here to be fully normalized by interactions among *all* types of particles. For our model the quasiparticle energies of momentum p are of the form, respectively,

$$\epsilon_e(p)=\frac{p_f}{m_e^*}(p-p_F)+\mu_e, \quad p\approx p_F \tag{1a}$$

$$\epsilon_{\rm ph}(p)=sp\ , \tag{1b}$$

and

$$\epsilon_i(p)=\frac{p^2}{2m_i^*}\ . \tag{1c}$$

In Eqs. (1), m_e^* and m_i^* are the fully renormalized-electron and impurity-like excitation effective masses (we simply take $m_e^*\approx m_e$ and $m_i^*\approx 2m_d$), p_F is the Fermi momentum [$p_F=\hbar(3\pi^2 n)^{1/3}$, where n is the number density of each species], and μ_e is the electron chemical potential. The fully renormalized sound speed s may be estimated from the Bohm-Staver relation: $s=(p_F^2/3m_e^* m_d)^{1/2}\approx 4\times 10^6/r_s$ cm/sec. We note that at $r_s=1.6$ the electron Fermi temperature is roughly $T_{F_e}\approx 2.3\times 10^5$ K and the Debye temperature is $T_D\approx 5.5\times 10^3$ K.

The transport properties are governed, of course, by the interactions among quasiparticles. There are nine relevant independent relaxation rates, each appropriate to a quasiparticle of a given type decaying through interaction with the distribution of one of the three types of quasiparticles. The assumption "Bose-condensed but otherwise normal" means that the quasiparticle interactions within the electron-phonon subsystem are qualitatively the same as those encountered in the usual treatment of low-temperature quantum plasmas. Accordingly, we expect a $\sim T^5$ behavior for the phonon contribution to the electrical resistivity.[12] Collisions among electron quasiparticles in a *liquid* do *not* degrade the electrical current, and therefore do not contribute to the electrical resistivity. It remains, therefore, to consider the new impurity-like excitations and their contribution to the electrical resistivity.

To solve three coupled Boltzmann equations describing the three quasiparticle distribution functions, allowing collisions among *all* possible pairs of quasiparticles and including superfluid effects, would be a formidable task and will not be attempted here. Instead, our approach is to obtain reasonable estimates and to use the variational formulation for the transport coefficients. Furthermore, we initially assume that for purposes of calculating the phonon and impurity-like contributions to the resistivity, the phonons and impurity-like excitations are themselves effectively in equilibrium. The legitimacy of this assumption will be examined further in the Appendix.

The effect of possible superfluid behavior of the screened ions upon the electrical resistivity is not considered, and we specifically ignore possible phonon- or impurity-like excitation—"drag" effects[12] associated with electrical current flow. In paper I it was shown that within a Bogoliubov approximation the impurity-like mode is a linear superposition of bare-particle modes in different spin states. We expect a very similar spin coherence aspect to be evident in the impurity-like modes renormalized beyond the Bogoliubov approximation. It therefore seems physically reasonable that impurity-like drag will be a small effect: One can easily imagine that on a time scale set by the electron-impurity—like excitation scattering time, the interaction between an impurity-like excitation and a wall will be such that the wall will be unable to generate properly spin-superposed combinations at the end of the "wire" at which electrons enter. In other words, too few *new* impurity-like excitations will be generated to sustain a significant drag.

It remains, however, to estimate the matrix elements describing the scattering of impurity-like excitations by electrons or other impurity-like excitations. The spin-coherence aspect of the impurity-like excitations notwithstanding, the relevant physical picture for collisions between impurity-like excitations and electrons or other impurity-like excitations is still one of the two "particle-like" quasiparticles colliding with a short-range interaction $\mathscr{V}$. An example is the case of an electron which can scatter off a spin component (with wave function proportional to $e^{i\vec{k}\cdot\vec{r}}$) in such a way as to preserve the spin coherence. We expect both the impurity-like excitation—electron and impurity-like excitation—impurity-like excitation cross sections to be of order $4\pi/k_{\rm TF}^2$ where $k_{\rm TF}$, the effective Thomas-Fermi wave vector, is given by $k_{\rm TF}=(4p_F m_e^* e^2/\pi\hbar^3)^{1/2}$. Note that the effect of a long-wavelength phonon on an impurity-like excitation can be regarded as a simple modification of the effective mass m_i^* through the local-density dependence of m_i^* (see the Appendix).

III. ELECTRICAL RESISTIVITY

We first treat the contribution to the electrical resistivity ($\rho_{e\text{-}i}$) from scattering between impurity-like excitations and electrons. The well-known variational principle for the linearized Boltzmann equation[12,13] leads to the following expression:

$$\rho_{e\text{-}i}\simeq\frac{(V/8k_BT)\int d^3p_1\,d^3p_2\,d^3p_3\,d^3p_4(\Phi_{\vec{p}_1}+\tilde{\Phi}_{\vec{p}_2}-\Phi_{\vec{p}_3}-\tilde{\Phi}_{\vec{p}_4})^2 P(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4)}{\left|[2eV/(2\pi\hbar)^3]\int d^3p\,\vec{v}_{\vec{p}}\Phi_{\vec{p}}(\partial n^0_{\vec{p}}/\partial\epsilon_{\vec{p}})\right|^2}\ , \tag{2a}$$

with

$$P(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4)\equiv 4\left[\frac{V}{(2\pi\hbar)^3}\right]^4\frac{2\pi}{\hbar}\delta(\epsilon_{e1}+\epsilon_{i2}-\epsilon_{e3}-\epsilon_{i4})$$
$$\times|\langle\vec{p}_3,\vec{p}_4|\mathscr{V}|\vec{p}_1,\vec{p}_2\rangle|^2 n^0(\epsilon_{e1})\tilde{n}^0(\epsilon_{i2})[1-n^0(\epsilon_{e3})][1+\tilde{n}^0(\epsilon_{i4})]\ , \tag{2b}$$

and with

$$|\langle \vec{p}_3,\vec{p}_4|\mathscr{V}|\vec{p}_1,\vec{p}_2\rangle|^2 \equiv \frac{1}{V^3}(2\pi\hbar)^3\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\,|A(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4)|^2 . \tag{2c}$$

Here, V is the volume of the system, k_B is Boltzmann's constant, e is the electron charge, and T is temperature. The initial and final electron (impurity-like) momenta are denoted by $\vec{p}_1$ and $\vec{p}_3$ ($\vec{p}_2$ and $\vec{p}_4$), respectively, and $\vec{v}_{\vec{p}}$ denotes the electron velocity at momentum $\vec{p}$. The $\epsilon_{\alpha k}$ denote quasiparticle energy for type α and momentum $\vec{p}_k$. Furthermore, $n^0(\epsilon)$ and $\tilde{n}^0(\epsilon)$ denote, respectively, the ideal-gas fermion and boson equilibrium distribution functions, the latter with a chemical potential set equal to zero.[14] The overall factor of 4 in Eq. (2b) accounts for the electron and impurity-like excitation degeneracy.[15] The Φ_p are related to the nonequilibrium solutions of the linearized Boltzmann equations, $n(\epsilon)$ and $\tilde{n}(\epsilon)$, by

$$n(\epsilon_e)=n^0(\epsilon_e)-\Phi_{\vec{p}}\frac{\partial n^0(\epsilon_e)}{\partial\epsilon_e} \tag{3a}$$

and

$$\tilde{n}(\epsilon_i)=\tilde{n}^0(\epsilon_i)-\tilde{\Phi}_{\vec{p}}\frac{\partial\tilde{n}^0(\epsilon_i)}{\partial\epsilon_i} . \tag{3b}$$

Finally, the integrations in (2) are over all momentum space.

