Sum rules and the momentum distribution in Bose-condensed systems

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The momentum distribution $\tilde{n}_p$ of atoms in a Bose-condensed system is computed in the long-wavelength limit $p \to 0$. We use frequency-moment sum rules for the single-particle Green's function, including a generalized version of Wagner's sum rule appropriate to hard-core potentials. We show that at finite temperatures ($cp \ll k_B T$) the correction to $\tilde{n}_p = n_0 m^2/\rho_S p^2$ involves the off-diagonal self-energy $\Sigma_{+-}(\vec{p}, \omega=0)$. In calculating $\tilde{n}_p$ in the limit $cp \gg k_B T$, we include first-sound as well as second-sound contributions.

I. INTRODUCTION

One of the few rigorous results we have for an interacting Bose-condensed system is that at zero frequency, the difference between the diagonal and off-diagonal single-particle Green's functions is given by

$$\lim_{p\to 0}[G_{++}(\vec{p},\omega=0)-G_{+-}(\vec{p},\omega=0)]=-\frac{2n_0 m}{\rho_S p^2}+\cdots . \tag{1}$$

Here $n_0 \equiv |\langle\Psi\rangle|^2$ is the condensate fraction involving the square of the macroscopic wave function $\langle\Psi\rangle$, and ρ_S is the two-fluid model superfluid density. This result is valid at arbitrary temperatures below T_λ. This long-wavelength expression was first derived by Bogoliubov[1] and by Hohenberg and Martin.[2] We shall refer to it as the "$1/p^2$ sum rule."

A stronger version of this sum rule is often used, namely,[1-6]

$$\lim_{p\to 0} G_{++}(\vec{p},\omega=0)=-G_{+-}(\vec{p},\omega=0)=-\frac{n_0 m^2}{\rho_S p^2}+\cdots . \tag{2}$$

One may derive (2) by a simple physical argument[4] based on the phase fluctuations associated with the normal modes of oscillation (phonons) in a Bose-condensed system. It is argued that the neglected amplitude fluctuations will give rise to less divergent contributions. We are not aware of a rigorous proof of (2). In contrast, (1) can be derived rigorously[1-3,7] as a direct consequence of the continuity equation in a Bose system and the next term on the right-hand side (rhs) shown to be independent of p.

In this paper we give a more formal derivation of the sum rule (2) and discuss the related momentum distribution $\tilde{n}_p$ of excited atoms in the $p \to 0$ limit. We are able to show with some rigor that for $cp \ll k_B T$,

$$\tilde{n}_p=-k_B T G_{++}(\vec{p},\omega=0)=\frac{k_B T m^2 n_0}{\rho_S p^2}+\frac{k_B T}{4\Sigma_{+-}(\vec{p},\omega=0)}+\cdots , \tag{3}$$

where $\Sigma_{+-}(\vec{p},\omega)$ is the off-diagonal self-energy. The remaining nondivergent terms are independent of the momentum $\vec{p}$ in the $p \to 0$ limit. The second term on the rhs of (3) is of the some interest because it offers the hope that one can obtain information about the momentum dependence of $\Sigma_{+-}(\vec{p},\omega=0)$ by measurements of $\tilde{n}_p$.[8] General arguments have been given[9] that $\Sigma_{+-}(\vec{0},0)=0$, but these have been criticized.[10] Moreover, within the one-loop approximation[7] which is consistent with (1), we show in Sec. IV that $\Sigma_{+-}(0,0)$ is finite.

We also point out that one can obtain the leading order divergent contribution to $\tilde{n}_p$ at all temperatures outside the critical region by using the well-known expression for $G_{++}(\vec{p},\omega)$ given by the two-fluid model.[1,2] This automatically includes both first and second sound, the latter being important at finite temperatures. At $T=0$ K, our expression reduces to (c is the sound velocity)

$$\tilde{n}_p=\frac{n_0 mc}{2np}+\cdots , \tag{4}$$

in agreement with the exact result of Gavoret and Nozières.[11]

As emphasized by Ferrell *et al.*,[4] sum rules such as (2) break down in the critical region when $p\xi(T) \gtrsim 1$ [$\xi(T)$ is the correlation length which becomes infinite at T_λ]. We shall not be concerned with this critical region in this paper.

