

## Models of cluster growth on the Cayley tree

J. Vannimenus

*Groupe de Physique des Solides de l'Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris, Cedex 05, France*

B. Nickel\*

*Schlumberger Doll Research, Ridgefield, Connecticut 06877*

V. Hakim

*Laboratoire de Physique Théorique et Hautes Energies, 91405 Orsay, France*

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We study the diffusion-controlled process of cluster growth, introduced by Witten and Sander, on a Cayley tree. We show that it is then equivalent to the Eden model where growth occurs at any boundary site with equal probability. The mean level number and the square gyration radius of an  $N$ -particle aggregate both increase as  $[K/(K-1)]\ln N$  on a tree of branching ratio  $K$ . The case of biased diffusion is studied numerically: an attractive bias does not change the logarithmic behavior of the size, but a repulsive bias leads to a different behavior, presumably with a mean level number of order  $N$ .

### I. INTRODUCTION

A model of cluster growth that captures some of the essential physics of floc formation has recently been proposed by Witten and Sander,<sup>1,2</sup> and studied by them and by Meakin<sup>3</sup> using numerical simulations. The most striking feature they discovered is that clusters generated by diffusion-controlled particle aggregation are fractal objects, i.e., their mass scales as a nontrivial power law of their size:

$$N \sim R^D. \quad (1)$$

$N$  is the number of particles in the aggregate,  $R$  is its gyration radius, and  $D$  is its fractal dimension ( $D < d$ , the dimension of space). Relation (1) holds asymptotically for large clusters.

This scaling property is apparently not shared by other models such as the Eden model,<sup>4</sup> where cluster growth occurs by adding particles at random with uniform probability on the boundary of the cluster, and one finds numerically that the cluster mass is proportional to its volume.

Also,  $D$  is larger by a significant amount than the fractal dimension corresponding to lattice animals,<sup>5</sup> where one simply averages  $R$  over all possible cluster shapes for given  $N$  with no kinetic constraints. This means that Witten-Sander aggregates are *more* compact than typical lattice animals. The reason for this difference may be understood qualitatively by considering both models on the Cayley tree: There are many ways to obtain a given compact cluster by a dynamic growth process, while a cluster with a very elongated filament can be obtained in very few ways. For the animal problem, all clusters have the same weight, and the elongated ones dominate the average size,<sup>6</sup>

while for the aggregate problem the weight of compact clusters is so large that they dominate, as is explicitly shown by the calculations presented below.

Several theories have already been proposed to interpret the simulation results. Gould, Family, and Stanley<sup>7</sup> have included kinematic effects in a real-space renormalization-group approach, and they find that clusters grown by a diffusion process indeed belong to a universality class different from lattice animals, in dimension  $d=2$ . However, their approximations are not controlled and the results are not satisfactory for the Eden model; therefore, their approach is not yet quantitatively reliable. A classical continuum approximation has been studied by Ball, Nauenberg, and Witten,<sup>8</sup> who find, neglecting fluctuations around a spherical shape, the simple result

$$D = d - 1. \quad (2)$$

This relation might be asymptotically valid for large  $d$ , or it might hold above some upper critical dimensionality  $d_c$ ,<sup>9</sup> by analogy with the well-known result  $D=4$  for lattice animals above  $d_c=8$ . A different mean-field-type theory has been presented by Muthukumar,<sup>10</sup> who obtains

$$D = (d^2 + 1)/(d + 1), \quad (3)$$

for all  $d$ , with no critical dimension. This prediction is in agreement with available numerical results,<sup>1-3</sup> but the derivation does not seem consistent, at least in its present form.

In view of the present situation, it seems worthwhile to investigate the problem by other approaches. In particu-

lar, one would like to understand in detail the limit of large dimensions, where some type of mean-field theory is expected to be valid.

In this paper we study models of cluster growth on the Cayley tree, with the usual rationale that the problem may then be solvable exactly and that the results may be relevant to their behavior on regular lattices in the limit of infinite dimension. We indeed find that an analytic solution exists for diffusion-controlled aggregation on the tree, and, in fact, the diffusion model becomes equivalent to the Eden model. We show that our results are consistent with the expectation that for the latter model the generated clusters are compact, in the sense that  $D=d$ . This suggests that a more elaborate theory of aggregation might start from the Eden model as a zeroth-order approximation, and include, as a perturbation, the "screening" of the interior which is absent on a tree.

