Universality of static properties near the superfluid transition in ^{4}He

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We present the results of new analyses of static properties near the superfluid transition in ^{4}He which incorporate recent accurate entropy measurements. They yield a superfluid density exponent $\zeta=0.6717\pm0.0004$ at vapor pressure, where the quoted error is 1 standard deviation. This yields a specific-heat exponent $\alpha=-0.015$ via scaling. We assumed these exponents to be universal (i.e., independent of pressure) in the remainder of the analysis. Their systematic errors are difficult to estimate; therefore we analyzed data at higher pressures with the three values $\zeta=0.669$, 0.672, and 0.675, corresponding to $\alpha=-0.007$, -0.016, and -0.025, respectively, which surely bracket the true values. Regardless of the exponent pair used, pressure-independent results were obtained for all experimentally accessible amplitude combinations that have been predicted to be universal. Specifically, the problem of apparent deviations from two-scale-factor universality along the λ line has been resolved by the use of the new entropy measurements and this analysis.

I. INTRODUCTION

The renormalization-group theory (RGT) of critical phenomena[1] predicts that certain exponents and dimensionless combinations of amplitudes which describe the singularities of various properties near a critical point are universal[2] in the sense that they depend only upon such general properties of a system as its spatial dimensionality and the number of degrees of freedom of its order parameter. The theory also predicts exact relationships, called scaling laws, between the exponents,[3] and gives good numerical estimates[4] of the values of the universal quantities. A number of extensive and high-precision measurements of static properties of liquid ^{4}He near the superfluid transition have provided several opportunities in recent years to test these predictions at a quantitative level.[5-15]

While the original analysis of most of these experiments supported the predictions of scaling[1,3] and universality[1,2] and agreed well with the RGT,[1] some experimental results have initially appeared to contradict the theoretical predictions.[6-8] Part of the difficulty is attributable to the fact that until very recently the theory has primarily made asymptotic predictions,[4] whereas experiments necessarily are performed a finite distance from the phase transition. This problem was resolved in part when it was realized over a decade ago that it is necessary to include confluent *singular* terms in the analysis of both static[11,16] and transport[17] properties in order to fit the data and to obtain universal leading exponents, and by the derivation for static properties of these confluent singularities from the RGT.[18] It will be eased even further when the theory of static properties is developed to the point where it not only predicts the asymptotic properties in the form of power-law expansions, but also gives the complete behavior of the system well away from the critical point. Much progress towards this goal has indeed been made for both the statics[19] and the dynamics,[20-22] but nonetheless, for the present analysis of static properties we are still limited primarily to comparisons based on power-law fits, albeit with the inclusion of a confluent singular term.

The remaining difficulties have to some extent been eliminated by obtaining more suitable experimental data. A notable example here is the determination of the amplitude ratio A/A' of the heat capacity at constant *pressure* C_p from measurements of the isobaric thermal-expansion coefficient[14] β_p rather than more indirectly from the heat capacity at constant *volume* C_v.[10]

There remained one major apparent discrepancy between the predicted universality of the phase transition along the λ line and experimental data for static properties. This discrepancy involves the free energy of an amount of fluid contained in a volume equal to the cube of the correlation length for fluctuations of the order parameter. The universality of the associated parameter R_ξ^- is a consequence of the existence of only two independent, system-specific scales that determine the actual magnitudes of all static properties of a given system and is often referred to as two-scale-factor universality.[23] Previous analyses of measurements near $T_\lambda(P)$ have yielded values of R_ξ^- which appeared to have a significant pressure dependence.[6]

The analysis and comparison of experiment near $T_\lambda(P)$ rest heavily upon the determination of the superfluid fraction ρ_s/ρ from the second-sound velocity u_2,[11] and upon the calculation of C_p from β_p.[14] Both of these require a knowledge of the entropy S of the system. For that reason we recently remeasured $S(P,t)$ near $T_\lambda(P)$ with substantially improved accuracy.[24] In the present paper we reexamine a large amount of available data on the static properties of ^{4}He near $T_\lambda(P)$ using our new entropy data. We reevaluate a number of universal parameters, but were particularly motivated in both the entropy measurements and the reanalysis in the present paper by the need for a reexamination of the two-scale-factor—universality problem. We now find that R_ξ^- varies by less than 2% over the entire pressure range of the superfluid transition. This variation is well within the expected experimental errors, and the result is thus in excellent agreement with the theoretical prediction of universality. Therefore the largest remaining discrepancy between predictions of the RGT and measurements of static properties near the superfluid transition is removed, and the universal character of the transition is strongly supported by experiment.

As we shall discuss in Sec. II A, at the present time the comparison of experiment to theory still requires fitting data to functions with many adjustable parameters. It turns out that a meaningful comparison is possible only if some of the parameters are either fixed or constrained by some of the theoretical predictions. Predictions as yet unused in the data analysis can then be tested by comparison with fits to the data. This approach to the experimental study of critical phenomena that we have adopted here has been discussed elsewhere.[6] In particular, we have assumed in this paper the validity of the relevant scaling laws[3] and the universality[2] of the exponents, and have tested the universality of various amplitude combinations. This enabled us to determine the exponent values from the superfluid fraction at vapor pressure[11] which give the most accurate information. The amplitude combinations are then examined as a function of pressure on the basis of data for the superfluid fraction[11] ρ_s/ρ, the heat capacity[9,10,25,26] C_p, and the isobaric thermal-expansion coefficient[14] β_p.

In Sec. II A we outline specifically and in detail our procedure for analyzing the experimental data and discuss the constraints taken from theory. The remainder of Sec. II is devoted to the analysis of ρ_s/ρ, β_p, and C_p, and to comparisons with measurements obtained by different authors. The analysis proceeds as follows: In Sec. II B we obtain the exponent ζ from the superfluid density at vapor pressure,[11] where the necessary data are the most extensive and accurate. Here we give careful consideration to the influence of a temperature-dependent leading amplitude on the exponent.[8] We then use scaling laws to obtain the specific-heat exponent α, and in Sec. II C we reanalyze the data for the expansion coefficient[14] β_p using this value for α and the other restrictions outlined in Sec. II A. Since there is some uncertainty in the experimental values of ζ and α, we repeat the analysis with two other values of α in order to explore any effect of the particular choice for the leading exponents upon other universal parameters. Heat-capacity measurements[9,10,25,26] are reanalyzed in Sec. II D using the same restrictions as for β_p from Sec. II C. The specific heat from Sec. II D and that derived from β_p differ from each other significantly at the higher pressures. They are each used in Sec. II E to rederive the superfluid density under pressure from the second-sound velocity. The correlation-length amplitudes ξ_0 obtained[27] from ρ_s/ρ for the two cases are compared to each other and to other independent measurements in Sec. II F.[13] The results of Sec. II will be used in Sec. III where we reexamine the question of two-scale-factor universality. In Sec. III we also derive values of a number of other universal parameters. In Sec. IV we summarize our results.