II. EXACT FREQUENCY MOMENTS OF $G_{++}(\vec{p},\omega)$

We recall that the diagonal single-particle Green's function is given by[12]

$$G_{++}(\vec{p},\omega)=\frac{\omega+\epsilon_p-\mu+\Sigma_{++}(\vec{p},-\omega)}{D(\vec{p},\omega)} , \tag{5a}$$

where

$$D(\vec{p},\omega)\equiv(\omega-\tilde{A})^2-(\epsilon_p-\mu+\tilde{S}-\Sigma_{+-})(\epsilon_p-\mu+\tilde{S}+\Sigma_{+-}) . \tag{5b}$$

Here $\epsilon_p = p^2/2m$ is the free-particle energy, μ is the chemical potential, and

$$\tilde{A} \equiv \tfrac{1}{2}[\Sigma_{++}(\vec{p},\omega) - \Sigma_{++}(\vec{p},-\omega)] ,$$
$$\tilde{S} \equiv \tfrac{1}{2}[\Sigma_{++}(\vec{p},\omega) + \Sigma_{++}(\vec{p},-\omega)] . \tag{5c}$$

In terms of the single-particle diagonal spectral density $A(\vec{p},\omega) = 2\,\mathrm{Im}G_{++}(\vec{p},\omega - i0^+)$, one has

$$G_{++}(\vec{p},\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{A(\vec{p},\omega')}{\omega - \omega'} . \tag{6}$$

At zero frequency, we have

$$G_{++}(\vec{p},\omega=0) = -\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{A(\vec{p},\omega')}{\omega'} = \frac{\epsilon_p - \mu + \Sigma_{++}(\vec{p},\omega=0)}{D(\vec{p},\omega=0)} , \tag{7}$$

where

$$D(\vec{p},\omega=0) = -[\epsilon_p - \mu + \Sigma_{++}(\vec{p},0) - \Sigma_{+-}(\vec{p},0)] \times [\epsilon_p - \mu + \Sigma_{++}(\vec{p},0) + \Sigma_{+-}(\vec{p},0)] . \tag{8}$$

Recently Talbot and Griffin[7] proved that ($\rho_S \equiv m n_S$)

$$\Sigma_{++}(\vec{p},0) - \Sigma_{+-}(\vec{p},0) = \mu + \left[\frac{n_S}{n_0} - 1\right]\epsilon_p + O(p^4) , \tag{9}$$

which generalized the $T=0$ K result of Gavoret and Nozières.[11] Using this in (8), we obtain after a little rearranging

$$G_{++}(\vec{p},\omega=0) = -\frac{n_0}{n_S \epsilon_p}\left[1 - \frac{\Sigma_{+-}(\vec{p},0)}{2\Sigma_{+-}(\vec{p},0) + (n_S/n_0)\epsilon_p}\right] . \tag{10}$$

If we assume that

$$2\Sigma_{+-}(\vec{p},0) \gg \frac{n_S p^2}{n_0 2m} \tag{11}$$

in the $\vec{p} \to 0$ limit, we can expand (10) to obtain a generalization of the $1/p^2$ sum rule, namely,

$$\lim_{p\to 0} G_{++}(\vec{p},\omega=0) = -\frac{n_0 m}{n_S p^2} - \frac{1}{4\Sigma_{+-}(\vec{p},\omega=0)} + \cdots . \tag{12}$$

All remaining terms are nondivergent in the $p \to 0$ limit. The condition (11) will be satisfied if $\Sigma_{+-}(\vec{p},\omega=0)$ does not vanish faster than p^2.

We note that a similar analysis for the off-diagonal (anomalous) Green's function gives

$$\lim_{p\to 0} G_{+-}(\vec{p},\omega=0) = \frac{n_0 m}{n_S p^2} - \frac{1}{4\Sigma_{+-}(\vec{p},0)} + \cdots . \tag{13}$$

Combining this with (12) gives the sum rule in (1). Unfortunately the $\Sigma_{+-}(\vec{p},\omega=0)$ terms cancel out in (1) so that it does not contain any information about the momentum dependence of $\Sigma_{+-}(\vec{p},\omega=0)$, in contrast to (12) and (13).

Summarizing the preceding analysis, we have given the first satisfactory proof of the results in (2), a derivation which makes clear that the inequality (11) must be satisfied.

