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Percolative conduction and the Alexander-Orbach conjecture in two dimensions

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Alexander and Orbach have recently proposed that the ratio of the fractal dimensionality of the incipient infinite cluster in percolation to the fractal dimensionality of a random walk on the cluster is $\frac{2}{3}$, independent of the spatial dimensionality of the system. As a consequence, they predict that the electrical conductivity exponent $t/\nu = 0.9479\ldots$ in two dimensions, where ν is the correlation-length exponent. Our numerical data, which are obtained from large-lattice finite-size scaling calculations, give a value $t/\nu = 0.973^{+0.005}_{-0.003}$, in disagreement with the conjecture by 2.6%.

The properties of fractals, and their application to percolation, are a subject of growing interest.[1,2] Recently, Alexander and Orbach[1] (AO) considered diffusion on fractals, and pointed out that, to very good accuracy, the critical exponent t for conductivity in the percolation problem could be related to the static exponents β and ν via

$$t = \frac{1}{2}[\nu(3d-4)-\beta] \quad , \tag{1}$$

which, using generaly accepted and probably exact[3-5] values $\nu = \frac{4}{3}$ and $\beta = \frac{5}{36}$ gives

$$\frac{t}{\nu} = \frac{91}{96} = 0.947\,916\,6 \ldots \quad . \tag{2}$$

In this Rapid Communication we present data which, although close to (1), are probably inconsistent with it.

We use a large-cell renormalization-group or finite-size scaling approach, which has been discussed elsewhere.[6-9] We consider a square lattice composed of links which have unit conductance with probability p, and zero conductance with probability $1-p$. Since the bulk conductance G of an infinite sample varies as $(p-p_c)^t$ near the percolation threshold p_c, it can be argued that the average conductance $\langle G \rangle$ of finite samples at $p = p_c$ is given by

$$\langle G \rangle = b^{-t/\nu}[c_1 + c_2 f_2(b) + \cdots] \quad , \tag{3}$$

where $f_2(b) \rightarrow 0$ as b, the sample size measured in units of the lattice constant, approaches infinity, and where c_1 and c_2 are constants.[8-10] In order to determine t/ν, then, $\langle G \rangle$ must be determined accurately for large samples, and a $b \rightarrow \infty$ extrapolation performed. In order to determine $\langle G \rangle$, we have developed an exact algorithm using only series, parallel, Y-∇, and ∇-Y transformations to reduce each realization to a single conductance.[10] The algorithm is extremely efficient, allowing all of the data in Table I to be obtained on a DEC LSI 11/2 and a VAX 11-780. At $b = 200$, an average realization took 11.5 sec on the VAX, and took 1.04 sec in test runs on an IBM 3081. The entire data set of Table I could be obtained in under two hours on an IBM 3081. A Monte Carlo program using the algorithm was written which kept track of arithmetic, geometric, and harmonic means of conductances. The data thus obtained are listed in Table I. Our Monte Carlo calculations were done on self-dual lattices; the advantages of this have been described elsewhere.[11,12,8,9] From (3), we see that plotting $\ln(\langle G \rangle^{-1})/\ln b$ against $1/\ln b$ will give t/ν as the y intercept; this has been done in Fig. 1. We note that, *even without extrapolation,* the AO conjecture (2), which is indicated by the arrow, seems to be much too low for the data.

This disagreement becomes even more striking when the data is analyzed. To do this, a number of choices for $f_2(b)$ in (3) were tried. Our first choice was $f_2(b) = b^{-\Delta/\nu}$; that is, the correction is a power law, by analogy to the standard percolation case.[13] A least-squares fitting was done, varying t/ν, Δ/ν, and the number of points included in the fit, to determine the minimum χ^2 for various mean conductances, as well as the parameter values which give the smallest sum of the three χ^2. The results are summarized in Table II. The values of t/ν thus obtained are between 0.975 and 0.979, with Δ/ν varying between 1.4 and 2.

Another alternative is that $f_2(b) = 1/\ln b$, as suggested by Kirkpatrick.[14] When this was tried, t/ν had a narrower spread for the different means, varying between 0.9725 and 0.973, but the resulting χ^2 were slightly larger, as seen in Table II.