For the present we may evaluate $\rho_{e\text{-}i}$ under the assumption that the thermalization rate for the impurity-like excitation distribution function is much greater than the impurity-like excitation—electron relaxation rate for resistivity. This allows us to set

$$\tilde{\Phi}_{\vec{p}_2}=\tilde{\Phi}_{\vec{p}_4}=0 \tag{4}$$

in Eq. (2). The validity of this assumption is established in the Appendix.

According to the variational principle, the true $\rho_{e\text{-}i}$ will always be less than or equal to the value computed from Eq. (2) when an approximate trial function $\Phi_{\vec{p}}$ is used. Here we use the standard ansatz

$$\Phi_{\vec{p}}=\vec{p}\cdot\hat{u} , \tag{5}$$

where $\hat{u}$ is a unit vector parallel to the electric field. For this choice the denominator D of Eq. (2a) is readily evaluated, namely

$$D=\left[\frac{ek_F^3}{3\pi^2}\right]V^2 , \tag{6}$$

where k_F is the Fermi wave vector.

Our estimate of the numerator N of Eq. (2a) is based on a number of approximations and observations as follows: Note first that the major purpose here is to obtain the leading-order temperature dependence and its order of magnitude. Accordingly, we do the following:

(1) We replace the boson final-state factor $1+\tilde{n}^0(\epsilon_{i4})$ by unity. Inclusion of $1+\tilde{n}^0(\epsilon_{i4})$ cannot contribute any overall factor of T as it does not restrict scattering (rather, it enhances it).

(2) From the well-known form for $\tilde{n}^0(\epsilon_i)$ we observe that most of the impurity-like excitations in the system have energy $\vec{k}_B T$. We therefore approximate the actual $\tilde{n}^0(\epsilon_{i2})$ by a δ-function form for which all particles have energy $k_B T$, i.e.,

$$\tilde{n}^0(\epsilon_{i2})\rightarrow N^*\delta(p_2-(2m_i^*k_BT)^{1/2}) . \tag{7}$$

Here, N^* is chosen so that the total number of bosons represented by the right-hand side of Eq. (7) is the same as that described by $\tilde{n}^0$. This gives

$$N^*=2.31\left[\frac{m_i^*k_BT}{2}\right]^{1/2} . \tag{8}$$

It should be noted that the total number of Bose excitations below T_B is temperature dependent. The approximation (7) should at least preserve the leading-order temperature dependence.

(3) We next observe that at low temperatures a typical electron interacting with an impurity-like excitation will have an energy very near the Fermi energy ϵ_F. Because the possible fermion final states are restricted by a factor $1-n^0(\epsilon_{e3})$, the scattered fermion must have a final energy $\gtrsim\epsilon_F-k_BT$. (We note, incidentally, that the electron scatters almost elastically at the low temperatures of interest; however, the impurity-like excitations *do* suffer large energy changes during collisions.)

We next ask whether the requirements of energy and momentum conservation, together with the requirement that $\epsilon_{e3}\gtrsim\epsilon_f-k_BT$, limits the number of impurity-like targets [all now having energy k_BT by Eq. (7)] available as scatterers. More specifically, for an impurity-like target of energy $\epsilon_{i2}=k_BT$, and an electron of energy $\epsilon_{e1}=\epsilon_F$, do these requirements actually limit the angle of incidence between electron and impurity-like propagation vectors?

To address this we consider the surface $S(\vec{p}_3)$ described by the vector $\vec{p}_3$ on the momentum and energy shell for such values of ϵ_{e1} and ϵ_{i2}. Using the notation of Fig. 1 we conclude, first, that for $\theta_3=0$, $|p_3|>p_F$ for all angles θ_2 between $\vec{p}_1$ and $\vec{p}_2$. This conclusion follows from parameter values for LMD at $r_s=1.6$: $p_1=p_F=1.2$ a.u.

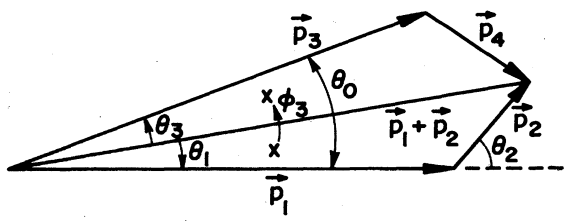

FIG. 1. Scattering geometry and notation used for evaluation of $\tilde{\tau}_{e\text{-}i}^{-1}$. Diagram exaggerates impurity excitation momenta $\vec{p}_2,\vec{p}_4$ in comparison to electron momenta $\vec{p}_1,\vec{p}_3$: $p_2,p_4\ll p_1\approx p_3\approx p_F$.

and $p_2=(2m_i^* k_B T)^{1/2}\approx 0.215 T^{1/2}$ a.u. Next we determine the maximum value of θ_3 (denoted by $\bar{\theta}_3$) for which, at a given θ_2, the conservation and phase-space restrictions can be satisfied. The value $\bar{\theta}_3$ corresponds to setting $\epsilon_{e3}=\epsilon_F-k_B T\,(=\epsilon_1-\epsilon_2)$. Using $p_2 \ll p_1$ we then find

$$X\equiv\cos\bar{\theta}_3=1-\left[\frac{p_2}{p_1}\right]^2(1-\tfrac{1}{2}\cos^2\theta_2)\ . \tag{9}$$

Note that $\bar{X}\approx 1$ and also that the expression for $|X|$ is less than unity for all θ_2. This means that in spite of the requirement that there must exist a maximum angle $\bar{\theta}_3$ (which is itself θ_2 dependent), *all* θ_2 are again admissible. The surface $S(p_3)$ as a function of θ_3 (for fixed θ_2) then interpolates between the value $p_3>p_F$ for $\theta_3=0$ and the value $p_3=[2m_i^*(\epsilon_F-k_B T)]^{1/2}$ at $\theta_3=\bar{\theta}_3(\theta_2)$. Thus a surface $S(\vec{p}_3)$ exists for all θ_2, i.e., there are no kinematic or phase-space restrictions on θ_2. All impurity-like targets are therefore available as scatterers under the conditions assumed here.

(4) We approximate the quantity A of Eq. (2c), related to the matrix element, by a suitable average value $\bar{A}$ (estimated below).