Taking the *high*-frequency limit of (6), we obtain

$$\lim_{\omega\to\infty} G_{++}(\vec{p},\omega) = \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} A(\vec{p},\omega') + \frac{1}{\omega^2}\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \omega' A(\vec{p},\omega') + \cdots . \tag{14}$$

Comparing this with the high-frequency limit of (5b), using $D(\vec{p},\omega\to-\infty) = \omega^2$, we obtain the additional sum rules

$$\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} A(\vec{p},\omega') = 1 , \tag{15}$$

$$\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \omega' A(\vec{p},\omega') = \epsilon_p + \Sigma_{++}(\vec{p},\omega=-\infty) - \mu . \tag{16}$$

The first frequency moment (16) was derived by Wagner[13] some years ago for interatomic potentials without a hard core. His derivation can be easily generalized. The only terms which contribute to the self-energy $\Sigma_{++}(\vec{p},\omega)$ in the high-frequency limit are those given by the t-matrix approximation to multiple scattering. Making use of standard discussions of multiple scattering,[12] one finds that the high-frequency limit simplifies to

$$\Sigma_{++}(\vec{p},\omega\to-\infty) = \frac{n_0}{m}\tilde{f}(\tfrac{1}{2}\vec{p},\tfrac{1}{2}\vec{p}) + \int \frac{d\vec{k}}{(2\pi)^3}\frac{\tilde{n}_k}{m}\tilde{f}(\tfrac{1}{2}(\vec{p}-\vec{k}),\tfrac{1}{2}(\vec{p}-\vec{k})) + \frac{n_0}{m}\tilde{f}(-\tfrac{1}{2}\vec{p},\tfrac{1}{2}\vec{p}) + \int \frac{d\vec{k}}{(2\pi)^3}\frac{\tilde{n}_k}{m}\tilde{f}(-\tfrac{1}{2}(\vec{p}-\vec{k}),\tfrac{1}{2}(\vec{p}-\vec{k})) . \tag{17}$$

Here $\tilde{n}_k$ is the momentum distribution of excited atoms and $\tilde{f}(\vec{p},\vec{p}\,')$ is the scattering amplitude for two atoms given by the integral equation (11.14) of Ref. 12. For $p,p' \to 0$, one has $\tilde{f}(\vec{p},\vec{p}\,') = 4\pi a$, where a is the s-wave phase shift. For a given potential, solving for $\tilde{f}(\vec{p},\vec{p}\,')$ reduces to a numerical problem. If one is dealing with a soft-core interatomic potential with a Fourier transform, the lowest order approximation is $\tilde{f}(\vec{p},\vec{p}\,') = mV(|\vec{p}-\vec{p}\,'|)$. Using this in (17), we obtain the Hartree-Fock (HF) approximation

$$\Sigma_{++}(\vec{p},\omega\to-\infty)=n_0V(\vec{0})+\int\frac{d\vec{k}}{(2\pi)^3}\tilde{n}_kV(\vec{0})+n_0V(\vec{p})+\int\frac{d\vec{k}}{(2\pi)^3}\tilde{n}_kV(\vec{k}+\vec{p})\ . \tag{18}$$

Wagner's original derivation of (16) used this HF approximation.[13] Our result in (17) is applicable to potentials with a hard core.[14]

Using (17) in (16), the long-wavelength limit of the first frequency-moment sum rule is

$$\lim_{p\to0}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\omega'A(\vec{p},\omega')=\frac{2n_0}{m}\tilde{f}(\vec{0},\vec{0})+\int\frac{d\vec{k}}{(2\pi)^3}\frac{\tilde{n}_k}{m}[\tilde{f}(\tfrac{1}{2}\vec{k},-\tfrac{1}{2}\vec{k})+\tilde{f}(-\tfrac{1}{2}\vec{k},-\tfrac{1}{2}\vec{k})]-\mu\ . \tag{19}$$

III. MOMENTUM DISTRIBUTION $\tilde{n}_p$

The momentum distribution of excited atoms can be expressed[12] exactly in terms of the diagonal single-particle Green's function (we set $\hbar=1$)

$$\tilde{n}_p=-\frac{1}{\beta}\sum_n G_{++}(\vec{p},\omega_n)e^{\omega_n0^+},\quad \vec{p}\neq\vec{0} \tag{20}$$

where $\omega_n=in2\pi/\beta$ $(n=0,\pm1,\pm2,\ldots)$ are the Bose Matsubara frequencies. Performing the usual analytic continuation to real frequencies, this result is equivalent to

$$\tilde{n}_p=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}A(\vec{p},\omega)N^0(\omega)\ , \tag{21}$$

where $N^0(\omega)$ is the Bose distribution function.