The paper is organized as follows. In Sec. II a model of diffusion-controlled aggregation is formulated on the tree, and is shown to be equivalent to the Eden model. Analytic results for the particle distribution and for the radius of gyration of the aggregates are obtained in Sec. III. An interesting point is that the "surface" of a cluster may be defined in two ways that give different answers: most particles lie in a well-defined surface region, but the density of occupied sites has a sharp boundary at a different distance from the origin.

Section IV contains a discussion of the results and their relation to the systems in finite dimensions. Finally, we study numerically a situation with biased diffusion, which crudely models the effect of long-range forces. An attractive bias appears to have no strong effect on the aggregation process, while a repulsive bias drastically changes the shape of the clusters. This suggests that the effect of weak long-range forces may be important for realistic situations.

## II. EQUIVALENCE BETWEEN THE ISOTROPIC DIFFUSION MODEL AND THE EDEN MODEL

We consider a tree of coordination number  $K+1$ , with one root site on level 0 and  $K^m$  sites on level (or generation)  $m$  (see Fig. 1). In the diffusion model, a seed particle is initially placed at the root. A second particle is introduced at a large distance and it diffuses on the tree, i.e., it makes random jumps to neighboring sites until it is either lost to infinity or it reaches one of the  $K$  sites on level  $m=1$ , where it stops. New particles are similarly added to the cluster when they reach a site adjacent to an occupied site belonging to the aggregate previously formed.

In the Eden model, the growth mechanism is purely local, and the new particle is added with uniform probability at a randomly chosen site on the boundary of the existing cluster. We now show that the two models are equivalent.

As a first step, let us calculate the probability  $P_1(m)$  that a diffusing particle starting at a given site on level  $m$  will reach the parent site on level  $m-1$ . The probability, that at any step the particle goes towards the root, is denoted by  $a$ , to allow for biased diffusion (see Sec. V),

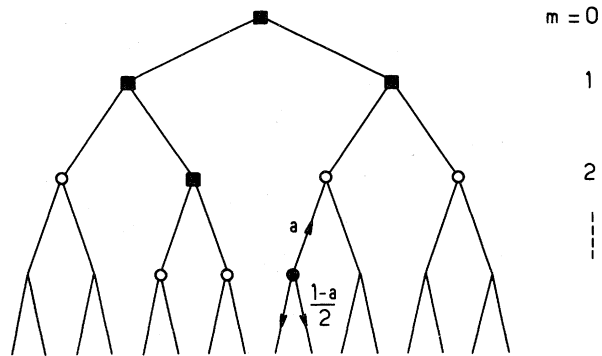


FIG. 1. Particle (solid circle) diffuses on the tree until it reaches an empty boundary site (open circles) and becomes part of the cluster (solid squares). Here the branching ratio of the tree is  $K=2$ .

and it has identical probabilities  $(1-a)/K$  to jump away to any one of the  $K$  possible sites. If the particle first goes to level  $m+1$  it has probability  $P_1(m+1)$  to return later to level  $m$ , and hence it has probability  $P_1(m+1)P_1(m)$  to eventually reach level  $m-1$ . This gives

$$P_1(m) = a + K[(1-a)/K]P_1(m+1)P_1(m). \quad (4)$$

This relation is independent of  $K$ , and may be simply viewed as a one-dimensional random walk between the generations, with probability  $a$  in one direction and  $1-a$  in the other. One then directly obtains

$$P_1 = \begin{cases} a/(1-a), & a < \frac{1}{2} \\ 1, & a > \frac{1}{2}. \end{cases} \quad (5)$$

The probability of escape to infinity is  $1-P_1$ , and is finite as long as the diffusion process is not too strongly biased towards the root. For the case of isotropic diffusion,  $a=(1+K)^{-1}$  and

$$P_1 = K^{-1}. \quad (6)$$

The probability  $P_l$  that the particle will reach a site at generation  $m-l$  is

$$P_l = 1/K^l. \quad (7)$$

For the second step, we note that any given empty boundary site situated on level  $l$  can be reached from  $K^{m-l}$  starting sites on level  $m$  (see Fig. 1). Averaging over many different samples, the probability that an isotropically diffusing particle will reach this boundary site is proportional to  $K^{m-l}P_{m-l}$  and is independent of  $l$  by Eq. (7).