With approximations (1)—(4) now made, the twelvefold integration in N is carried out at follows: Momentum conservation is used to eliminate the p_4 integration. In view of Eqs. (4) and (5) we have a factor $|(\vec{p}_1-\vec{p}_3)\cdot\hat{u}|^2$ in the integrand. Since $\rho_{e\text{-}i}$ does not depend on $\hat{u}$, we may replace this factor by its angular average:

$$|(\vec{p}_1-\vec{p}_3)\cdot\hat{u}|^2\rightarrow\tfrac{1}{3}|\vec{p}_1-\vec{p}_3|^2\approx\tfrac{2}{3}p_F^2(1-\cos\theta_0)\ , \tag{10}$$

where in the last step we have used the quasielastic nature of the electron scattering, and where θ_0 is the electron scattering angle. Using approximations (1), (2), and (4), and Eq. (10), we so far have

$$N=\frac{2.31V^2}{6(2\pi)^2}\frac{1}{\hbar^{10}k_B T}|A|^2 p_F^2(2m_i^* k_B T)^{1/2}\int d^3p_1\,d^3p_2\,d^3p_3\,\delta(\epsilon_{e1}+\epsilon_{i2}-\epsilon_{e3}-\epsilon_{i4}(\vec{p}_1,\vec{p}_2,\vec{p}_3))$$
$$\times n^0(\epsilon_{e1})\delta(p_2-(2m_i^* k_B T)^{1/2})[1-n^0(\epsilon_{e3})][1-\cos\theta_0(\vec{p}_1,\vec{p}_3)]\ , \tag{11}$$

where the arguments of ϵ_{i4} and θ_0 are now specifically indicated. We carry out the ϵ_{e3} integration by exploiting the energy δ-function. Note that because $\epsilon_{e1}\approx\epsilon_{e3}\gg\epsilon_{i2},\epsilon_{i4}$, this energy δ-function is comparable in effect to the presence of a factor $\sim\delta(\epsilon_2-\epsilon_3)$. Implementation of this approximation would, however, be premature at this point, because it does *not* take into account the X_3 $(=\cos\theta_3)$ dependence in ϵ_{e3}; as seen in approximation (3), X_3 is restricted to a small range $\bar{X}\le X_3\le 1$ because of the final-state fermion factor. Thus, after the ϵ_{e3} integration an additional overall factor of $m_e^* p_F$ is introduced, and the fermion final-state and angular weight factors become, respectively,

$$1-n^0(\epsilon_{e3}(p_1,p_2,X_3))\ \text{ and }\ 1-\cos\theta_0(\vec{p}_1,\vec{p}_2,X_3,\phi_3)\ ,$$

with ϕ_3 the dihedral angle between the plane containing $\vec{p}_3$ and $\vec{p}_1+\vec{p}_2$, and that containing $\vec{p}_1$ and $\vec{p}_1+\vec{p}_2$.

In view of approximation (3) we are next led to the following additional simplifications used in the remaining X_3 integration. For $f(X_3)$ an arbitrary but reasonably smooth function,

$$\int_{-1}^{1}dX_3[1-n^0(\epsilon_{e3}(p_1,p_2,X_3))]f(X_3)\ , \tag{12a}$$
$$\sim\int_{\bar{X}}^{1}dX^3[1-n^0(\epsilon_{e3}(p_1,p_3,X_3))]f(X_3)\ , \tag{12b}$$
$$\sim\int_{\bar{X}}^{1}dX_3[1-n^0(\epsilon_{e1})]f(X_3)\ . \tag{12c}$$

Next, with $\vec{p}_1+\vec{p}_3$ taken as the polar axis (Fig. 1), we have[16]

$$1-\cos\theta_0=\cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_3\cos\phi_3\ . \tag{13}$$

Clearly, the contribution from the second term of Eq. (13) vanishes under the ϕ_3 integration. Furthermore, we may express $\cos\theta_1$ in terms of p_1,p_2 and $X_2=\cos\theta_2$, keeping terms only to order $(p_2/p_1)^2$. These two observations then lead to the replacement in the integrand,

$$1-\cos\theta_0\rightarrow\Omega(X_2,X_3,p_1,p_2)$$
$$\equiv 1-\left[1-\frac{1}{2}\left[\frac{p_2}{p_1}\right]^2(1-X_2^2)\right]X_3\ . \tag{14}$$

In view of Eqs. (11), (12), and (14), we now readily perform the X_2,X_3 integrations,

$$\int_{-1}^{1}dX_2\int_{\bar{X}(X_2)}^{1}dX_3\,\Omega(X_2,X_3,p_1,p_2)=\frac{79}{60}\left[\frac{p_2}{p_1}\right]^4$$
$$=\frac{79}{60}\frac{(2m_i^* k_B T)^2}{p_F^4}\ . \tag{15}$$

From Eq. (12c) we are now led to the following factor occurring in N:

$$\int d^3p_1\,n^0(\epsilon_{e1})[1-n^0(\epsilon_{e1})]=4\pi p_F m_e^* k_B T\ . \tag{16}$$

Performing the remaining trivial azimuthal and $|\vec{p}_2|$ integrations, and using the simple estimate

$$|\bar{A}|^2\sim\left[\frac{4\pi e^2}{k_{\rm TF}^2}\right]^2\ , \tag{17}$$

we finally arrive at

$$\rho_{e\text{-}i}\sim 1.62\times 10^2\frac{e^2 m_e^{*2}(m_i^* k_B T)^{7/2}}{\hbar^{10}k_{\rm TF}^4 k_F^6}\quad (T\ll T_B)\ . \tag{18}$$

The $\sim T^{7/2}$ behavior is interesting and unusual.

For LMD at $r_s=1.6$ we estimate (with T in K)

$$\rho_{e\text{-}i}\sim 9.8\times 10^{-4}T^{7/2}\ \mu\Omega\,\text{cm}\quad (T\lesssim 5\ \text{K})\ . \tag{19}$$

A resistivity relaxation rate for electron-impurity—like ex-

citation scattering is given by $\tilde{\tau}_{e\text{-}i}^{-1} = ne^2\rho_{e\text{-}i}/m_e$, or

$$\tilde{\tau}_{e\text{-}i}^{-1} \sim 5.48 \frac{e^4 m_e^{*2}(m_i^* k_B T)^{7/2}}{\hbar^{10} k_{TF}^4 k_F^3 m_e} \quad (T \ll T_B), \tag{20}$$

which for LMD at $r_s = 1.6$ has the approximate value

$$\tilde{\tau}_{e\text{-}i}^{-1} \sim 1.06\times 10^{11} T^{7/2} \text{ sec}^{-1} \quad (r_s = 1.6,\ T \ll 5 \text{ K}). \tag{21}$$

The contribution to the low-temperature electrical resistivity from the electron-phonon interaction, $\rho_{e\text{-ph}}$, is well known. Ziman[12] discusses a variational estimate for $\rho_{e\text{-ph}}$ in a solid starting with a form analogous to Eq. (2a). This result is based on the assumption that the phonon equilibration rate is rapid compared to the electron-phonon—resistivity relaxation rate. (The applicability of this assumption is examined below.) Transcribing this result to the present case of LMD we arrive at[17]

$$\rho_{e\text{-ph}} \sim 9.30\times 10^2 \frac{(k_B T)^5}{e^2 \hbar^3 m_d k_F^5 s^6} \quad (T \ll T_D), \tag{22a}$$

or

$$\rho_{e\text{-ph}} \sim 3.2\times 10^{-16} T^5 \ \mu\Omega\,\text{cm} \quad (r_s = 1.6,\ T \ll 5.5\times 10^3 \text{ K}). \tag{22b}$$