One can use the frequency sum rules derived in Sec. II to calculate $\tilde{n}_p$ at finite temperatures. More specifically, if the important contributions to (21) are from frequencies $\omega\ll k_BT$, then one can use the expansion

$$N_0(\omega)=\frac{k_BT}{\omega}-\frac{1}{2}+\frac{1}{12}\frac{\omega}{k_BT}+\ldots \tag{22}$$

and the sum rules give immediately

$$\lim_{p\to0}\tilde{n}_p=\frac{k_BTmn_0}{n_Sp^2}+\frac{k_BT}{4\Sigma_{+-}(\vec{p},0)}-\frac{1}{2}+\frac{E_0}{12k_BT}+\cdots\ , \tag{23}$$

where E_0 is defined by the rhs of (19). Recalling our result in (12), the divergent terms in (23) are those given by the $\omega_n=0$ term in (20), as one might have expected. The next correction to (3) involves $G_{++}(\vec{p},\omega_1=i2\pi k_BT)$, which is well behaved (nondivergent) in the $p\to0$ limit.

The exact result in (23) is of interest because it has been suggested that $\Sigma_{+-}(\vec{0},0)=0$. More specifically, if at finite temperatures one has[9] $\Sigma_{+-}(\vec{p}\to\vec{0},\omega=0)=Bp$, then our exact result in (23) implies that $\tilde{n}_p$ will have a divergent contribution which goes as $1/p$ in addition to the $1/p^2$ term. In contrast, if $\Sigma_{+-}(\vec{p}\to\vec{0},\omega=0)\neq0$, then the only divergent contribution to $\tilde{n}_p$ will be the first term on the rhs of (23). In Sec. IV we use the one-loop approximation within the dielectric formalism for Bose systems[7] to calculate $\Sigma_{+-}(\vec{0},0)$ and show that it is finite.

Our result for $\tilde{n}_p$ in (23) is restricted to the high-temperature region, where $\omega_n=0$ is the dominant term in (20). A more general approach is to make use of the two-fluid equations, which predict that[1,2] for low frequencies and long wavelengths (for example, $p\lesssim0.1$ Å^{-1} in ^{4}He),

$$A(\vec{p},\omega)=2\pi\frac{n_0m}{n_S}\left[\frac{\rho_S}{\rho}c_1^2\delta(\omega^2-c_1p^2)+\frac{\rho_N}{\rho}c_2^2\delta(\omega^2-c_2^2p^2)\right]\mathrm{sgn}\omega\ . \tag{24}$$

Here c_1 (c_2) is the first- (second-) sound velocity and we have neglected thermal expansion ($C_P=C_V$) as well as damping. Strictly speaking, all that microscopic theory[1-3] tells us that the longitudinal part of the superfluid velocity correlation is related to a particular combination of single-particle correlation functions, namely,

$$\lim_{p\to0}\tfrac{1}{2}[G_{++}(\vec{p},\omega)-G_{+-}(\vec{p},\omega)]=\frac{m^2n_0}{p^2}\chi^I_{v_sv_s}(\vec{p},\omega)\ . \tag{25}$$

In writing down (24), we have made the additional assumption that $G_{++}(\vec{p},\omega)=-G_{+-}(\vec{p},\omega)$ in the long-wavelength, low-frequency limit. While we know of no general proof valid at finite temperatures, this equality holds in most model calculations and it has been proven at $T=0$ K by Gavoret and Nozières.[11] In this limit, c_2 is finite but this mode has zero weight in $A(\vec{p},\omega)$, and (24) reduces to[11]

$$A(\vec{p},\omega)=2\pi\frac{n_0m}{n}c_1^2\delta(\omega^2-c_1^2p^2)\mathrm{sgn}\omega\ . \tag{26}$$

Finally we note that using (24) in (6), we only find the leading-order $1/p^2$ divergent term in (12). Two-fluid hydrodynamics does not include the next correction involving $\Sigma_{+-}(\vec{p},\omega=0)$.