It is thus possible to entirely forget the diffusion process and consider that growth occurs directly by addition of particles on the boundary sites with a uniform proba-

bility  $p$ . The number of boundary sites is  $(K-1)N+1$  for a cluster of  $N$  sites on a rooted tree, and thus this probability is

$$p = 1/[(K-1)N+1].$$

### III. SOLUTION OF THE EDEN MODEL

#### A. Generating function and mean level number

We first derive a recurrence relation for the average number  $A_N^m$  of particles on level  $m$  for a cluster of  $N$  particles. The average number  $B_N^m$  of empty boundary sites is

$$B_N^m = 2A_N^{m-1} - A_N^m$$

for a branching ratio  $K=2$ . Since any of the  $N+1$  boundary sites may be occupied with equal probability, this gives

$$A_{N+1}^m - A_N^m = B_N^m/(N+1),$$

hence

$$(N+1)A_{N+1}^m = NA_N^m + 2A_N^{m-1}. \quad (8)$$

For a general branching ratio  $K$ , one has similarly

$$[(K-1)N+1]A_{N+1}^m = (K-1)NA_N^m + KA_N^{m-1}. \quad (9)$$

The initial conditions on  $A_N^m$  are

$$A_N^0 = 1,$$

$$A_1^m = 0, \quad m \geq 1.$$

The linear character of relation (8) makes it possible to obtain an exact analytic solution. Introducing the generating functions

$$F^{(m)}(\beta) = \sum_n \beta^n A_n^m,$$

one sees that they obey the relation

$$(1-\beta) \frac{dF^{(m)}}{d\beta} = 2F^{(m-1)}.$$

The first  $F^{(n)}$  are

$$F^{(0)} = \beta/(1-\beta),$$

$$F^{(1)} = 2 \left[ \frac{\beta}{1-\beta} + \ln(1-\beta) \right],$$

$$F^{(2)} = 4 \left[ \frac{\beta}{1-\beta} + \ln(1-\beta) - \frac{1}{2!} \ln^2(1-\beta) \right],$$

and introducing the double generating function

$$F(\alpha, \beta) = \sum_{m,n} \alpha^m \beta^n A_n^m,$$

one obtains

$$\begin{aligned} F &= \frac{1}{1-2\alpha} \left[ \frac{\beta}{1-\beta} + 2\alpha \ln(1-\beta) \right. \\ &\quad \left. - \frac{[2\alpha \ln(1-\beta)]^2}{2!} + \dots \right] \\ &= \frac{1}{1-2\alpha} \left[ \frac{1}{1-\beta} - \left[ \frac{1}{1-\beta} \right]^{2\alpha} \right]. \end{aligned} \quad (10)$$

For general  $K$ , the recurrence relation for  $F^{(m)}$  is

$$(K-1)(1-\beta) \frac{d}{d\beta} F^{(m)} + \frac{2-K}{\beta} F^{(m)} = K F^{(m-1)}.$$

Defining

$$G^{(m)}(\beta) = \left[ \frac{\beta}{1-\beta} \right]^{(2-K)/(K-1)} F^{(m)}(\beta),$$

one obtains

$$(1-\beta) \frac{dG^{(m)}}{d\beta} = \frac{K}{K-1} G^{(m-1)}.$$

The final result is

$$\begin{aligned} F(\alpha, \beta) &= \left[ \frac{\beta}{1-\beta} \right]^{(K-2)/(K-1)} G(\alpha, \beta), \\ G(\alpha, \beta) &= \frac{(1-\beta)^{-K\alpha/(K-1)}}{K-1} \\ &\quad \times \int_1^{1/(1-\beta)} \left[ \frac{(x-1)^{2-K}}{x^{K\alpha}} \right]^{1/(K-1)} dx, \end{aligned}$$

and the subsequent calculations are somewhat involved, so in the following we give the details only for  $K=2$ .

The mean level number  $m_N$  is directly obtained from the derivative of  $F$ :

$$m_N = \frac{1}{N} \sum_m m A_N^m,$$

so

$$\begin{aligned} \frac{dF}{d\alpha} \Big|_{\alpha=1} &= \sum_n n m_n \beta^n \\ &= -\frac{2\beta}{(1-\beta)^2} - 2 \frac{\ln(1-\beta)}{(1-\beta)^2} \\ &= 2 \sum_n \beta^n \left[ \left(1 + \frac{1}{2} + \dots + 1/n\right)(n+1) - 2n \right]. \end{aligned}$$

This gives

$$\begin{aligned} m_N &= 2 \left[ \frac{N+1}{N} [\psi(N+1) + \gamma] - 2 \right] \\ &\sim 2(\ln N + \gamma - 2), \end{aligned} \quad (11)$$

where  $\psi(n)$  is the digamma function and  $\gamma$  is Euler's constant ( $\gamma=0.5772\dots$ ).

The leading term in  $m_N$  can also be derived through a recurrence relation. One has for general  $K$ ,

$$m_{N+1} \sim [1 - O(1/N^2)]m_N + \left[ \frac{K}{K-1} \right] \frac{1}{N},$$

and hence

$$m_N \sim \left[ \frac{K}{K-1} \right] \ln N, \tag{12}$$

which can be checked using the explicit expression of  $F(\alpha, \beta)$ .