The corresponding relaxation rate is then

$$\tilde{\tau}_{e\text{-ph}}^{-1} \sim 3.14\times 10^1 \frac{(k_B T)^5}{\hbar^3 m_e m_d k_F^2 s^6} \quad (T \ll T_D), \tag{23a}$$

$$\sim 3.5\times 10^{-2} T^5 \text{ sec}^{-1} \quad (r_s = 1.6,\ T \ll 5.5\times 10^3 \text{ K}). \tag{23b}$$

Note that $\tilde{\tau}_{e\text{-ph}}^{-1} \sim r_s^8 m_{\text{ion}}^2$. Thus electron-phonon scattering is *much* weaker in a high-density "light-ion" system such as LMD than in, say, metallic Na. (In fact, for $r_{s,\text{Na}} = 4$, $m_{\text{Na}} \sim 10 m_d$, we find $\rho_{e\text{-ph}}^{\text{LMD}}/\rho_{e\text{-ph}}^{\text{Na}} \sim 10^6$.)

IV. DISCUSSION AND CONCLUSIONS

A summary of the results of the Appendix for the relaxation and scattering rate appears in Table I.

The impurity-like excitation total scattering rate τ_i^{-1} is seen to receive two very comparable and dominating contributions from scattering between impurity-like excitations and electrons or other impurity-like excitations (scattering between impurity-like excitations and phonons is utterly negligible in comparison),

$$\tau_i^{-1} \approx \tau_{i\text{-}e}^{-1} + \tau_{i\text{-}i}^{-1} \quad (T \ll T_B), \tag{24a}$$

$$\sim 1.7\times 10^{10} T^2 \quad (r_s = 1.6,\ T \lesssim 5 \text{ K}). \tag{24b}$$

We draw particular attention to the fact that $\tilde{\tau}_{e\text{-}i}^{-1}/\tau_i^{-1} \sim 10 T^{3/2}$. Thus at very low temperatures ($T \lesssim 0.05$ K) the assumption underlying the result for $\rho_{e\text{-}i}$ [Eq. (18)], namely that the impurity-like excitation may be regarded, as in thermal equilibrium, on the time scale of the relevant impurity-like excitation—electron scattering time, would seem to be fully justified.

The phonon total scattering rate τ_{ph}^{-1} is overwhelmingly dominated by the phonon-electron contribution, with linear temperature dependence, i.e.,

$$\tau_{\text{ph}}^{-1} \sim \tau_{\text{ph-}e}^{-1} \quad (T \ll T_B), \tag{25a}$$

$$\sim 2\times 10^9 T \quad (T \lesssim 5 \text{ K}). \tag{25b}$$

We then have $\tilde{\tau}_{e\text{-ph}}^{-1}/\tau_{\text{ph}}^{-1} \sim 1.8\times 10^{-11} T^4$, a very small quantity for all temperatures of interest; this supports the assumption in the derivation of $\rho_{e\text{-ph}}$ [Eq. (22a)] that the phonons can be treated as in equilibrium.

Having justified the preliminary results for $\rho_{e\text{-}i}$ and $\rho_{e\text{-ph}}$ and given the numerical estimates in the Appendix, we conclude that under the assumed conditions, the electrical resistivity of LMD, ρ_e^{LMD}, is dominated by scattering between impurity-like excitations and electrons and is given by

$$\rho_e^{\text{LMD}} \approx \rho_{e\text{-}i} \sim 1.6\times 10^2 \frac{e^2 m_e^{*2}(m_i^* k_B T)^{7/2}}{\hbar^{10} k_{\text{TF}}^4 k_F^6} \quad (T \ll T_B), \tag{26a}$$

$$\sim 10^{-3} T^{7/2} \ \mu\Omega\,\text{cm} \quad (T \lesssim 5 \text{ K}). \tag{26b}$$

The $T^{7/2}$ behavior may be physically argued to arise as follows: (a) The electrons scatter off all available

TABLE I. Relaxation and collision rates in LMD. Numerical values (in sec^{-1}) are for LMD at $r = 1.6$. Note the provisos on the applicability of these formulas in the text.

$\tau_{e\text{-}e} \sim 1.7 \dfrac{m_e^*(k_B T)^2}{\hbar^3 k_{\text{TF}}^2}$	$\tau_{e\text{-}i}^{-1} \sim 5.5 \dfrac{e^4 m_e^{*2}(m_i^* k_B T)^{7/2}}{\hbar^{10} k_{\text{TF}}^4 k_F^3 m_e}$	$\tau_{e\text{-ph}}^{-1} \sim 3.1\times 10^1 \dfrac{(k_B T)^5}{\hbar^3 m_e m_d k_F^2 s^6}$
$(4.6\times 10^5 T^2)$	$(1.1\times 10^{11} T^{7/2})$	$(3.5\times 10^{-2} T^5)$
$\tau_{i\text{-}e}^{-1} \sim 4.2 \dfrac{e^4 m_e^{*2} m_i^*(k_B T)^2}{\hbar^7 k_{\text{TF}}^4}$	$\tau_{i\text{-}i}^{-1} \sim 5.9 \dfrac{m_i^*(k_B T)^2}{\hbar^3 k_{\text{TF}}^2}$	$\tau_{i\text{-ph}}^{-1} \sim 4.6 \dfrac{m_i^{*2}(k_B T)^9(\Delta_1^2+\Delta_2^2)}{\hbar^7 s^{10}}$
$(5.4\times 10^9 T^2)$	$(1.2\times 10^{10} T^2)$	$(1.6\times 10^{-18} T^9)$
$\tau_{\text{ph-}e}^{-1} \sim 0.52 \dfrac{k_F k_B T}{m_d s}$	$\tau_{\text{ph-}i}^{-1} \sim 8.0\times 10^{-2} \dfrac{m_i^{*7/2}(k_B T)^{15/2}(\Delta_1^2+\Delta_2^2)}{\hbar^7 s^7}$	$\tau_{\text{ph-ph}}^{-1} \sim 3.1 \dfrac{1}{\hbar^7 \rho_0^2 \gamma s} \left[\dfrac{k_B T}{s}\right]^7$
$(1.9\times 10^9 T)$	$(1.4\times 10^{-11} T^{15/2})$	$(5.4\times 10^{-10} T^7)$

impurity-like targets whose number is proportional to $\sim T^{3/2}$. (b) However, only a fraction $\sim k_B T/\epsilon_F$ of all electrons can scatter because of the fermion final-state restrictions. (c) For resistivity, the differential scattering rate must be weighted by the standard angular factor $1-\cos\theta_0$, which, since the electron scattering angle is small, introduces a factor of T. Thus $\tau_{e\text{-}i}^{-1}\sim T^{7/2}$.