II. DATA ANALYSIS

A. Procedure

When analyzing critical properties near a continuous phase transition, one generally wishes to fit the data from experiments to functions of a form that permits as much contact as possible with theoretical predictions. For this purpose we will start with

$$C_p^+(t)=(A/\alpha)t^{-\alpha}[1+D_c t^{\Delta}+O(t^{2\Delta})]+B \tag{2.1a}$$

and

$$C_p^-(t)=(A'/\alpha')\,|\,t\,|^{-\alpha'}[1+D_c'\,|\,t\,|^{\Delta'}+O(|\,t\,|^{2\Delta'})]+B' , \tag{2.1b}$$

for the specific heat above (C_p^+) and below (C_p^-) T_λ, and with similar functions with coefficients A_β, B_β, D_β etc. for the isobaric thermal-expansion coefficient $\beta_p^{\pm}(t)$. Here,

$$t\equiv T/T_\lambda(P)-1 . \tag{2.2}$$

The superfluid fraction below T_λ will be written in the form

$$\rho_s/\rho=k\,|\,t\,|^{\zeta}[1+D_\rho\,|\,t\,|^{\Delta}+O(|\,t\,|^{2\Delta})] . \tag{2.3}$$

The amplitudes in Eqs. (2.1) and (2.3) are regular functions of the pressure P and of the absolute temperature T. Equations (2.1) and (2.3) thus contain a large number of adjustable parameters, and the high correlation between them results in experimental uncertainties which are prohibitively large unless some of the parameter values are constrained by theoretical predictions. Our approach will be to restrict the critical exponents α, ζ, and Δ in Eqs. (2.1) and (2.3) to universal (pressure-independent) values which satisfy relevant scaling laws. We find that the data are consistent with these constraints in the sense that the constraints do not cause systematic deviations from the fitting function. The remaining free parameters can then be used to test additional theoretical predictions which have not yet been employed as input in the data analysis.

Specifically, we imposed the following constraints:

(i) The leading exponents of ρ_s/ρ and C_p^- are assumed to obey the Josephson scaling law[28]

$$\zeta=(2-\alpha')/3 . \tag{2.4}$$

The value of ζ is assumed to be universal.

(ii) The leading critical exponents α and α' above and below T_λ for C_p and β_p [see Eqs. (2.1) and (2.12)] are assumed to obey the scaling law[3]

$$\alpha=\alpha' . \tag{2.5}$$

(iii) The exponents of the lowest-order confluent singular terms above (Δ) below (Δ') T_λ are assumed to be equal and to have the value

$$\Delta=\Delta'=0.5 \tag{2.6}$$

for all relevant properties. The value 0.5 was chosen to agree with an earlier analysis of experimental data.[11] Our analysis is not very sensitive to the exact value of Δ, and our results would remain essentially unaltered if we used, for instance, the theoretical estimate[29] 0.522.

(iv) The additive constants above (B or B_β) and below (B' or B'_β) T_λ are assumed to be equal:[30]

$$B = B' \text{ and } B_\beta = B'_\beta \; . \tag{2.7}$$

Additive constants are absent in the case of ρ_s/ρ.

(v) We neglect all terms of $O(t^{2\Delta})$ and higher. In the analysis of β_p and C_p we also neglect all regular terms [of $O(t)$ and higher], but in the analysis of ρ_s/ρ we investigate the effect of such terms on the exponent ζ.

The influence of constraints (i)—(v) on the quality of the fit of the data to the function has been investigated on numerous occasions.[5–8] As before, we again find that no systematic deviations are induced by these restrictions.

In Sec. II C we also show on the basis of our analysis that the amplitude ratios for the leading and confluent singularities for β_p, i.e., A_β/A'_β and D_β/D'_β, respectively [see Eq. (2.12)], are independent of pressure, as predicted by universality.[2,31] We then perform a second analysis with A_β/A'_β and D_β/D'_β set equal to their average (universal) values at all pressures.

A final restriction is on the data itself. Except for some analyses of measurements at vapor pressure, data are limited to the reduced temperature range $-0.01<t<0.01$, where the exclusion of higher-order confluent singular terms [of $O(t^{2\Delta})$, etc.] and of regular terms has, at most, a small influence on the remaining parameters. At vapor pressure, where precise data over a wide range of t exist, the effect of neglecting these terms is investigated. In all the fits the appropriate weights defined in the original publications are used.

B. Superfluid density at vapor pressure

The superfluid fraction ρ_s/ρ at vapor pressure can be obtained from the second-sound—velocity measurements[11] using the relation[5]

$$u_2^2 = \left[\frac{S^2T}{C_p}\right]\left[\frac{\rho_s}{\rho_n}\right]\left[1+O\left[\frac{u_2^2}{u_1^2}\right]\right] , \tag{2.8}$$

which is based on two-fluid hydrodynamics.[32] The term in large square brackets may be set equal to unity for all temperatures and pressures of interest to us.[5] We redetermined ρ_s/ρ from u_2 using the entropy S from Ref. 24 and the specific heat C_p from Ref. 9. Following Ref. 8, we fitted it to Eq. (2.3) with

$$k = k_0(1+k_1|t|) \; . \tag{2.9}$$

In addition to k_0, k_1, D, and ζ, we also treated T_λ as an adjustable parameter; however, its adjusted value in this and all subsequent fits differed from the experimental value only by a fraction of a micro-Kelvin The data covered the relatively wide range $|t|<t_{max}$ with $t_{max}=0.05$. The parameter values are given in the first row of Table I. Because the entropy used here differs from that used in Ref. 8 essentially by an additive constant, we obtain a slightly different value for the leading amplitude k_0 in Eq. (2.9), but the other parameters have essentially the same values as in Ref. 8. For the leading exponent we have

$$\zeta = 0.6717 \pm 0.0004 \; . \tag{2.10}$$

TABLE I. Parameter values obtained by fitting ρ_s/ρ data with $|t| \le t_{max}$ to Eqs. (2.3) and (2.9). The numbers in parentheses are the standard errors expressed as a variation in the last quoted digit. When T_λ was fixed at its experimental value, nearly the same parameter values were obtained and all standard errors were somewhat smaller.

t_{max}	k_0	k_1	D_ρ	ζ
0.05	2.403(8)	−1.46(5)	0.33(2)	0.6717(4)
0.03	2.414(10)	−2.24(8)	0.28(3)	0.6722(5)
0.011	2.390(20)	−1.80(30)	0.40(8)	0.6710(10)

There were no obvious systematic deviations from the fit described above. Nonetheless, we explored the effect of greater flexibility in the fitting function by introducing a term k_2t^2 in Eq. (2.9). This resulted in values of ζ, k_0, k_1, and D_ρ which were within the range of their original standard errors, but with their standard errors roughly doubled. The standard error of k_2 was larger than the value of k_2. This procedure yielded $\zeta=0.6719\pm0.0007$.

With regard to the other omitted terms in the truncated expansion (2.3), we note here that the term $k_1|t|$ effectively includes any contribution from terms of $O(|t|^{2\Delta})$ since 2Δ is essentially equal to 1. Thus these higher-order singular terms cannot be distinguished from an analytic dependence of k upon T and are already included in our fit.

As a further test of the validity of the result (2.10), we returned to Eqs. (2.3) and (2.9) and restricted more severely the range of the data. For $t_{max}=0.03$ and 0.011 we obtained the parameters given in Table I. It is apparent that there is no significant systematic variation of the parameters with t_{max}.

On the basis of the above analysis, we favor the result (2.10) as our best estimate of ζ, but caution that the standard error quoted there does not include *systematic* errors from the experiment or due to truncations of the fitting function. These errors are extremely difficult to estimate, but even a generous subjective guess of the total uncertainty would result in a number less than ±0.002. The result is in good agreement with the theoretical estimates for the correlation-length exponent ν which are based on a $d=3$ field theory (ζ is expected[28] to be equal to ν). These estimates are 0.669 ± 0.002 (Ref. 33) and 0.672 ± 0.002 (Ref. 34).

With Eq (2.4), our value for ζ yields $\alpha=-0.015$. This result is somewhat lower than that produced by the original analysis[14] of β_p; however, that analysis did not take analytic temperature dependences of the amplitudes into consideration. Furthermore, the difference is only slightly larger than the sum of the standard errors of α determined both from β_p and from Eqs. (2.10) and (2.4). Our new estimate of α is in very good agreement with the recent measurements of C_p over the range[35] $2\times10^{-7} \lesssim |t| \lesssim 10^{-3}$, both above and below T_λ, by Lipa and Chui,[15] which gave $\alpha=-0.013$ [the standard error of α produced by a fit of these data to Eqs. (2.1) is not known since the authors fixed the values of D_c and D_c' in their analysis]. It also agrees well with analyses[14] of earlier specific-heat measurements at vapor pressure[9] and at pressures up to about 15 bars (Ref. 10) (the specific-heat measurements at high pressures[10] appear to be influenced by systematic errors).