Following the approach of Baym[3] and others, we can calculate $\tilde{n}_p$ using (24) in (21) to obtain

$$\tilde{n}_p=\frac{n_0m}{n_Sp}\frac{\rho_S}{\rho}c_1[N^0(c_1p)+\tfrac{1}{2}]+\frac{n_0m}{n_Sp}\frac{\rho_N}{\rho}c_2[N^0(c_2p)+\tfrac{1}{2}]+\cdots\ . \tag{27}$$

This explicitly shows the contribution to $\tilde{n}_p$ from both first and second sound, properly weighted. At finite temperatures, such that $c_ip\ll k_BT$, this reduces to the expected result $\tilde{n}_p=k_BTmn_0/n_Sp^2$. Several previous authors[3-5] have used

$$A(\vec{p},\omega)=2\pi\frac{n_0m}{n_S}c_1^2\delta(\omega^2-c_1^2p^2)\mathrm{sgn}\omega \tag{28}$$

instead of the correct two-fluid hydrodynamic expression

given by (24), and consequently they found

$$\tilde{n}_p = \frac{n_0 m}{n_S p} c_1 [N^0(c_1 p) + \tfrac{1}{2}] + \cdots \tag{29}$$

instead of (27). Both (24) and (29) give the same high-temperature limit. The zero-point energy is canceled out by the second term on the rhs of (22). Similarly, at $T=0$ K (where $\rho_S = \rho$, $\rho_N = 0$), both expressions lead to the correct Gavoret-Nozières result in (4). However, at intermediate temperatures, there are differences and (27) seems preferable. Of course, one must remember that (24) and hence (27) are only valid in the hydrodynamic domain $c_i p \tau \ll 1$ (where τ is some appropriate quasiparticle lifetime) and outside the critical region.

Martin[5] has written down a result based on (29) which is equivalent to only keeping the first term on the rhs of (22),

$$\tilde{n}_p = \frac{k_B T m n_0}{n_S p^2} + \frac{n_0 m c}{n_S 2p} + \cdots . \tag{30}$$

However, the zero-point energy contribution in (30) which goes as $1/p$ is canceled out when one keeps the second term in (22).[15] It should not be confused with the correction term in (23) involving the reciprocal of $\Sigma_{+-}(\vec{p},0)$.

To conclude this section, we recall that the momentum distribution $\tilde{n}_p$ is the Fourier transform of $\Lambda_1(\vec{R}) \equiv \langle \psi^\dagger(\vec{r})\psi(\vec{r}\,')\rangle$ with respect to $\vec{R} = \vec{r} - \vec{r}\,'$. It is easily checked[4,16] that at large R, these results imply (with $\hbar$ put back in)

$$\Lambda_1(R) = \frac{k_B T m n_0}{4\pi n_S \hbar^2 R} + \cdots \quad \text{for } R \gg \frac{\hbar c}{k_B T}, \lambda_c , \tag{31}$$

$$\Lambda_1(R) = \frac{n_0 m c}{4\pi^2 n_S R^2} + \cdots \quad \text{for } \lambda_c \ll R \ll \frac{\hbar c}{k_B T} , \tag{32}$$

where $\lambda_c \equiv \hbar/2mc$.

At the present time, the only experimental way of measuring $\tilde{n}_p$ is to extract it from high-momentum inelastic neutron-scattering data (see Ref. 8). Unfortunately, there is no simple way to distinguish the peak in $\tilde{n}_p$ from the condensate peak.

IV. ONE-LOOP APPROXIMATION FOR $G_{++}(\vec{p}, \omega=0)$

In Sec. II assuming the validity of (11), we proved that $G_{++}(\vec{p}, \omega=0)$ was given by (12) in the $p \to 0$ limit. In Sec. III we found that two-fluid hydrodynamics gave no information about the second term on the rhs of (12). We now discuss a specific microscopic calculation which reproduces the first term on the rhs of (12) correctly and which further implies that $\Sigma_{+-}(\vec{p}=\vec{0}, \omega=0)$ is finite. This means the second term in (12) is nondivergent in the limit $\vec{p} \to \vec{0}$.