**B. Particle distribution**

To study the shape of the distribution  $A_N^m$  around the mean level number  $m_N$ , it is convenient to expand Eq. (10) as

$$F = \frac{1}{1-2\alpha} \sum_n \beta^n \left[ 1 - \frac{\Gamma(2\alpha+n)}{\Gamma(2\alpha)\Gamma(n+1)} \right],$$

so that

$$\begin{aligned} A_N^m &= \frac{1}{2\pi i} \oint \frac{d\alpha}{\alpha^{m+1}} \frac{1}{1-2\alpha} \left[ 1 - \frac{\Gamma(2\alpha+N)}{\Gamma(2\alpha)\Gamma(N+1)} \right] \\ &= 2^m \frac{1}{2\pi i} \oint \frac{dx}{x^{m+1}} \frac{1}{1-x} \left[ 1 - \frac{\Gamma(x+N)}{\Gamma(x)\Gamma(N+1)} \right]. \end{aligned} \tag{13}$$

Different limits have to be studied separately.

**1.  $m > N$**

If  $m > N$ , the contour can be stretched to infinity and runs through the saddle point at  $x_0 = m/\ln N$ . The pole term cancels the 1 in the brackets and obtains

**2.  $m < N$**

If  $m < N$ , we write (13) as

$$A_N^m = 2^m \left[ 1 - \frac{1}{2\pi i} \oint \frac{dx}{x^{m+1}} \frac{1}{1-x} \frac{\Gamma(x+N)}{\Gamma(x)\Gamma(N+1)} \right]. \tag{14}$$

(i) If  $m \geq \ln N$ , one can deform the contour so that it runs through the saddle point at  $x_0 = m/\ln N$ . The pole term cancels the 1 in the brackets and one obtains

$$A_N^m \sim 2^m \left[ \frac{1}{2\pi i} \int_C \frac{dx}{x^m} \frac{\Gamma(x+m)}{\Gamma(N+1)} \right] \frac{1}{x_0(x_0-1)\Gamma(x_0)},$$

where the contour  $C$  is parallel to the imaginary axis and runs through  $x_0$ . This gives

$$\begin{aligned} A_N^m &\sim \frac{2^m}{\sqrt{2\pi m} [(m/\ln N) - 1] \Gamma(m/\ln N)} \\ &\times \exp \left[ -m \ln \left[ \frac{m}{\ln N} \right] + m - \ln N \right]. \end{aligned} \tag{15}$$

In the vicinity of the mean level  $m_N \sim 2\ln N$ , one can expand this expression and obtain

$$A_N(\delta) \sim \frac{N}{2\sqrt{\pi \ln N}} \exp(-\delta^2/4 \ln N), \tag{16}$$

with  $\delta = m - 2\ln N$ . The particle distribution is therefore Gaussian for large clusters, and there is a ‘‘surface region’’ of half-width  $2(\ln N)^{1/2}$  that contains essentially all the particles when  $N$  goes to infinity. This is a peculiar feature of the tree model, but in three dimensions the surface region already contains a number of particles of order  $N^\lambda$ , with  $\lambda \sim 0.74$ .<sup>9,11</sup>

(ii) For  $m < \ln N$ , the contour is not taken past the pole at  $x=1$  when deformed, and

$$A_N^m = 2^m - \bar{A}_N^m, \tag{17}$$

where  $\bar{A}_N^m$  is the absolute value of expression (15).

(iii) For  $m \ll \ln N$ , expression (14) for  $A_N^m$  can be usefully written using the definition of Stirling numbers  $S_N^{(m)}$ :

$$X(X-1)(X-2) \cdots (X-n+1) = \sum_m S_n^{(m)} X^m.$$

Thus one obtains

$$\begin{aligned} A_N^m &= 2^m \left[ 1 - \sum_{k=1}^m (-1)^{n-k} \frac{S_N^{(k)}}{N!} \right] \\ &= 2^m \left[ 1 - \sum_{k=1}^m \frac{|S_N^{(k)}|}{N!} \right]. \end{aligned}$$

Now, using the asymptotic form of Stirling numbers gives

$$A_N^m \sim 2^m \left[ 1 - \frac{1}{N} \sum_{k=1}^m \frac{(\gamma + \ln N)^{k-1}}{(k-1)!} \right]$$

for  $m = o(\ln N)$ .