We emphasize that the $T^{7/2}$ dependence is a direct consequence of the important quantum behavior in the ionic degrees of freedom. In particular, it is due both to the boson character of the ions *and* the nonzero boson spin. For example, if the boson spin were zero there would be no impurity-like excitations and the resistivity would follow the familiar $\rho_e\sim\rho_{e\text{-ph}}\sim T^5$. Furthermore, if the ions were (degenerate) fermions rather (condensed) bosons, as in liquid metallic hydrogen (LMH), then $\rho_e\sim T^2$ as a consequence of electron-proton scattering.[5] In fact, we have estimated, for LMH,[5]

$$\rho_e^{\rm LMH}\sim 1.1\frac{m_p^{*2}(k_B T)^2}{e^2\hbar^3 k_F^5}\quad (T\ll T_{F_p})\,, \tag{27a}$$

$$\sim 10^{-3}T^2\ \mu\Omega\,{\rm cm}\quad (r_s=1.6,\ T\lesssim 10^2\ {\rm K})\,. \tag{27b}$$

Here, m_p^* is the proton effective mass and T_{F_p} is the proton Fermi temperature. We note that $\rho_e^{\rm LMD}/\rho_e^{\rm LMH}\sim T^{3/2}$.

We observe that for a corresponding normal (and perfect) crystalline metallic phase of highly compressed deuterium, the ultimate low-temperature resistivity would vary as $\sim T^2$ as a result of umklapp-mediated electron-electron scattering.[12] However, this fermion contribution, though of the same form as that encountered in electron-proton scattering in LMH, would be $\sim 10^8$ times smaller than in Eq. (27b), a consequence of, mainly, the smaller final-state phase space available to the scattered electrons as compared to protons.[5] The boson character and nonzero boson spin will have no significant implications for the resistivity in an assumed *normal* crystalline phase.

Further effort might well be directed to understanding the role of superfluidity in the low-temperature resistivity. It would be desirable to carry out a hydrodynamic-kinetic approach[18] of the kind used for ^{4}He, together with an attempt to solve the corresponding three coupled Boltzmann equations for the three types of quasiparticles. The possibility of simultaneous superfluid and normal flows (with zero net deuteron current) taking place as electric current passes through the sample and other novel superfluid effects might also be examined using this approach.

ACKNOWLEDGMENTS

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APPENDIX: QUASIPARTICLE RELAXATION RATES

We now examine the legitimacy of the assumption used in Eq. (4) and the corresponding assumption for phonons underlying the result (22a). In particular, we evaluate collision rates of impurity-like excitations and of phonons against the other quasiparticle distributions ($\tau_{i\text{-}\alpha}^{-1}$ and $\tau_{\rm ph\text{-}\alpha}^{-1}$, α denotes e,ph,i). For order-of-magnitude estimates of these quantities we assume in the formal expression for the scattering rates that all quasiparticle types are in equilibrium.[19]

1. $\tau_{i\text{-}i}^{-1}$

The impurity-like excitation—impurity-like excitation scattering rate may be roughly estimated as (ignoring boson final-state factors)

$$\tau_{i\text{-}i}^{-1}\sim v_i\sigma_0 n_i\,, \tag{A1}$$

where $v_i=(2k_BT/m_i^*)^{1/2}$ is a typical impurity-like velocity, where $\sigma_0\sim 4\pi/k_{\rm TF}^2$ impurity-like excitation cross section, and where $n_i\approx 0.33(m_i^* k_B T)^{3/2}/\hbar^3$ is the density of impurity-like excitations. Thus,

$$\tau_{i\text{-}i}^{-1}=5.86\frac{m_i^*(k_BT)^2}{\hbar^3 k_{\rm TF}^2}\quad (T\ll T_B)\,, \tag{A2a}$$

$$\sim 1.2\times 10^{10}T^2\ {\rm sec}^{-1}\quad (r_s=1.6,\ T\lesssim 5\ {\rm K})\,. \tag{A2b}$$

2. $\tau_{i\text{-}e}^{-1}$

The characteristic scattering rate of an impurity-like excitation of energy k_BT against the electrons may be estimated from

$$\tau_{i\text{-}e}^{-1}=2\left[\frac{V}{(2\pi\hbar)^3}\right]^3\int d^3\vec{p}_2 d^3\vec{p}_3 d^3\vec{p}_4\frac{2\pi}{\hbar}\,|\langle\vec{p}_3\vec{p}_4|\mathscr{V}|\vec{p}_1\vec{p}_2\rangle|^2\delta(\epsilon_{i1}+\epsilon_{e2}-\epsilon_{i3}-\epsilon_{e4})n^0(\epsilon_{e2})[1+\tilde{n}^0(\epsilon_{i3})][1-n^0(\epsilon_{e4})]\,. \tag{A3}$$

In Eq. (A3), $p_1=(2m_i^*k_BT)^{1/2}\hat{p}_1$ and $\vec{p}_3$ ($\vec{p}_2$ and $\vec{p}_4$) are the initial and final impurity-like excitation (electron) momenta, respectively. The matrix element is the same as that appearing in Eq. (2c) (with properly interchanged momentum indices). Note that for a simple estimate it is not necessary to include an angular weighting factor of the form $1-\hat{p}_1\cdot\hat{p}_3$. This is because the impurity-like excitations, unlike the electrons, suffer large angular deflections when scattered by the electrons.

We again approximate the boson final-state factor $1+\tilde{n}^0(\epsilon_{i3})$ by unity, and replace A in the matrix element [Eq. (2c)] by an appropriate average value $\bar{A}$. From the analysis of approximation (3) (Sec. III), we recall that the initial and final electron momenta are both very close to

p_F and that all electron targets of momentum $\vec{p}_2 = p_F |\hat{p}_2|$ are available as scatterers. Furthermore, from approximation (3), but now with $X_4 = \cos\theta_4$ playing the role of the X_3 of Sec. III (see Fig. 2), we see that momentum and energy conservation and the final-state restriction for the scattered electron (approximated by the statement $\epsilon_4 \geq \epsilon_F - k_B T$) leads to the restriction $\bar{X} \leq X_4 \leq 1$. Note that $\bar{X}$ is given by Eq. (9) with θ_0 here, as in Sec. III, denoting the angle between $\vec{p}_1$ and $\vec{p}_2$. To evaluate $\tau_{i\text{-}e}^{-1}$ we first use momentum conservation [Eq. (2c)] to eliminate the $\vec{p}_3$ integration. To this point we then have

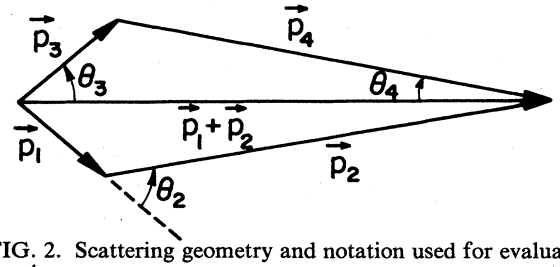

FIG. 2. Scattering geometry and notation used for evaluation of $\tau_{i\text{-}e}^{-1}$. Diagram exaggerates impurity excitation momenta $\vec{p}_1, \vec{p}_3$ in comparison to electron momenta $\vec{p}_2, \vec{p}_4$: $p_1, p_3 \ll p_2 \approx p_4 \approx p_F$.