To explore the correction term more systematically, a form which interpolates between the two forms above was tried. Since $\ln x = (x^\alpha - 1)/\alpha + \cdots$ as $\alpha \rightarrow 0$, we use $f_2(b) = 1/(b^{\tilde{\Delta}/\nu} - 1)$, which approaches $1/\ln b$ for $\tilde{\Delta} \rightarrow 0$ and the power law form for $b^{\tilde{\Delta}/\nu}$ large. This procedure revealed a "valley" in $\chi^2(t/\nu, \tilde{\Delta}/\nu)$ which had two local minima corresponding to the minima found above. We thus see that they are the only two choices suggested by the data.

On the basis of χ^2 alone, it would be difficult to distinguish between power law and log corrections. We favor the log correction for two reasons. First, the various means give a t/ν which varies by 0.4% when a power law correction is used; this is reduced to 0.05% when the log correction is used. Second, the value determined here of $\Delta/\nu \sim 1.4$–2 is greater than one. In such a case, we have ignored a possibly stronger analytic correction; i.e., $f_2(b) = 1/b +$ higher-order terms. In fact, the $1/b$ correction does not give too bad a fit, with $t/\nu \simeq 0.974$. On the basis of this, we suggest that

$$\frac{t}{\nu} = 0.973^{+0.005}_{-0.003} \quad , \tag{4}$$

where the value chosen reflects our preference for the log

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TABLE I. Average conductances at $p=\frac{1}{2}$ as a function of lattice size b. N is the total number of realizations considered; N_c are the numbers which conduct, and the subscripts a, g, and h stand for arithmetic, geometric, and harmonic means.

b	N	N_c	$\langle G\rangle_a$	$\langle G\rangle_g$	$\langle G\rangle_h$
2	all	$\frac{1}{2}$	0.566. . .	0.5424. . .	0.5217. . .
3	all	$\frac{1}{2}$	0.3845. . .	0.3629. . .	0.3443. . .
4	110 000	55 166	0.2886 ±0.000 47	0.2707	0.2554
5	110 000	55 094	0.2315 ±0.000 58	0.2168	0.2041
6	110 000	54 994	0.1939 ±0.000 32	0.1812	0.1703
7	30 000	15 063	0.166 69 ±0.000 53	0.1557	0.1462
8	110 000	55 295	0.1455 ±0.000 24	0.1359	0.1276
9	40 000	19 923	0.1294 ±0.000 36	0.1207	0.1132
10	110 000	55 067	0.117 14 ±0.000 20	0.109 26	0.102 45
12	40 000	20 005	0.0981 ±0.000 28	0.0915	0.0857
14	20 000	10 075	0.0846 ±0.000 34	0.0788	0.0737
16	40 000	20 094	0.074 11 ±0.000 21	0.069 05	0.064 66
17	20 000	9 946	0.0702 ±0.000 28	0.0652	0.0609
18	40 000	20 091	0.066 117 ±0.000 19	0.061 65	0.057 72
20	20 000	9 978	0.059 78 ±0.000 24	0.055 55	0.051 92
22	40 000	19 935	0.054 35 ±0.000 15	0.050 61	0.047 36
25	20 000	10 105	0.0478 ±0.000 19	0.049 46	0.041 60
30	20 000	9 992	0.039 87 ±0.000 16	0.037 09	0.034 69
40	20 002	10 032	0.030 20 ±0.000 12	0.028 08	0.026 24
50	20 000	9 938	0.024 30 ±0.000 10	0.022 59	0.021 11
60	2 704	1 368	0.0203 ±0.000 23	0.0188	0.0175
100	2 400	1 197	0.012 42 ±0.000 14	0.011 54	0.010 78
200	1 300	661	0.006 329 ±0.000 10	0.005 876	0.005 481