Recently, Talbot and Griffin[7] have given an extensive discussion of the one-loop approximation to proper, irreducible quantities in Bose-condensed systems. In particular, they were able to show this approximation was consistent with the exact zero-frequency result given in (9). This fact immediately tells us that the one-loop approximation will lead to (12). Turning to the question of whether $\Sigma_{+-}(0,0) \neq 0$ in this approximation, we recall that this self-energy can be split into irreducible and reducible parts:

$$\Sigma_{+-}(\vec{p},\omega) = \bar{\Sigma}_{+-}(\vec{p},\omega) + \frac{V(\vec{p})\bar{\Lambda}_+(\vec{p},\omega)\bar{\Lambda}_-(\vec{p},\omega)}{1 - V(\vec{p})\bar{\chi}^R_{nn}(\vec{p},\omega)} . \tag{33}$$

Here $\bar{\Lambda}_\mu(\vec{p},\omega)$ is a proper, irreducible anomalous density vertex function and $\bar{\chi}^R_{nn}(\vec{p},\omega)$ is the proper, irreducible density response function (we follow the notation of Ref. 7).

We first discuss the reducible self-energy in (33). In the one-loop approximation, $\bar{\chi}^R_{nn}$ is given by Eq. (4.1) of Ref. 7. One sees that $\bar{\chi}^R_{nn}(\vec{p}, \omega=0)$ diverges at $p=0$ and thus care must be taken in its evaluation for small p. The dominant contribution to the $\vec{k}$ integral involved comes from the phonon region $k \ll p$. In the $p \to 0$ limit, we can assume that $\omega_{\vec{k}}, \omega_{\vec{p}+\vec{k}} \ll k_B T$ and we find after some calculation

$$\lim_{p\to 0} \bar{\chi}^R_{nn}(\vec{p}, \omega=0)$$

$$= -\int \frac{d\vec{k}}{(2\pi)^3} \frac{k_B T}{\omega_{\vec{k}} \omega_{\vec{p}+\vec{k}}} \frac{2(mc^2)^2}{\omega^2_{\vec{p}+\vec{k}}} \frac{\omega_{\vec{k}}}{\epsilon_{\vec{k}}} \frac{\epsilon_{\vec{p}+\vec{k}}}{\omega_{\vec{p}+\vec{k}}} + \cdots . \tag{34}$$

This integral can be easily evaluated:

$$\lim_{p\to 0} \chi^R_{nn}(\vec{p}, \omega=0) = -k_B T \int \frac{d\vec{k}}{(2\pi)^3} \frac{1}{2} \frac{1}{\epsilon_{\vec{k}} \epsilon_{\vec{k}+\vec{p}}}$$

$$= -\frac{m^2 k_B T}{4p} + \cdots . \tag{35}$$

Taking into account that $\bar{\Lambda}_\mu(\vec{p}, \omega=0)$ is a constant in the $\vec{p} \to \vec{0}$ limit, we see that the second term on the rhs of (33) is of order p and thus vanishes in the $p \to 0$ limit. Note that if we use $\bar{\Lambda}_\mu(\vec{p}, \omega=0) = n_0^{1/2}$, then the reducible self-energy contribution to $\Sigma_{+-}(\vec{p}, \omega=0)$ in (33) is precisely the same as that given by the shielded potential approximation[17,18] (SPA). In the SPA, this is the *only* contribution to $\Sigma_{+-}(\vec{p}, \omega=0)$ and thus it will lead to a term of order p^{-1} in the high-temperature value of $\tilde{n}_p$. Such a term was indeed found in Ref. 17 and also in the equivalent perturbative calculations of Morita and Hara.[16,19]

However, in the one-loop approximation, we also have an irreducible contribution $\bar{\Sigma}_{+-}(\vec{p}, \omega=0)$ given by Eq. (3.2) of Ref. 7. Evaluating this, we obtain

$$\lim_{p\to 0}\bar{\Sigma}_{+-}(\vec{p},\omega=0)=\int\frac{d\vec{k}}{(2\pi)^3}V(\vec{k})\tilde{m}_{\vec{k}}+n_0\int\frac{d\vec{k}}{(2\pi)^3}V^2(\vec{k})(u_{\vec{k}}-v_{\vec{k}})^2\times\left[\frac{\partial N^0(\omega_{\vec{k}})}{\partial\omega_{\vec{k}}}(u_{\vec{k}}-v_{\vec{k}})^2+\frac{1+2N^0(\omega_k)}{2\omega_k}4u_{\vec{k}}v_{\vec{k}}\right], \tag{36}$$

where $\tilde{m}_{\vec{k}}=-u_{\vec{k}}v_{\vec{k}}[2N^0(\omega_{\vec{k}})+1]$. The terms in (36) may be combined to give