$$\tau_{i\text{-}e}^{-1} = \frac{2}{(2\pi)^5} \frac{|\bar{A}|^2}{\hbar^7} \int d^3\vec{p}_2 d^3\vec{p}_4 \delta(\epsilon_{i1} + \epsilon_{e2} - \epsilon_{i3}(\vec{p}_1, \vec{p}_2, \vec{p}_4) - \epsilon_{e4}) n^0(\epsilon_{e2})[1 - n^0(\epsilon_{e4})] \,. \tag{A4}$$

We next perform the ϵ_{e4} integration; this brings in a factor of $m_e^* p_F$ and sets $\epsilon_{e4} = \epsilon_{e4}(\vec{p}_1, \vec{p}_2, X_4)$ in the final-state factor. Proceeding as in Eqs. (12), but utilizing the characteristic of X_4, we have

$$\int_{-1}^{1} dX_4 [1 - n^0(\epsilon_{e4}(\vec{p}_1, \vec{p}_2, X_4))] \sim \int_{\bar{X}(X_2)}^{1} dX_4 [1 - n^0(\epsilon_{e2})] \,. \tag{A5}$$

The ϕ_2, ϕ_4 and X_2, X_4 integrals are then easily performed. From the integrand of the right-hand side of Eq. (A5) and the result

$$n^0(\epsilon)[1 - n^0(\epsilon)] \approx k_B T \delta(\epsilon - \epsilon_F) \,,$$

the remaining ϵ_{e2} integration is also seen to be simple. With approximation Eq. (17) for $\bar{A}$ we obtain

$$\tau_{i\text{-}e}^{-1} \sim 4.24 \frac{e^4 m_e^{*2} m_i^* (k_B T)^2}{\hbar^7 k_{\mathrm{TF}}^4} \quad (T \ll T_B) \,, \tag{A6a}$$

$$\sim 5.4 \times 10^9 T^2 \ \mathrm{sec}^{-1} \quad (r_s = 1.6,\ T \ll 5\ \mathrm{K}) \,. \tag{A6b}$$

3. $\tau_{i\text{-ph}}^{-1}$

For the interaction between impurity-like excitations and phonons we use an idea similar to the "Landau quantum hydrodynamics" method which is suitable when the phonon wavelength λ_{ph} is very large compared to the impurity-like excitation wavelength λ_i.[18] Note that for LMD at $r_s = 1.6$, $\lambda_{\mathrm{ph}}/\lambda_i \sim 8 \times 10^2 / T^{1/2}$. The spirit of this method is to regard the phonon as the source of effectively uniform density fluctuations on the scale of the impurity-like packet. This density-fluctuation affects the impurity-like excitation energy through the density dependence of m_i^*. For a simple estimate we may ignore possible superfluid effects or local flow effects associated with passage of the phonon. For the change in energy $\delta\epsilon_{i\vec{p}}$ of an impurity-like excitation of momentum $\vec{p}$, associated with a density fluctuation $\delta\rho(\vec{r}, t)$, we have

$$\delta\epsilon_{i\vec{p}} = \frac{p^2}{2} \frac{\partial m_i^{*-1}}{\partial n} \delta\rho(\vec{r}, t) + \frac{1}{2} \frac{p^2}{2} \frac{\partial^2 m_i^{*-1}}{\partial n^2} [\delta\rho(\vec{r}, t)]^2 + \cdots \,. \tag{A7}$$

Note that unlike the ^{3}He-phonon interaction in ^{3}He-^{4}He mixtures,[20] there are no terms in $\delta\epsilon_{i\vec{p}}$ associated with the density dependence of the "impurity" chemical potential: Here the "impurity" chemical potential vanishes. In fact, the presence of the overall factor of p^2 [Eq. (A7)], which will typically be a very small quantity for the temperatures here, means that the interaction between impurity-like excitations and phonons will be considerably weaker than the ^{3}He-phonon interaction.

The density fluctuation may be expressed in terms of the phonon creation and destruction operators $b_{\vec{q}}^\dagger$ and $b_{\vec{q}}$ as[20]

$$\delta\rho_4(r) = \sum_{\vec{q}} \left[\frac{qn}{2m_d s V} \right]^{1/2} (b_{\vec{q}} e^{i\vec{q}\cdot\vec{r}} + b_{\vec{q}}^\dagger e^{-i\vec{q}\cdot\vec{r}}) \,. \tag{A8}$$

When Eq. (A8) is introduced into Eq. (A7) the first-order term is of the form of the usual particle-phonon Hamiltonian. From this term we easily compute the matrix element for impurity-like excitations to scatter from $\vec{p}$ to $\vec{p}\,'$, and at the same time absorbing a phonon of momentum $\vec{q}$,

$$\langle \vec{p}\,' | T_1 | \vec{p}, \vec{q} \rangle = \delta_{\vec{p}\,', \vec{p}+\vec{q}} \left[\frac{qn}{2m_d s V} \right]^{1/2} \frac{p^2}{2} \frac{\partial m_i^{*-1}}{\partial n} \,. \tag{A9}$$

Note that though the typical impurity-like velocity $\bar{v}_i$ is much smaller than the sound speed; in fact, $\bar{v}_i / s \sim 10^{-3} T^{1/2}$ for $r_s = 1.6$. It is also easily shown that this makes it kinematically impossible for a typical impurity-like excitation to absorb or emit a phonon.[21] Thus we must consider second and higher-order processes to understand interactions between impurity-like excitations and phonons.

We consider then binary *scattering* processes of the type where one impurity-like excitation and one phonon are

present, both in the initial and final states. We analyze processes (a)—(c) of Fig. 3, which are of this type. For our simple estimate we need not consider phonon-mediated processes of this class (these involve the three-point phonon vertex). Such processes contribute at the same order as processes (a)—(c), but do not affect the form or order of magnitude of the scattering amplitude.[20]

Process (a) arises in the first-order perturbation treatment of the second term in Eq. (A7), while processes (b) and (c) arise from a second-order perturbation treatment of the first term of Eq. (A7). We find, for the total scattering amplitude from processes (a)—(c),

$$\langle \vec{p}\,',\vec{q}\,' \mid T_2 \mid \vec{p},\vec{q}\rangle = \delta_{\vec{p}+\vec{q},\vec{p}\,'+\vec{q}\,'} \frac{qp^2}{4sV} \left[\frac{1}{m_i^*} \frac{\partial m_i^{*-1}}{\partial n} \cos\theta + \frac{n}{m_d} \frac{\partial^2 m_i^{*-1}}{\partial n^2} \right] , \tag{A10}$$

where θ is the angle between $\vec{q}$ and $\vec{q}\,'$. In deriving Eq. (A10) we have used $\bar{v}_i \ll s$, ignored boson enhancement factors in the intermediate states, and have used the fact that on the energy shell, $q' \approx q$. This last approximation follows from the fact that typical phonon energies are much smaller than $m_d s^2$ (or, in other words, $m_d s^2/k_B T \sim 3\times 10^5/T$ at $r_s = 1.6$).