In order to consider possible systematic errors in the value of ζ, we carried out three separate analyses of the ρ_s/ρ and β_p data at each pressure with the three values $\zeta=0.669$, 0.672, and 0.675, corresponding to $\alpha=-0.007$, -0.016, and -0.025. Surely this range includes the exact exponent values. This multiple analysis establishes the influence of the imposed leading-exponent values upon the other universal parameters. None of our main conclusions are found to be sensitive to the choice of ζ, but some of the experimental values for the universal parameters are significantly dependent upon ζ.

C. Thermal-expansion coefficient β_p

The heat capacity C_p along isobars near T_λ is related to $\beta_p \equiv V^{-1}(\partial V/\partial T)_p$ by the thermodynamic relation[5]

$$C_p = VT\left[\frac{\partial P}{\partial T}\right]_\lambda \beta_p + T\left[\frac{\partial S}{\partial T}\right]_t . \qquad (2.11)$$

The parameters VT and $T(\partial S/\partial T)_t$ are less singular than C_p and β_p. For $|t|<0.01$ they are nearly constant, and with vanishing t they approach their finite values at T_λ.[5] Thus C_p is an asymptotically linear function of β_p, and the two variables have the same exponents and amplitude ratios. Specifically, we write

$$\beta_p^+ = (A_\beta/\alpha)t^{-\alpha}(1+D_\beta t^\Delta)+B_\beta \qquad (2.12a)$$

for $t>0$, and

$$\beta_p^- = (A_\beta'/\alpha)|t|^{-\alpha}(1+D_\beta'|t|^\Delta)+B_\beta' \qquad (2.12b)$$

for $t<0$. Equation (2.11) then implies on thermodynamic grounds that

$$A_\beta/A_\beta' = A/A' , \qquad (2.13)$$

and that

$$D_\beta = D_c \text{ and } D_\beta' = D_c' . \qquad (2.14)$$

Thus we will at times refer to the leading amplitude ratio of β_p as A/A' and to the confluent singularity amplitudes as D_c and D_c'.

TABLE II. Thermodynamic λ-line parameters used to convert β_p to C_p.

P (bars)	T_λ (K)	V_λ (cm^3/mol)	$-T_\lambda(\partial P/\partial T)_\lambda$ (bars)	$T_\lambda(\partial S/\partial T)_\lambda$ (J/mol K)
5.05	2.1211	25.84	191.01	13.09
10.09	2.0625	24.78	166.24	9.66
15.24	1.9962	23.94	146.80	7.92
20.10	1.9268	23.28	130.25	6.90
20.18	1.9265	23.28	130.23	6.88
25.24	1.8486	22.70	114.42	6.12
25.28	1.8479	22.70	114.39	6.12
28.76	1.7876	22.32	102.97	5.68
30.05	1.7642	22.19	98.44	5.54

TABLE III. Parameters for fits of thermal-expansion-coefficient data (Ref. 14) to Eq. (2.12) using the constraints $\alpha=\alpha'=-0.016$, $\Delta=\Delta'=0.5$, and $|t|\le 0.01$.

P (bars)	A_β/A_β'	D_β/D_β'	$-10^2A_\beta'$ (K^{-1})	$-D_\beta'$	$-B_\beta$ (K^{-1})
5.05	1.066	0.69	1.143	0.053	0.707
10.09	1.065	0.91	1.375	0.069	0.853
15.24	1.066	0.77	1.541	0.109	0.957
20.10	1.067	0.89	1.842	0.123	1.142
20.18	1.069	1.32	1.811	0.077	1.126
25.24	1.073	2.61	2.195	0.054	1.363
25.28	1.070	1.07	2.213	0.132	1.369
28.76	1.072	1.21	2.602	0.136	1.608
30.05	1.067	0.95	2.973	0.261	1.817

We also note that $B_\beta=B_\beta'$ if $B=B'$, as already implied in Eq. (2.7). We used the parameters in Table II (Refs. 24 and 36) for conversions of β_p to C_p via Eq. (2.11).

The data of each pressure were fitted to Eqs. (2.12) with Δ fixed at 0.5 and for each of the three values $\alpha=-0.025$, -0.016, and -0.007. The adjustable parameters were A_β, A_β', D_β, D_β', and $B_\beta=B_\beta'$. We give explicitly in Table III the results for the case $\alpha=-0.016$ in order to demonstrate the pressure independence of A/A' and D_c/D_c' (for D_c/D_c' at small P, one should consider the small size of D_c and D_c' which leads to relatively large errors in their ratio). The results were similar for the other values of α, and no obvious significant trends with pressure existed for A/A' and D_c/D_c'. Weighted averages of A/A' and D_c/D_c' are given in Table IV. In order to obtain better values of the parameters A_β, D_c, and B_β, the data were reanalyzed with the additional constraint of equating A/A' and D_c/D_c' with the values in Table IV. This yielded the parameters in Table V for the case $\alpha=-0.016$. The results for D_c' are plotted for each value of α in Fig. 1 as a function of pressure. At a given pressure, D_c' depends significantly upon α. The dependence upon α of D_c' is a consequence of using Eq. (2.1), from which it follows that D_c' must be proportional to α for small α if the contribution to C_p from the confluent singularity is to remain finite as α vanishes. The values of A_β and B_β are relatively insensitive to α. In order to facilitate the calculation of quantities involving β_p, the parameters A', D', and B were fitted, separately for each α over the range of pressure 5 to 30 bars, to the polynomials

$$A' = a_0 + a_1P + a_2P^2 + a_3P^3 + a_4P^4 , \qquad (2.15a)$$

TABLE IV. Experimental estimates of universal or pressure-independent parameters.

ν	0.669	0.672	0.675
α	-0.007	-0.016	-0.025
A/A'	1.029	1.068	1.111
$\mathcal{P}$	4.14	4.25	4.42
R_ξ^-	0.86	0.85	0.84
u^*	0.0342	0.0343	0.0341
D_c/D_c'	0.78	1.03	1.37
D_c'/D_ρ	-0.028	-0.068	-0.10
D_c/D_u	0.032	0.080	0.138

TABLE V. Parameters for fits of the thermal-expansion—coefficient data of Ref. 14 to Eq. (2.12) using the constraints $\alpha=\alpha=-0.016$, $\Delta=\Delta'=0.5$, $A_\beta/A'_\beta=1.068$, $D_\beta/D'_\beta=1.03$, and $|t|\leq 0.01$.

P (bars)	$-10^2A'_\beta$ (K^{-1})	$-D'_\beta$	$-B_\beta$ (K^{-1})
5.05	1.118	0.025	0.693
10.09	1.326	0.042	0.827
15.24	1.523	0.077	0.948
20.10	1.829	0.103	1.135
20.18	1.817	0.099	1.128
25.24	2.205	0.132	1.363
25.28	2.252	0.145	1.391
28.76	2.707	0.177	1.664
30.05	2.948	0.235	1.804

$$D'=d_0+d_1P+d_2P^2\ , \qquad (2.15b)$$

and

$$B-A'/\alpha=b_0+b_1P+b_2P^2+b_3P^3+b_4P^4\ .$$

The coefficients a_n, b_n, and d_n are given in Table VI for $\alpha=-0.016$.

