Duality and Potts critical amplitudes on a class of hierarchical lattices

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By using the duality transformation on a class of hierarchical lattices, I show that the Potts critical amplitudes above and below the critical temperature are equal. Logarithmic modifications of the power-law singularity occur when the exponent $2-\alpha$ is an even integer, but do not occur when $2-\alpha$ equals an odd integer.

A numerical study of the q-state Potts critical amplitude was recently performed[1] for the diamond hierarchical lattice.[2] The model is defined by associating with each bond $\langle i,j\rangle$ an energy: $-\mathscr{H}_{i,j}/kT=2J\delta_{S_i,S_j}$, where $S_i=1,2,\ldots,q$. Close to the critical point, the singular part of the free energy per bond behaves as

$$f_{\rm sing}=A_{\pm}|t|^{2-\alpha} , \tag{1}$$

where $t=J_c-J$ and $\pm$ refers to $t>0$ and $t<0$, respectively. When, by varying q, $2-\alpha$ approaches an integer m, it is expected[1] that $A_{\pm}$ diverge signaling the occurrence of a logarithmic modification of the power law at $2-\alpha=m$:

$$f_{\rm sing}=at^m\ln|t|+c_{\pm}|t|^m . \tag{2}$$

It was found numerically[1] that for all q values analyzed $A_+=A_-$ and that the amplitudes diverge as $2-\alpha$ approaches 2 and 4, but they do not diverge as $2-\alpha$ approaches 3 and 5. In this note I explain and generalize these results.

I consider here the hierarchical lattices associated with the Migdal-Kadanoff renormalization group[3] for dimension $d=2$ and rescaling factor $b=2,3,4,\ldots$. These lattices are iteratively constructed as shown in Fig. 1. The diamond lattice corresponds to $b=2$. Their duals[4] are shown in Fig. 2. It is readily seen by inspecting Figs. 1 and 2 that

$$\frac{1}{b}f(bJ)=\bar{f}(J) , \tag{3}$$

where f is associated with the lattice in Fig. 1 and $\bar{f}$ is the dual free energy associated with the lattice in Fig. 2. Moreover, the duality transformation[5] relates f and $\bar{f}$ evaluated at J and $\bar{J}$ (the dual coupling), respectively:

$$f(J)=\bar{f}(\bar{J})+\text{regular contribution} \tag{4}$$

and

$$(e^{2J}-1)(e^{2\bar{J}}-1)=q . \tag{5}$$

Equations (3) and (4) imply the following functional equation for the singular part of the free energy:

$$f_{\rm sing}(J)=\frac{1}{b}f_{\rm sing}(b\bar{J}) . \tag{6}$$

Equation (6) is the main result of this note and I next discuss its consequences.

First, the critical couplings J_c and $\bar{J}_c$ are the coordinates of the intersection point of the line $J=b\bar{J}$ and the duality curve [Eq. (5)] on the $(J,\bar{J})$ plane:

$$(e^{2J_c}-1)(e^{2J_c/b}-1)=q . \tag{7}$$

Then by expressing $f_{\rm sing}$ as a function of t, Eq. (6) can be rewritten, for t small, as

$$f_{\rm sing}(t)=\frac{1}{b}f_{\rm sing}\left(b\left.\frac{d\bar{J}}{dJ}\right|_{J=J_c}t\right) . \tag{8}$$

Since $d\bar{J}/dJ<0$, Eq. (8) is a relationship between values of $f_{\rm sing}$ above and below the critical point. In view of Eq. (1), Eq. (8) can be written for $t>0$ and $t<0$ as

$$\begin{aligned} A_+|t|^{2-\alpha}&=\frac{1}{b}A_-\left|b\left.\frac{d\bar{J}}{dJ}\right|_{J=J_c}\right|^{2-\alpha}|t|^{2-\alpha} ,\\ A_-|t|^{2-\alpha}&=\frac{1}{b}A_+\left|b\left.\frac{d\bar{J}}{dJ}\right|_{J=J_c}\right|^{2-\alpha}|t|^{2-\alpha} , \end{aligned} \tag{9}$$

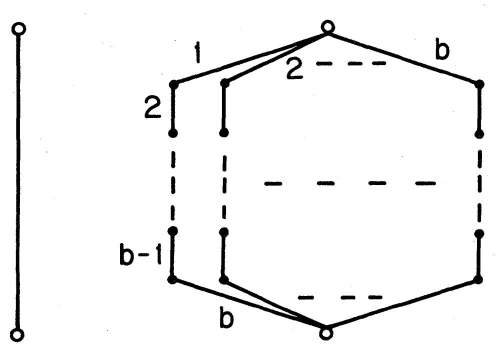

FIG. 1. Construction of the hierarchical lattice corresponding to the Migdal-Kadanoff renormalization group for $d=2$ and $b=2,3,4,\ldots$.

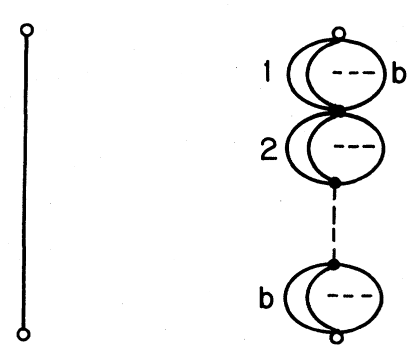

FIG. 2. Dual of the lattice in Fig. 1.

which imply

$$2-\alpha=\ln b/\ln\left|b\frac{d\bar{J}}{dJ}\bigg|_{J=J_c}\right| \qquad (10)$$

and

$$A_+=A_- \ . \qquad (11)$$

Lastly, when $2-\alpha=m$, integer, f_{sing} is given in Eq. (2), and Eq. (8) in conjunction with the observation that $d\bar{J}/dJ<0$ implies

$$a=0 \text{ for } m=\text{odd integer} \ , \qquad (12)$$

and

$$c_+=c_- \ . \qquad (13)$$

Hence, there is no logarithmic modification of the power law, and the amplitudes $A_\pm$ do not diverge when $2-\alpha$ approaches an odd integer. The amplitudes $A_\pm$ and $c_\pm$ are each equal to a constant plus a numerically small periodic function of $\ln|t|$.[6] *My discussion and Eqs. (11) and (13) regard the constant contribution.*

It is worth noting that the location of the critical point and the results concerning the amplitudes, Eqs. (11)–(13), can also be obtained for the square lattice, by using self-duality, and for the triangular and honeycomb lattices, by using duality and star-triangle transformations. More curious is that for the hierarchical lattices of Figs. 1 and 2, the exponent $2-\alpha$ could also be determined from the duality transformation, albeit supplemented by the relationship in Eq. (3).

After completing this manuscript I received an unpublished paper[7] which also discusses the consequence of the duality transformation for these hierarchical lattices. David Andelman co-authored the work in Ref. 1 which provides the motivation for the present paper. I am grateful to him for numerous discussions. This research was supported by the U.S. Joint Services Electronics Program under Contract No. DAAG29-83-K0003. I also acknowledge support at MIT from the Bantrell Trust.

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