The rate $\tau_{i\text{-ph}}^{-1}$ at which a typical impurity-like excitation of momentum $p_1 = (2m_i^* k_B T)^{1/2}$ is scattered by the phonons is estimated by

$$\tau_{i\text{-ph}}^{-1} \sim \sum_{\vec{p}\,',\vec{q},\vec{q}\,'} \frac{2\pi}{\hbar} \mid \langle \vec{p}\,',\vec{q}\,' \mid T_2 \mid \vec{p},\vec{q}\rangle \mid^2 \delta(\epsilon_{i\vec{p}} + \epsilon_{\text{ph}\vec{q}\,'} - \epsilon_{i\vec{p}\,'} - \epsilon_{\text{ph}\vec{q}}) [1 - n^0(\epsilon_{ep'})] \tilde{n}^0(\epsilon_{iq}) [1 + \tilde{n}^0(\epsilon_{iq'})] . \tag{A11}$$

We use momentum conservation [Eq. (A10)] to eliminate the $\vec{p}\,'$ sum. The effect of the energy δ-function in the q' integration is to set $q' \approx q$ and to bring in a factor of $1/s$. Moreover, $q' \approx q \ll p$ implies $\epsilon_{\vec{p}+\vec{q}-\vec{q}\,'} \approx \epsilon_{\vec{p}}$ in the integrand. After some straightforward algebra we find

$$\tau_{i\text{-ph}}^{-1} \sim 4.59 \frac{m_i^{*2} (k_B T)^9}{\hbar^7 s^{10}} (\Delta_1^2 + \Delta_2^2) , \tag{A12}$$

where

$$\Delta_1 \equiv \frac{1}{3m_i^*} \frac{\partial m_i^{*-1}}{\partial n} \tag{A13a}$$

and

$$\Delta_2 \equiv \frac{n}{m_d} \frac{\partial^2 m_i^{*-1}}{\partial n^2} . \tag{A13b}$$

We may obtain a crude estimate of the requisite effective-mass density derivatives by assuming that m_i^* $(r_s = 2\times 1.6) = 0.5 m_i^*$ $(r_s = 1.6)$, and m_i^* $(r_s = 0.5\times 1.6)$ $= 1.5 m_i^*$ $(r_s = 1.6)$. This then gives

$$m_i^{*-1}(n) = (1.08 - 11.1n + 2.00n^2) m_d^{-1} \quad (n \text{ in a.u.}) . \tag{A14}$$

Using Eqs. (A14) and (A13) in Eq. (A12) we then estimate

$$\tau_{i\text{-ph}}^{-1} \sim 1.6\times 10^{-18} T^9 \text{ sec}^{-1} . \tag{A15}$$

4. $\tau_{\text{ph-ph}}^{-1}$

In a study of phonon-phonon interaction in superfluid ^{4}He, Khalatnikov gives an approximate expression for the phonon-phonon scattering rate.[18,22] For a typical phonon of momentum $p = k_B T/s$,

$$\tau_{\text{ph-ph}}^{-1} \sim 3.1 \frac{1}{\hbar^7 \rho_0^2 \gamma s} \left[\frac{k_B T}{s} \right]^7 . \tag{A16}$$

Here, ρ_0 is the total mass density and γ is related to the deviation of the phonon dispersion from $\sim p$ via

$$\bar{\epsilon}_{\text{ph}}(p) = s(p - \gamma p^3) . \tag{A17}$$

A rough estimate of γ for condensed LMD may be made from a consideration of the polarization propagator in a joint random-phase approximation for both electrons and deuterons, and by assuming weak condensate depletion. The collective modes are determined by the van-

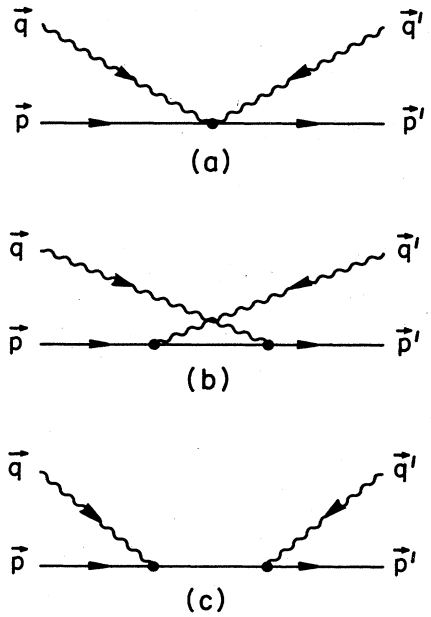

FIG. 3. Three process included in impurity-like excitation—phonon scattering matrix element. Initial and final impurity-like excitation (phonon) momenta are denoted by $\vec{p}$ and $\vec{p}\,'$ ($\vec{q}$ and $\vec{q}\,'$), respectively.

ishing of

$$V_0(k)\mathrm{Re}[\Pi_e^R(k,w)+\Pi_b^R(k,w)]-1\ , \tag{A18}$$

where $V_0(k)=4\pi e^2/k^2$, and Π_e^R (Π_b^R) is the retarded electron (deuteron) polarization propagator. In our approximation $\Pi_e^R(k,w)$ is given by the usual Lindhard result,[10] whereas the boson propagator is given by[10]

$$\Pi_b^R(k,w)=\frac{-2n_b\epsilon_b(k)}{\epsilon_b^2(k)-\hbar^2w^2}\ , \tag{A19}$$

where $\epsilon_b(k)=\hbar^2k^2/2m_d$. Here, n_b is the condensate density, which we approximated by the total density $n_b\approx n$.

We are interested in the root of Eq. (A18) with $w\sim k$ as $k\to 0$. Thus, setting $w(k)=\bar{\epsilon}_{\mathrm{ph}}(\hbar k)/\hbar$ in Eq. (A18) and expanding to low order in k, we find, after some algebra,

$$\gamma\sim\frac{\pi}{24}\frac{\hbar^2k_F}{e^2m_dm_e^{*2}s^2} \tag{A20a}$$

$$\sim 8\times 10^{36}\ \mathrm{g}^{-2}\,\mathrm{cm}^{-2}\,\mathrm{sec}^2\ \ (r_s=1.6)\ . \tag{A20b}$$

This leads to

$$\tau_{\text{ph-ph}}^{-1}\sim 5.4\times 10^{-10}T^7\ \mathrm{sec}^{-1}\ \ (r_s=1.6)\ . \tag{A21}$$

The corresponding value for ^{4}He is $\tau_{\text{ph-ph}}^{-1}(^4\mathrm{He})\sim 9\times 10^7T^7$ sec^{-1}. The enormous contrast between ^{4}He and LMD stems from the facts that (a) the γ for ^{4}He is roughly of the same order as the for LMD, and (b) $\tau_{\text{ph-ph}}^{-1}\sim s^{-7}$ (for a typical phonon); $s_{^4\mathrm{He}}/s_{\mathrm{LMD}}\sim 10^{-2}$.