It follows from Eqs. (2.11) and (2.12) that the amplitudes per unit volume A'/V of the leadings singularity of C_p below T_λ and the amplitude A'_β of β_p at a given pressure are related by

$$A'/V=T_\lambda\left[\frac{\partial P}{\partial T}\right]_\lambda A'_\beta\ . \qquad (2.16)$$

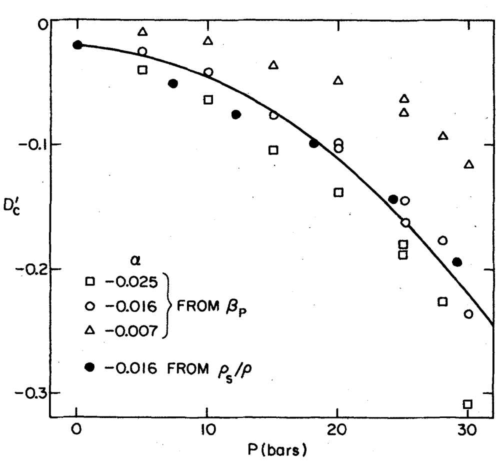

FIG. 1. Amplitude D'_c of the first confluent singularity of C_p^- ($T<T_\lambda$). Open symbols were derived from measurements of the thermal-expansion coefficient using three different values of the leading specific-heat exponent α. Solid circles were obtained from the confluent singularity amplitude D_ρ of the superfluid fraction ρ_s/ρ by multiplying by the universal value $D'_c/D_\rho=-0.068$ given in Table IV; they correspond to $\zeta=0.672$ ($\alpha=-0.016$). To the extent that the solid circles agree with the open circles, the universality of D'_c/D_ρ is established. The line corresponds to Eq. (2.15b) with the parameters in Table VI.

TABLE VI. Values of the coefficients of Eq. (2.15) for the choice $\alpha=-0.016$. The unit of the pressure is bars and β_p is measured in K^{-1}. The table entries are to be multiplied by the power of ten in the first row.

n	0 (10^{-2})	1 (10^{-4})	2 (10^{-5})	3 (10^{-6})	4 (10^{-7})
a_n	−1.0491	0.852 72	−5.8428	2.992	−0.6126
b_n	1.4029	−34.814	38.814	−18.389	3.4004
d_n	−2.10	−5.094	−20.21	0	0

The results for this quantity, obtained with $\alpha=-0.016$, are shown as the solid circles in Fig. 2.

D. Heat capacity at constant pressure, C_p

The heat-capacity of Refs. 9 and 10 were fitted to Eq. (2.1) using the same constraints as for β_p, including the fixed values of A/A' and D/D' derived from β_p (Table III). This analysis was done for all three values of α. The parameters A', D', and B for $\alpha=-0.016$ are given in Table VII. The amplitude of the leading singularity per unit volume A'/V is shown as the open circles in Fig. 2. Also shown in Fig. 2 is A'/V obtained from the reanalysis, using our procedure and constraints, of the heat-capacity data of other authors.[25,26] The agreement between A'/V from all sources is excellent for all pressures up to about 20 bars and still within the error estimates of the various authors even at the higher pressures. The general trend with P of A'/V, and the agreement between different data sets, does not depend upon α.

Note that the agreement between the amplitude A'/V of the leading singularity for the heat capacity as determined from direct measurements and that derived from β_p via Eq. (2.16) does not mean that C_p *itself* derived from β_p [see Eq. (2.11)] agrees with C_p derived from measurements of C_v. In fact, the percent difference between smoothed values of C_p from Ref. 11 for $T<T_\lambda$ and that calculated from Eq. (2.11) varies from near zero at pressures $P\lesssim 15$ bars to several percent at the highest pressures.[37]

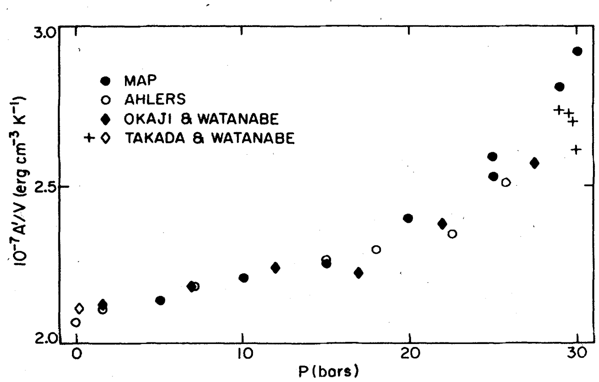

FIG. 2. Amplitude per unit volume A'/V obtained by analyzing data for β_p from Ref. 14 (solid circles) and for C_p from Refs. 9 and 10 (open circles); from Ref. 25 (open diamond and pluses) and from Ref. 26 (solid diamonds).

TABLE VII. Parameters determined by fitting the C_p data of Refs. 9 and 10 to Eq. (2.1) using the constraints $\alpha=\alpha'=-0.016$, $\Delta=\Delta'=0.5$, $A/A'=1.068$, $D_c/D_c'=1.03$, and $|t|\leq 0.01$.

P (bars)	A' ($\mathrm{J\,mol^{-1}\,K^{-1}}$)	$-D_c'$	B ($\mathrm{J\,mol^{-1}\,K^{-1}}$)
0.05	5.594	−0.047	364.1
1.65	5.532	−0.003	359.1
7.33	5.477	0.017	352.2
15.03	5.386	0.068	343.2
18.18	5.363	0.097	340.2
22.53	5.355	0.147	337.7
25.86	5.586	0.150	350.2

E. Superfluid fraction ρ_s/ρ under pressure

In subsection B we obtained ρ_s/ρ at vapor pressure from Eq. (2.8) using the direct measurements of C_p at vapor pressure from Eq. (4.41) of Ref. 10 and the entropy S from Ref. 24. We then fitted ρ_s/ρ to Eqs. (2.3) and (2.9), with the results shown in Table I. In this section we extend this analysis for the case $\alpha=-0.016$ and $\nu=0.672$ to higher pressures, but limit the data to reduced temperatures $|t|\leq 0.01$, where experimental results for S^2T/C_p exist.

For these fits we used ρ_s/ρ data obtained from u_2 with C_p from Eq. (4.41) of Ref. 10 and with S from Ref. 24. The results are listed in Table VIII. The values of k_1 and D_ρ do not differ significantly from those given in Ref. 8, but k_0 is slightly altered (no more than 3%) because we used the improved entropy values. We repeated the above analysis using the more restricted range $|t|\leq 0.003$, where terms of $O(t)$ should be negligible. Over this range, we set $k_1=0$ in Eq. (2.9). The results based on $\nu=0.672$ and $\alpha=-0.016$ are also given in Table VIII. We see that k_0 and D_ρ are not very sensitive to the range of the data.

The C_p data used in the preceding analysis for pressures up to 15 bars are reliable and thermodynamically consistent with the β_p results and Eq. (2.11). At higher pressures there is a significant inconsistency. We therefore wish to reanalyze ρ_s/ρ based on u_2, β_p, and the new entropy measurements. We note that the entropy enters into this problem not only in Eq. (2.8), where it appears explicitly, but also in Eq. (2.11) where its derivative with respect to T_λ is needed. The values of $(\partial S/\partial T)_\lambda$ based on Ref. 24 differ as much as 15% from the older estimates and this change has a significant influence on k_0.[37] We use the fits of subsection C for the analysis based on β_p, but, since the original β_p data (see Fig. 6 of Ref. 14) for $T<T_\lambda$ do not extend all the way to $|t|=0.01$, we restrict the fit to $|t|\leq 0.003$ and set $k_1=0$. The results for all three values of ζ and α are given in Table IX. Those for $\zeta=0.672$ can be compared to the results in Table VIII which are based on C_p. As anticipated, there is a significant difference only at the higher pressures.