$$\lim_{p\to 0}\bar{\Sigma}_{+-}(\vec{p},\omega=0)=n_0\int\frac{d\vec{k}}{(2\pi)^3}V^2(\vec{k})\frac{\epsilon_{\vec{k}}^2}{\omega_{\vec{k}}^2}\times\left[\frac{\partial N^0(\omega_{\vec{k}})}{\partial\omega_{\vec{k}}}-\frac{1+2N^0(\omega_{\vec{k}})}{2\omega_{\vec{k}}}\right], \tag{37}$$

which is a well-defined integral not equal to 0. We conclude that in the one-loop approximation, $\Sigma_{+-}(\vec{0},0)$ is finite. In view of (12), this means the only divergent contribution to $G_{++}(\vec{p},\omega=0)$ is that given in (2). This result is in disagreement with the conclusion of Nepomnyashchii and Nepomnyashchii,[9] who argued that $\Sigma_{+-}(\vec{0},0)=0$ was necessary in order to remove divergencies which arose in the diagrammatic analysis of correlation functions.[11] However, Talbot and Griffin[10] have recently shown that these divergencies do not arise when one includes the necessary symmetry-breaking perturbation.

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[1]N. N. Bogoliubov (unpublished); N. N. Bogoliubov, *Lectures on Quantum Statistics* (Gordon and Breach, New York, 1970), Vol. 2, pp. 58 and 185.

[2]P. C. Hohenberg and P. C. Martin, Phys. Rev. Lett. **22**, 69 (1963); P. C. Hohenberg and P. C. Martin, Ann. Phys. (N.Y.) **34**, 291 (1965).

[3]G. Baym, in *Mathematical Methods in Solid State and Superfluid Theory,* edited by R. C. Clark and G. H. Derrick (Oliver and Boyd, Edinburgh, 1969), p. 151. When one uses a Hermitian broken-symmetry perturbation, Baym's linear-response argument gives (1) rather than (2) [E. Talbot (private communication)].

[4]R. A. Ferrell, N. Menyhard, H. Schmidt, F. Schwabl, and P. Szepfalusy, Ann. Phys. (N.Y.) **47**, 565 (1968); see also E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics* (Pergamon, Oxford, 1980), Part 2, p. 108.

[5]P. C. Martin, Physica **96A**, 70 (1979).

[6]B. D. Josephson, Phys. Lett. **21**, 608 (1966).

[7]E. Talbot and A. Griffin, Ann. Phys. (N.Y.) **151**, 71 (1983).

[8]V. F. Sears, E. C. Svensson, P. Martel, and A. D. B. Woods, Phys. Rev. Lett. **49**, 279 (1982); further references are given here.

[9]Y. A. Nepomnyashchii and A. A. Nepomnyashchii, Zh. Eksp. Teor. Fiz. **75**, 976 (1978) [Sov. Phys.—JETP **48**, 493 (1978)].

[10]E. Talbot and A. Griffin, Phys. Rev. B **29**, 3952 (1984); see also Chap. 6 of E. Talbot, Ph.D. thesis, University of Toronto, 1983 (unpublished).

[11]J. Gavoret and P. Nozières, Ann. Phys. (N.Y.) **28**, 349 (1964).

[12]A. L. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems* (McGraw-Hill, New York, 1971).

[13]H. Wagner, Z. Phys. **195**, 273 (1966).

[14]The Wong-Gould sum rule discussed in Ref. 7 (see pp. 76ff) can be similarly generalized to hard-core potentials. See also Appendix D of V. K. Wong and H. Gould, Ann. Phys. (N.Y.) **83**, 252 (1974).

[15]The incorrect expression in (30) was used in Ref. 8. However, since the leading-order p^{-2} term is given correctly, one does not expect that the values of n_0 given in Ref. 8 will be altered significantly [E. Svensson (private communication)].

[16]T. Morita and H. Hara, Prog. Theor. Phys. **41**, 50 (1969).

[17]T. H. Cheung and A. Griffin, Can. J. Phys. **48**, 2135 (1970).

[18]P. Szépfalusy and I. Kondor, Ann. Phys. (N.Y.) **2**, 1 (1974).

[19]T. Morita, Prog. Theor. Phys. (Suppl.) **44**, 69 (1969).