5. $\tau_{\text{ph-}e}^{-1}$

From the standard result for the electron-phonon interaction in jellium with dynamic background,[10] we may readily estimate the rate at which a typical phonon of momentum q is absorbed due to interaction with electrons,

$$\tau_{\text{ph-}e}^{-1}\sim 2\left[\frac{2\pi}{\hbar}\right]\left[\frac{\pi^2\hbar^2}{m_e^*k_F}\right]^2\left[\frac{nq}{2m_dsV}\right]\sum_{\vec{p},\vec{p}\,'}\{\delta(\epsilon_{e\vec{p}}+\epsilon_{\mathrm{ph}\vec{q}}-\epsilon_{e\vec{p}\,'})\delta_{\vec{p}\,',\vec{p}+\vec{q}}n^0(\epsilon_{e\vec{p}})[1-n^0(\epsilon_{e\vec{p}\,'})]$$
$$+\delta(\epsilon_{e\vec{p}}-\epsilon_{\mathrm{ph}\vec{q}}-\epsilon_{e\vec{p}\,'})\delta_{\vec{p}\,'+\vec{q},\vec{p}}n^0(\epsilon_{e\vec{p}})[1-n^0(\epsilon_{e\vec{p}\,'})]\}\ . \tag{A22}$$

Here, $\vec{p}$ and $\vec{p}\,'$ denote initial and final electron momenta. The two terms in Eq. (A22) correspond to absorption and emission. After some straightforward alegebra we arrive at the very simple result ($q=k_BT/s$),

$$\tau_{\text{ph-}e}^{-1}\sim 0.524\frac{k_Fk_BT}{m_ds}\ \ (T\ll T_D)\ , \tag{A23a}$$

$$\sim 1.9\times 10^9T\,\mathrm{sec}^{-1}\ \ (r_s=1.6,\ T\ll 5.5\times 10^3\ \mathrm{K})\ . \tag{A23b}$$

6. $\tau_{\text{ph-}i}^{-1}$.

As noted in subsection 3, phonon absorption and emission for typical phonons due to interaction with impurity-like excitations is kinematically impossible. We thus require an estimate of the rate at which a typical phonon of momentum q is *scattered* by the impurity-like excitations,

$$\tau_{\text{ph-}i}^{-1}\sim 2\sum_{\vec{p},\vec{p}\,',\vec{q}\,'}|\langle\vec{p}\,',\vec{q}\,'|T_2|\vec{p},\vec{q}\rangle|^2\delta(\epsilon_{e\vec{p}}+\epsilon_{\mathrm{ph}\vec{q}}-\epsilon_{e\vec{p}\,'},-\epsilon_{\mathrm{ph}\vec{q}\,'})[1+\tilde{n}^0(\epsilon_{\mathrm{ph}\vec{q}\,'})]\tilde{n}^0(\epsilon_{i\vec{p}})[1+\tilde{n}^0(\epsilon_{i\vec{p}\,'})]\ , \tag{A24}$$

where the matrix element is given by Eq. (A10). Momentum conservation eliminates the $\vec{p}\,'$ sum, and under the conditions here, $\vec{p}\,'\to\vec{p}$ in the boson final-state factor. The effect of the energy δ-function in the q' integration is to set $q'\approx q$ while bringing out a factor of $1/s$. The remaining integration is easily performed and we arrive at (for $\epsilon_{\mathrm{ph}\vec{q}}=k_BT$)

$$\tau_{\text{ph-}i}^{-1}\sim 8.04\times 10^{-2}\frac{m_i^{*7/2}(k_BT)^{15/2}}{\hbar^7s^7}(\Delta_1^2+\Delta_2^2)\ \ (T\ll T_B)\ , \tag{A25a}$$

$$\sim 1.4\times 10^{-11}T^{15/2}\ \mathrm{sec}^{-1}\ \ (T\lesssim 5\ \mathrm{K})\ . \tag{A25b}$$

It should be pointed out that at very low temperatures the absorption of phonons due to "viscosity" in the impurity-like excitations (i.e., taking into account interactions between impurity-like excitations) may be an important contribution to τ_{ph}^{-1}. We know, for example, that viscosity effects in the ^{3}He subsystem in dilute ^{3}He in ^{4}He mixtures have been shown to play an important role in low-temperature phonon attentuation.[23] These effects are not considered here; hence Eq. (A25) should be regarded as an estimated lower bound on $\tau_{\text{ph-}i}^{-1}$.

7. $\tau_{e\text{-}e}^{-1}$

We quote here the standard result for electrons in jellium with an averaged Yukawa effective interaction,[24] namely

$$\tau_{e\text{-}e}^{-1} \sim 1.70 \frac{m_e^*}{\hbar^3 k_{\mathrm{TF}}^2} (k_B T)^2 \quad (T \ll T_{F_e}) , \tag{A26a}$$

$$\sim 4.6 \times 10^5 T^2 \ \mathrm{sec}^{-1} \quad (r_s = 1.6,\ T < 2.3 \times 10^5\ \mathrm{K}) , \tag{A26b}$$

where T_{F_e} is the electron Fermi temperature.

Note that in the rates $\tau_{j\text{-}k}^{-1}$ the effects of mediated interactions involving the remaining quasiparticle type(s), $l \neq j,k$, are not considered.

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[14]The number of bose *excitations* in the system must vanish as $T \to 0$; the lack of number conservation of the excitations corresponds to vanishing "chemical potential" in the Bose distribution function describing these excitations.

[15]For boson spin S the degeneracy of the impurity-like modes is $2S$.

[16]This easily follows when $\vec{p}_1$ and $\vec{p}_3$ are written in terms of components parallel and perpendicular to the axis $\vec{p}_1 + \vec{p}_2$.

[17]The quantity $C(q)$ appearing in Ziman's formula 9.5.17 may be transcribed to the jellium case by a comparison of Ziman's total transition probability (see his Eqs. 9.5.1 and 9.5.6) and the corresponding result appropriate for "jellium" with a dynamically responding background. For the latter, see, e.g., Ref. 10, Chap. 12; the result there involves the replacement of the bare Coulomb interaction $\sim e^2/k^2$ by e^2/k_{TF}^2. The replacement here is then $C^2 \to \pi^4 N e^4 a_0^2 n / V k_F^2$.

[18]I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965).

[19]D. Pines and P. Nozières, *The Theory of Quantum Liquids* (Benjamin, New York, 1966).

[20]G. Baym and C. Pethick, in *The Physics of Liquid and Solid Helium,* edited by K. A. Bennemann and J. B. Ketterson (Wiley, New York, 1978), Chap. 2.

[21]Only an exponentially small number of emission and absorption events are possible; these correspond to the tails of the impurity-like excitation distribution function.

[22]A dimensionless quantity u appears in Khalatnikov's formula. Here we set $u = 6$.

[23]G. Baym and C. Ebner, Phys. Rev. **164**, 235 (1967).

[24]D. Pines, *Elementary Excitations in Solids* (Benjamin, New York, 1963).