F. Correlation and healing-length amplitudes

The universal ratio R_ξ^- to be discussed in Sec. III B is proportional to the correlation-length amplitude ξ_0. We can obtain ξ_0 from ρ_s/ρ using the relation[27]

$$\xi_0=m_4^2k_BT_\lambda/\hbar^2\rho k_0\ , \qquad (2.17)$$

where m_4 is the mass of the ^{4}He atom. With k_0 from Table VIII, based on ρ_s derived from u_2 using C_p, we obtain the values for ξ_0 shown in Fig. 3(a). As noted before,[6] this analysis of the data results in values of ξ_0 that vary no more than 2% with pressure. Alternately, we can use the values of k_0 from Table IX which were obtained from the analysis of ρ_s/ρ derived from u_2 using β_p. These results are shown in Fig. 3(b) as solid circles. In this case, ξ_0 decreases by about 10% with increasing pressure.

We can compare ξ_0 with measurements by Ihas and Pobell[13] of a superfluid healing-length amplitude ξ_0^*. Ihas and Pobell measured the reduced temperature t_0 at which the second-sound amplitude vanished in resonators equipped with superleak transducers with several different channel diameters d. The correlation length ξ for $t=t_0$ is presumed to be equal to a given fraction of the pore diameter d. One can define $\xi_0^*\equiv d/t_0^{-\nu}$ as a healing-length amplitude[38] which is expected to be related to ξ_0 by a pressure-independent multiplicative constant of order unity. We show ξ_0^* multiplied by normalization factors of 0.865 and 0.717 as pluses ($d=0.4$ μm) and crosses ($d=0.6$ μm), respectively, in Fig. 3(b). The agreement

TABLE VIII. Parameters values obtained by fitting ρ_s/ρ data to Eqs. (2.3) and (2.9). The data were derived from (Ref. 11) using the fits to C_p described in Sec. II D and the entropy values from Ref. 24. The values $\zeta=0.672$ and $\Delta=0.5$ were used.

	$t_{max}=0.01$			$t_{max}=0.003$	
P (bars)	k_0	k_1	D_ρ	k_0	D_ρ
0.05	2.408	−1.6	0.32	2.412	0.22
7.27	2.150	−1.7	0.77	2.156	0.62
12.13	1.979	−3.5	1.34	1.985	1.12
18.06	1.851	−3.2	1.76	1.857	1.53
24.10	1.701	−3.5	2.61	1.709	2.32
24.17	1.722	−5.0	2.70	1.732	2.31
29.09	1.585	−3.2	3.57	1.592	3.32

TABLE IX. Parameter values obtained by fitting ρ_s/ρ data to Eqs. (2.3) and (2.9), with k_1 set equal to 0 and for $t \leq 0.003$. The data were derived from the second-sound velocity u_2 (Ref. 11) using the β_p measurements (Ref. 14) analyzed in Sec. II C. Entropies from Ref. 24 and the value $\Delta=0.5$ were used.

		ζ			ζ	
P	0.669	0.672	0.675	0.669	0.672	0.675
(bars)		k_0			D_p	
7.27	2.096	2.136	2.190	0.77	0.71	0.74
12.13	1.961	1.990	2.048	1.24	1.06	1.03
18.06	1.852	1.901	1.956	1.60	1.39	1.24
24.10	1.733	1.796	1.848	2.28	2.02	1.72
24.17	1.757	1.822	1.874	2.25	2.00	1.70
29.09	1.688	1.729	1.796	3.38	2.7	2.23

with ξ_0 is quite good and well within the experimental uncertainties. The dashed line in Fig. 3(b) is used as a guide in determining ξ_0 at various pressures for use in Sec. III B.

III. UNIVERSALITY

A. Specific-heat amplitudes

The analysis of Sec. II C showed that the thermal-expansion—coefficient measurements[14] lead, within experimental error, to pressure-independent values of A/A' and D/D'. This is apparent from the data in Table III, which is for $\alpha=-0.016$. The other values of α yield similar results. The universal values of the amplitude ratios depend somewhat upon the value of α used in the analysis, as seen in the third and seventh rows of Table IV. A parameter which is relatively insensitive to the value of α is the ratio

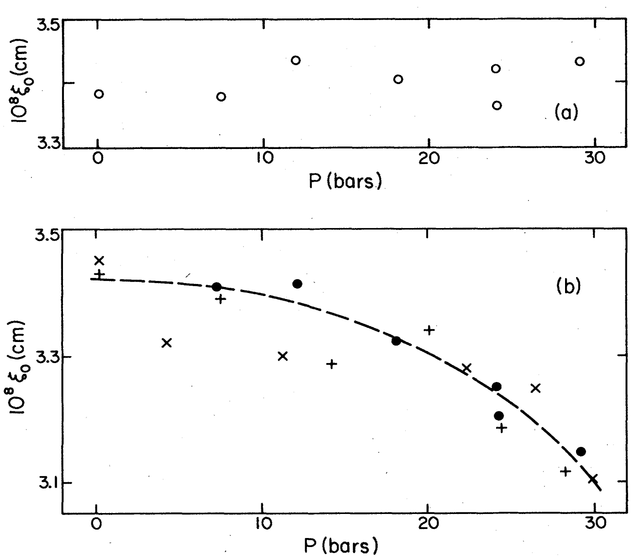

FIG. 3. Correlation-length amplitude ξ_0 from Eq. (2.17) derived using (a) C_p data from Ref. 10 (open circles) and (b) β_p from Ref. 14 (solid circles) in the analysis of ρ_s/ρ. Also shown in (b) as crosses and pluses is ξ_0^* derived from data of Ref. 13 and defined in the text.

$$\mathscr{P}=(1-A/A')/\alpha \ , \tag{3.1}$$

which was introduced by Barmatz, Hohenberg, and Kornblit[39] and has been examined for various experimental systems.[40] We find $4.1 \lesssim \mathscr{P} \lesssim 4.4$ as shown in the fourth row of Table IV.

Ratios of amplitudes of confluent singularities such as D/D' have been calculated from the RGT,[41–45] but not with high accuracy. For D/D' these calculations yield values close to 1, roughly consistent with the results in Table IV. The best theoretical value[41] is 1.17, but its uncertainty is difficult to estimate.

B. Two-scale-factor universality

Another experimentally accessible parameter that is expected to be universal is the singular contribution to the free energy, normalized by $k_B T_\lambda$, of an amount of fluid contained in a volume equal to the cube of the correlation length ξ for spatial fluctuations of the order parameter.[23] Since the singular part of the free energy is determined by the specific heat, one can define this universal parameter as

$$(R_\xi^-)^3 \equiv \alpha' t^2 (C_p^{\text{sing}}/k_B)\xi^3 \ , \tag{3.2}$$

where

$$C_p^{\text{sing}}=(A'/\alpha' V)\,|\,t\,|^{-\alpha'} \ .$$

Using the scaling law $3\nu=2-\alpha$, one obtains

$$(R_\xi^-)^3=(A'/Vk_B)\xi_0^3 \tag{3.3}$$

independent of t. The specific-heat amplitude A'/V is shown in Fig. 2 for several experimental measurements of C_p (Refs. 9, 10, 25, and 26) and for the β_p (Ref. 14) measurements. In the case of β_p, A'/V is obtained using Eq. (2.11) with parameters from Table II. The correlation-length amplitude ξ_0 derived from ρ_s/ρ is shown in Fig. 3 for the two different analyses involving C_p and β_p described in Sec. II E. In Fig. 4(a) we plot $(R_\xi^-)^3$ with ξ_0 taken from Fig. 3(a) and A'/V from Fig. 2. We see that $(R_\xi^-)^3$ is larger by almost 50% at 30 bars than its low-pressure value. This pressure dependence is, of course, essentially the same as that of A'/V in Fig. 2, since ξ_0