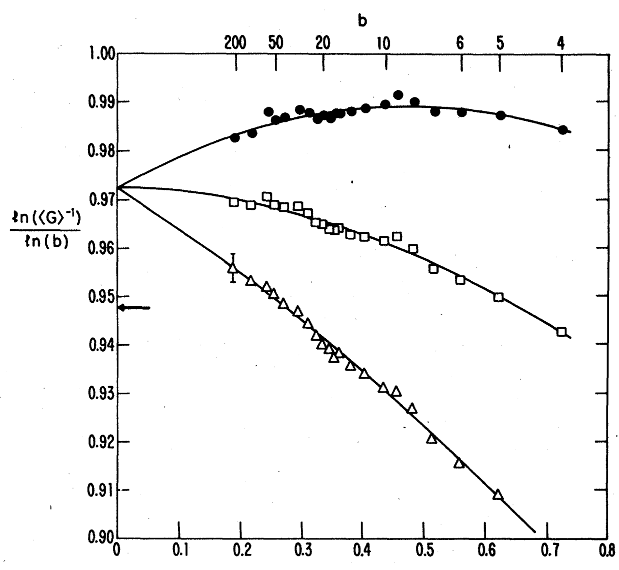

FIG. 1. Plot of $\ln(\langle G\rangle^{-1})/\ln b$ against $1/\ln b$. According to (3), the y intercept of this data gives t/ν. Symbols indicate arithmetic (Δ), geometric ($\Box$), and harmonic ($\bullet$) means. The lines are fits to (3) with $f_2(b)=1/\ln b$, and are three independent fits. The arrow indicates the AO conjecture, $t/\nu=0.9479\ldots$. The error bar at $b=200$ (the least accurately determined point) corresponds to one rms deviation from the mean.

correction and the error bars are chosen qualitatively to enclose most of the valley in $\chi^2(t/\nu,\tilde{\Delta}/\nu)$ space. (We have recently run 1000 realizations at $b=500$ and 400 cases at $b=1000$ which do not significantly alter this result.) Using $\nu=\frac{4}{3}$ exactly,[3,6,7,15,16] we obtain

$$t=1.297^{+0.007}_{-0.004}\ . \tag{5}$$

We thus see from (4) that our data are inconsistent with the AO conjecture. They are also inconsistent with the simple result $t=\nu$.[17-19] We know of no theoretically proposed value which is consistent with our value.

Our determination of different means gives information on the shape of the conductance distribution, as well as its scale invariance. We find that the ratios of the various means are roughly independent of b. This is expected at the critical point, where changing b should change only the scale of the distribution, but not its shape. For example, the ratio of the arithmetic mean to the geometric mean varies from 1.04 for $b=2$ to 1.08 for $b=200$, and extrapolates to ~ 1.08 at $b=\infty$. Similarly, the ratio of the harmonic mean to the geometric mean approaches ~ 0.93 at $b=\infty$, starting from 0.96 at $b=2$. This relatively weak dependence on sample size indicates that the distribution is scale invariant even for fairly small cells, and thus accounts for the close agreement between exponent values obtained from the different means.

While a longer version of this work[10] was being prepared, we became aware of a number of other manuscripts which report similar results from a variety of independent tech-

TABLE II. Fitting parameters for the various mean conductances using both power law and logarithmic corrections.

Mean	Power law correction $\frac{t}{\nu}$	$\frac{\Delta}{\nu}$	χ^2	Logarithmic correction t/ν	χ^2
Arithmetic	0.976	2.0	0.805	0.973	0.824
Harmonic	0.979	1.8	0.467	0.9725	0.486
Geometric	0.9775	1.8	0.593	0.973	0.613
All three	0.976	1.4	0.642	0.973	0.642

niques. Using a transfer-matrix method, Zabolitzky[20] obtained precisely the same value for t/ν as we have; this agreement from two independent methods is strong support for (4). Herrmann, Derrida, and Vannimenus[21] used a transfer-matrix technique to calculate s (which is identically equal to t in two dimensions); their result also disagrees with the AO conjecture. In addition, Hong, Havlin, Hermann, and Stanley[22] have studied a random walk on the percolation backbone; their result gives $t/\nu = 0.970 \pm 0.009$, again in violation of the AO conjecture. The range of applicability of the conjecture has still to be determined, but we believe that it is not true for two-dimensional percolation.[23]

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