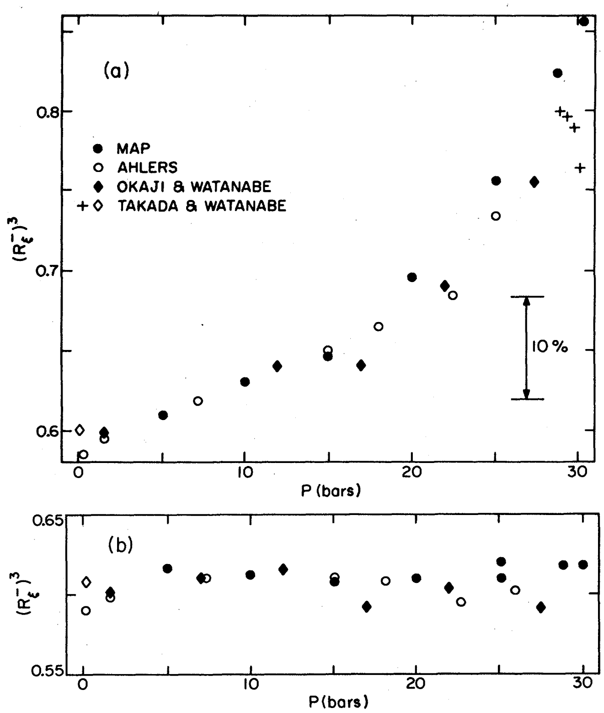

FIG. 4. Parameter $(R_\xi^-)^3$ derived from Eq. (3.3) using A'/V from Fig. 2, and (a) ξ_0 from Fig. 3(a) and (b) ξ_0 from Fig. 3(b).

taken from Fig. 3(a) is nearly independent of pressure. If we instead use ξ_0 from Fig. 3(b) in Eq. (3.3) together with A'/V from Fig. 2, we obtain the results shown in Fig. 4(b). We see that $(R_\xi^-)^3$ is constant to within about 5% over the entire range of pressures from 0 to 30 bars, in excellent agreement with the predictions of two-scale-factor universality.[23] The value calculated for R_ξ^- using the RGT (Ref. 46) is equal to 0.96 and is accurate to perhaps 20%. This is to be compared to the average experimental value determined here of 0.85. Thus the experimental R_ξ^- is consistent with the theoretical prediction both in its pressure independence and its numerical value. Analyses with $\alpha=-0.007$ and -0.025 give similar results. As can be seen in the fifth row of Table IV, the value of R_ξ^- is not very sensitive to the choice for α.

C. Amplitude ratio D_c'/D_ρ

Another parameter that is predicted to be universal[2] is the amplitude ratio D_c'/D_ρ, where D_c' is the amplitude of the confluent singular term for either β_p or C_p, and D_ρ is the amplitude of the confluent singularity for ρ_s/ρ [see Eqs. (2.1) and (2.3)]. We use interpolated values of D_ρ obtained from graphs of the results in Table IX and the value of D_c' from Table V (or their equivalent for other values of α). The average values of D_c'/D_ρ are given in Table IV for the three values of α. The result is strongly dependent upon the choice for α, because D_c' depends strongly upon α (see Fig. 1), whereas D_ρ does not. Regardless of the choice for α, the ratio D_c'/D_ρ is independent of pressure within the experimental resolution. This is illustrated in Fig. 1 for $\alpha=-0.016$ where the results for D_ρ, multiplied by the value of D_c'/D_ρ in Table IV, are shown as solid circles. They are consistent with the values of D_c' derived from β_p if one considers the low accuracy of the result for D_c' (reasonable guesses at the uncertainty of D_c' would yield about ± 0.02).

It is interesting to note that D_c'/D_ρ is quite small and not of $O(1)$. This is a consequence of using Eq. (2.1), which for small α implies that D_c' is proportional to α (see Sec. II C). We find that $D_c'/D_\rho \simeq 4\alpha$ regardless of the value used for α in the analysis.

The best theoretical value[44] for D_c'/D_ρ is $\frac{1}{6}$. Although roughly of the right magnitude, it is positive, whereas the experimental value is negative. The source of this discrepancy is not known at present.

D. Ratio R_D

Recently, Bagnuls and Bervillier[21] derived an expression for the heat capacity above T_c for systems of spin dimensionality $n=2$ from high-order perturbation theory[33] in dimension $d=3$ which has the form[47]

$$C_p^+(t)=g_0^3\theta^2 C_{\text{th}}(\tau)+C_{\text{reg}} , \tag{3.4}$$

where $C_{\text{th}}(\tau)$ is obtained from a fit of numerical theoretical results to the empirical function

$$C_{\text{th}}(\tau)=X_1\tau^{-\alpha}(1+X_2\tau^{\Delta})^{X_3}(1+X_4\tau^{\Delta})^{X_5}+X_6 . \tag{3.5}$$

Here $\tau=\theta t$, g_0 and θ are two nonuniversal scale factors, and the X_i are pure numbers quoted in Ref. 21. There is an analytical background contribution C_{reg} to the heat capacity of the physical system which, in addition to θ, g_0, and T_λ, must be determined from the experimental data. Defining $B_{\text{cr}}=B-C_{\text{reg}}$ and comparing Eqs. (3.4) and (3.5) to Eq. (2.1), Bagnuls and Bervillier note that the combination $A(-D_c)^{\alpha/\Delta}/(\alpha B_{\text{cr}})$ does not depend on the system-specific parameters g_0, θ, or C_{reg}, and therefore suggest that it should be pressure independent for ^{4}He. Specifically, we define

$$R_D \equiv -A(-D_c)^{\alpha/\Delta}/(\alpha B_{\text{cr}}) . \tag{3.6}$$

It follows[21] that

$$R_D=-X_1(-X_2X_3-X_4X_5)^{\alpha/\Delta}/X_6 . \tag{3.7}$$

The theoretical estimates[21] of X_i give

$$1.076 \geq R_D \geq 1.035 . \tag{3.8}$$

Experimental estimates of R_D based on Eq. (3.6) are given in Fig. 5 as a function of pressure for our three values of α. Except for the solid triangles, the data are based on the thermal-expansion parameters from Table V and its equivalent for the other values of α and on Table II. The solid triangles (for $\alpha=-0.016$ only) are from the heat-capacity parameters in Table VII, except that D was based on Eq. (2.15b) with the coefficients in Table VI. The results are not very sensitive to the exact value of the background term C_{reg}, and we simply used the value of C_v at its minimum above T_λ.[48] The theoretical result (3.8) is shown as the vertical bar near the lower left-hand corner of Fig. 5.

The difference between the data in Fig. 5 and the prediction seems to be only a few percent, and thus the agree-

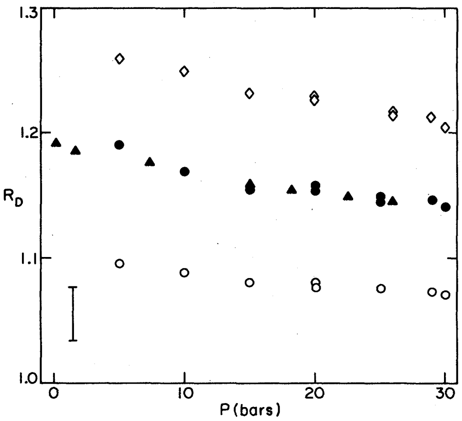

FIG. 5. Experimental values of the amplitude combination R_D given by Eq. (3.6). To obtain A, D, and B from β_p, we used Tables II and V (or their equivalent for other values of α) with Eq. (2.11). We set $B_{cr}=B-C_{reg}$ with C_{reg} equal to the minimum in C_p for $T>T_\lambda$. Open diamonds are based on β_p and $\alpha=-0.025$. Solid circles are based on β_p and $\alpha=-0.016$. Open circles are based on β_p and $\alpha=-0.007$. Solid triangles are based on C_p (Table VII) and $\alpha=-0.016$. Vertical bar near the left-hand ordinate indicates the theoretical prediction Eq. (3.8).

ment might be considered quite good. However, this is illusory and due to the high correlation between α and B_{cr}. This can be seen by writing Eq. (3.6) in terms of the parameters of the function

$$C_p^+=(A/\alpha)(t^{-\alpha}-1)+(A/\alpha)D_c t^{\Delta-\alpha}+\widetilde{B}_{cr}+C_{reg}\ , \quad (3.9)$$

which in the limit of α passing through zero has smoothly varying nondivergent coefficients (note that B_{cr} diverges as $\alpha\to 0^-$). We obtain

$$R_D=[1-(\widetilde{B}_{cr}/A)\alpha]^{-1}(-D_c)^{\alpha/\Delta}\ . \quad (3.10)$$

We observe that $(-D_c)^{\alpha/\Delta}$ will be very close to unity for almost any nonvanishing $|D_c|$ because α is so small. Furthermore, although $\widetilde{B}_{cr}$ is negative, it is of the same magnitude as A,[49] so that $-\widetilde{B}_{cr}/A$ is a number of order 1. Thus it follows that R_D will differ from unity only by an amount comparable to the size of α no matter what parameters (within reason) are use to compute it. We should therefore demand "good" agreement for, say, R_D-1, rather than for R_D itself. At that level the comparison between experiment and theory in Fig. 5 is not gratifying. We note that, as expected from Eq. (3.10), the results for R_D-1 vary roughly in proportion to α. The discrepancy between theory and experiment reflects the difference in the estimates for α. Whereas the theory yields[21] $-0.009\lesssim\alpha\lesssim-0.004$, the experiment favors more negative values. We note that the experimental data in Fig. 5 suggest a slight pressure dependence, although it is difficult to rule out systematic errors in the analysis as the cause.

An interesting qualitative feature of Eq. (3.6) is that a pressure-independent R_D requires D_c to have the same sign at all pressures. Indeed, the data in Fig. 1 are all negative and thus consistent with this aspect of Eq. (3.6). However, the results for D_c' in Table VII, based on the analysis of C_p, change sign for P near 2 bars. The positive values at the low pressures are not permitted by the high-pressure data and Eq. (3.6) since D_c'/D_c must be universal. The problem is associated with the specific choice for A/A' and D_c/D_c'. The values of D_c are quite sensitive to these parameters, and a slight adjustment could result in negative values for D_c at all pressures.

E. Effective coupling of the Landau-Ginzburg-Wilson Hamiltonian

The behavior away from the fixed point of the renormalized coupling constant $u(t)$ of the Landau-Ginzburg-Wilson Hamiltonian plays a role in the comparison of the second-sound damping[50] D_2 with the predictions of the dynamic renormalization-group theory.[19,51] Its dependence upon t influences the size of confluent singular contributions to D_2. A relation between the specific heat and $u(t)$ was recently derived by Dohm[22] and has the form

$$\frac{R_0(t)-1}{G(u(t))-1}=\frac{\alpha_e(t)}{Q(u(t))}\ . \quad (3.11)$$

Here,

$$R_0(t)\equiv C_p^-(\tfrac{1}{2}|t|)/C_p^+(t) \quad (3.12)$$

and

$$\alpha_e(t)\equiv-[t/C_p^+(t)]\left[\frac{\partial C_p^+(t)}{\partial t}\right]_p\ . \quad (3.13)$$

The functions $G(u)$ and $Q(u)$ have been obtained[22] from renormalized perturbation theory, albeit so far only to one-loop order. They are

$$G(u(t))=-\frac{1}{2nu(t)}+\frac{4}{n}+O(u(t)) \quad (3.14)$$

and

$$Q(u(t))=2-5\widetilde{\nu}(t)+O(u(t))\ , \quad (3.15)$$

where n is the spin dimensionality.

For $t=0$, the function $\widetilde{\nu}(t)$ is equal to the correlation-length exponent ν. For $|t|>0$, Dohm[52,22] writes $\widetilde{\nu}=[2-\zeta_r(t)]^{-1}$ with ζ_r approximated by[52]

$$\zeta_r(t)=(n+2)\{4u(t)-40[u(t)]^2\}+\{2-\nu^{-1}-(n+2)[4u^*-40(u^*)^2]\}[u(t)/u^*]^3\ . \quad (3.16)$$

Here, u^* is the value of u for $t=0$. We shall use this expression with ν equal to the experimental value of ζ given by the ρ_s/ρ analysis.

With Eqs. (3.14) and (3.15), Eq. (3.11) yields

$$u(t)=\{8-2n-2n[2-5\widetilde{\nu}(t)][R_0(t)-1]/\alpha_e(t)\}^{-1} \quad (3.17)$$

for the relationship between $u(t)$ and $C_p^\pm(t)$. Although Eq. (3.17), together with Eqs. (3.12) and (3.13), can be evaluated numerically[22] on the basis of experimental data for $C_p^\pm$ or $\beta_p^\pm$, it is instructive to evaluate R_0 and α_e

analytically from their definitions [Eqs. (3.12) and (3.13)] using Eq. (2.1). Substituting into Eq. (3.17) for $u(t)$, one obtains

$$u(t)=\{8-2n-2n(1/\alpha)(2-5\tilde{\nu})\times[2^{\alpha}A'/A-1+f_1(t)]/[1+f_2(t)]\}^{-1}, \quad (3.18)$$

with

$$f_1(t)=D_c(2^{\alpha-\Delta}A'D_c'/AD_c-1)t^{\Delta} \quad (3.19a)$$

and

$$f_2(t)=D_c(1-\Delta/\alpha)t^{\Delta}. \quad (3.19b)$$

We note that, as t vanishes Eq. (3.18) yields

$$u^*=[8-2n-2n(2-5\nu)(1/\alpha)(2^{\alpha}A'/A-1)]^{-1}. \quad (3.20)$$

Thus, *in the one-loop approximation* of the renormalization scheme used by Dohm, u^* is given entirely in terms of the spin dimensionality and the associated universal quantities α, ν, and A'/A. This explains why the experimental data for the expansion coefficient β_p near the λ line in liquid helium, when fitted to functions with a pressure-independent α and A/A', will also yield a pressure-independent u^* as observed by Dohm.[52] For $n=1$, 2, and 3 one obtains for u^* the values given in Table X from the representative experimental values of α and A/A' given there. The values of u^* corresponding to our analyses are given in Table IV.

Equations (3.17) or (3.18), in conjunction with typical experimental values of the specific heat, yields a significant dependence of u upon t, especially at the higher pressures.[52] Therefore, it is instructive to expand Eq. (3.18) to lowest order in t^{Δ}. This yields

$$u(t)=u^*[1+D_u t^{\Delta}+O(t^{2\Delta})], \quad (3.21)$$

with

$$D_u=\frac{2n}{\alpha}(2-5\nu)\left[\left(\frac{2^{\alpha}A'}{A}-1\right)\left(\frac{\Delta}{\alpha}-1-\frac{5\nu}{2-5\nu}\frac{D_{\tilde{\nu}}}{D_c}\right)+\frac{A'D'2^{\alpha-\Delta}}{AD}-1\right]u^*D_c, \quad (3.22)$$

where we wrote $\tilde{\nu}=\nu(1+D_{\tilde{\nu}}t^{\Delta})$. The universal ratio $D_{\tilde{\nu}}/D_c$ can be obtained in terms of D_u/D_c from Eq. (3.16), which yields

$$D_{\tilde{\nu}}=[3\nu(2-\nu^{-1})+8\nu u^*(n+2)(5u^*-1)]D_u. \quad (3.23)$$

Equations (3.22) and (3.23) yield D_c/D_u. It follows that, to this order of perturbation theory, the confluent singularity amplitude ratio D_c/D_u depends only on universal quantities, namely ν, α, Δ, A/A', and D/D', for a given n. Values for $n=1$, 2, and 3 are given in Table X. Values of D_c/D_u corresponding to our analyses of the ^{4}He data are given in Table IV. Our ^{4}He results for D_c/D_u are quite small because D_c is small. The smallness of D_c already manifested itself in the smallness of the ratio D_c'/D_ρ. Typical values of D_c for the λ line are near -0.1 and are sensitive to the exact value of α, whereas typically $D_u\approx-1$ and $D_\rho\approx1$, roughly independent of α. The results for D_c/D_u in Table IV and X are numerically close to -5α.

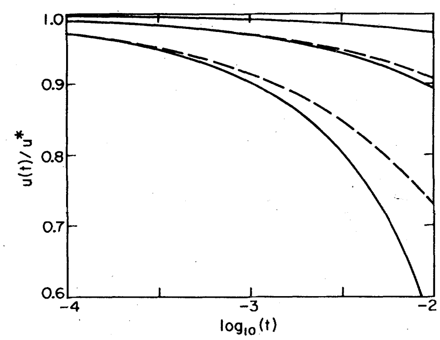

FIG. 6. Temperature dependence of the Landau-Ginzburg-Wilson effective coupling constant based on a one-loop calculation (Ref. 22) and the thermal-expansion measurements (Ref. 14). Solid lines, Eq. (3.18) to Eq. (3.19); dashed lines, Eqs. (3.21) to (3.23). We used $\alpha=-0.016$, $A/A'=1.068$, $D_c/D_c'=1.03$, and D_c' as given by Eq. (2.15b) and Table VI. The three pairs of lines, from top to bottom, are for 0, 15, and 30 bars, respectively.

In Fig. 6 we show as solid lines u/u^* calculated from Eqs. (3.16), (3.18) and (3.19) as a function of t for ^{4}He at 0, 15, and 30 bars. We used $\alpha=-0.016$ and the corresponding values of ν, A/A', and D/D' from Table IV. For D' we used Eq. (2.15b) with the parameters in Table VI. The dashed lines are the corresponding result based on Eqs. (3.21)–(3.23), which neglect terms of $O(t^{2\Delta})$.

The results for u^* given in Table IV are in excellent agreement with the value $u^*=0.0363$ derived[22] from a high-order ϵ expansion for $d=3$.[53] The agreement is remarkable because the results (3.14) and (3.15), on which the determination of u^* is based, are valid only to one-loop order. Presumably it is attributable to the use of experimental values for the universal quantities on the right-hand side of Eq. (3.20). It would be interesting to know if the dependence upon t of $u(t)$ [Eqs. (3.18)–(3.19) or (3.21)–(3.23)] is given with similar accuracy.

IV. SUMMARY

We have used recent accurate entropy measurements[24] in a reanalysis of a number of static properties near the

TABLE X. Reasonable estimates for α, ν, and A/A' for various spin dimensionalities and the corresponding values of u^* given by Eq. (3.20) and D_c/D_u given by Eq. (3.22). The scaling law $3\nu=2-\alpha$ and the relation $\mathcal{P}\equiv(1-A/A')/\alpha=4.2$ were assumed in order to obtain ν and A/A' from α. The values for α and $\mathcal{P}=4.2$ are typical experimental results. For D_c/D_c' we used 1.0.

n	α	ν	A/A'	u^*	D_c/D_u
1	0.11	0.630	0.538	0.0370	−0.55
2	−0.016	0.672	1.067	0.0347	0.080
3	−0.14	0.713	1.588	0.0325	0.70

superfluid transition line $T_\lambda(P)$ in ^{4}He. The entropy data were needed to obtain the superfluid fraction ρ_s/ρ from measurements of the second-sound velocity u_2 and to derive the heat capacity at constant pressure C_p from the isobaric thermal-expansion coefficient β_p. The heat capacity is needed not only for its own sake but also in obtaining ρ_s/ρ from u_2.

A fit of the data to appropriate functions, which is necessary for comparison with theory, requires the adjustment of many parameters and would lead to very large uncertainties if carried out independent of theory. Our approach has been to impose constraints derived from theory upon the critical exponents and to compare the values of the remaining free parameters with additional theoretical predictions, pertaining primarily to the amplitudes, that have not yet been used in the analysis. Specifically, we used the following theoretical predictions as constraints:

(1) We assumed the validity of the Josephson scaling law (2.4) and derived the specific-heat exponent from the superfluid density exponent ζ.

(2) We assumed the validity of the scaling law $\alpha=\alpha'$.

(3) We assumed all static properties to have the same exponent Δ for the leading confluent singularity and took the value of Δ from earlier analyses of data and from theoretical predictions to be equal to 0.5.

(4) We assumed all exponent values to be universal (i.e., independent of pressure).

Our strategy in utilizing the above constraints has been as follows. The superfluid-density exponent ζ was determined from the data at vapor pressure where the precision and accuracy is largest. Next, we carried out complete analyses for three values of ζ (and the corresponding values of α) which cover a range considerably wider than any possible systematic errors in ζ would suggest. We feel that the true values of the leading exponents are surely between the largest and smallest values used here. From an analysis of β_p at several pressures, we established the consistency of the data with the universality of the specific-heat—amplitude ratios A/A' and D_c/D_c', regardless of the values of α used in the analysis. Thereafter all data were analyzed once more assuming A/A' and D_c/D_c' to be universal. This additional constraint increased the precision of the amplitudes themselves and produced results of fair precision even for D_c' which is otherwise hard to determine. The ρ_s/ρ data were also analyzed at several pressures.

We used measurements of specific heat,[9,10,25,26] thermal expansion[14] and second-sound velocity,[11] typically over the range $-0.01 \lesssim t \lesssim 0.01$, in the above analysis. Our results, contingent upon the validity of constraints (1)—(4) above, are in good agreement with most theoretical predictions. Our best estimates of the leading exponents are

$$\zeta=0.6717, \quad \alpha=-0.015 .$$

Statistical errors are very small, but systematic errors may be larger and are difficult to estimate. Nonetheless, it is unlikely that the true values of ζ differ from the above result by more than ± 0.001. This corresponds to an uncertainty in α of ± 0.003. The result agrees well with the high-order perturbation calculation based on a $d=3$ field theory which yields[33] $\nu=0.669\pm0.002$ and $\alpha=-0.007 \pm 0.006$, or[34] $\nu=0.672\pm0.002$ and $\alpha=-0.016\pm0.006$, depending on the detail of the series resummation techniques. It does not agree within the estimated uncertainties with the result $\nu=0.665\pm0.001$ and $\alpha=0.005\pm0.003$ which was obtained recently[53] from an expansion to fifth order in $\epsilon=4-d$. Our experimental exponent values agree well with recent new specific-heat measurements by Lipa and Chui[15] which yielded $\alpha=-0.013$.

All experimentally accessible amplitude combinations that are expected on the basis of theory to be universal or pressure independent were indeed found to be independent of P within experimental resolution. Values of the amplitude combinations are summarized in Table IV. Some of them are sensitive to the choice for the leading exponents, while others are not. Most values are consistent with theoretical predictions, but most of them are not yet known with high accuracy from theory.

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