**C. Occupation density**

It is instructive to consider in detail the density of occupied sites on level  $m$ ,

$$\rho_N^m = 2^{-m} A_N^m.$$

According to Eqs. (15) and (17), this density varies abruptly around  $m = \ln N$ . Its variation between two successive generations is

$$\begin{aligned} \delta \rho_N^m &= 2^{-m} A_N^m - 2^{-(m-1)} A_N^{m-1} \\ &= -\frac{1}{2\pi i} \oint \frac{dx}{x^{m+1}} \frac{\Gamma(x+N)}{\Gamma(x)\Gamma(N+1)} \\ &\sim -\frac{1}{\sqrt{2\pi \ln N}} \exp \left[ -\frac{(m - \ln N)^2}{2 \ln N} \right]. \end{aligned}$$

For the density this gives

$$\rho_N^m \sim \frac{1}{2} \operatorname{erfc} \left[ \frac{m - \ln N}{\sqrt{2 \ln N}} \right]. \tag{18}$$

Numerical results obtained by growing aggregates on a computer are compared to these analytic results in Fig. 2, and the agreement is quite good. It is quite striking that

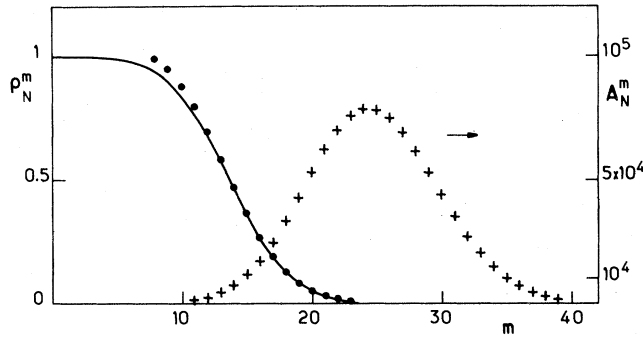


FIG. 2. Density profile and particle distribution function for  $N = 10^6$  particles. The circles and the pluses represent averages over 50 samples. The curve corresponds to Eqs. (15) and (17).

the density has a sharp boundary at a distance  $\ln N$ , while the particle distribution is concentrated in a region around  $2 \ln N$ . Since the width of both is of order  $(\ln N)^{1/2}$ , there is no overlap between these two characteristic regions for large  $N$ . Of course, the tree structure is quite special, but this may indicate that the surface of an aggregate has to be defined carefully, and that its location may depend on the property under study.

#### D. Gyration radius

The knowledge of the particle distribution function is not sufficient to obtain the gyration radius of the cluster, since within a given level the particles are correlated—if a site is empty on level  $m$ , the  $K$  corresponding sites are empty on level  $m + 1$ .

Following the classic work of Zimm and Stockmeyer,<sup>12</sup> we use a theorem due to Kramers for the average square radius of a cluster containing no loops:

$$\langle R^2 \rangle = \frac{1}{N^2} \sum_{l=1}^{N-1} \langle N_l(N - N_l) \rangle. \quad (19)$$

The summation in (19) is over the bonds (taken of unit length) between occupied sites of the cluster.  $N_l$  and  $N - N_l$  are the number of sites in the two connected components of the cluster when the  $l$ th bond is cut. This formula was originally derived for a Gaussian branched polymer, and rests on the assumption that

$$\langle \bar{u}_{ij} \cdot \bar{u}_{jk} \rangle = 0,$$

where  $\bar{u}_{ij}$  is the unit vector along the bond between sites  $i$  and  $j$ . It is therefore also valid for a cluster on a Cayley tree if the tree is locally embedded in a hypercubic lattice of very large dimensionality, so that any two consecutive bonds may be regarded as orthogonal.

In order to evaluate expression (19), we introduce three generating functions corresponding to different sets of clusters.

(i) Let  $g_N$  be the number of clusters of  $N$  sites on a tree where all sites have the same connectivity, including the

root. Then the number of empty boundary sites is  $(K - 1)N + 2$ , and since the order in which the sites are filled is important (in contrast with lattice animals), we have

$$g_N = \prod_{n=1}^{N-1} [(K - 1)n + 2] = \frac{1}{2} (1 - K)^N \prod_{n=0}^{N-1} \left[ \frac{2}{1 - K} - n \right],$$

and the generating function is

$$g(x) = \sum_{N=1}^{\infty} \frac{x^N}{N!} g_N = \frac{1}{2} [(1 - kx)^{-2/k} - 1], \quad (20)$$

with  $k = K - 1$ .

(ii)  $\tilde{g}_N$  is the number of clusters when the root has only  $K$  neighbors, as in Fig. 1:

$$\tilde{g}_N = \prod_{n=0}^{N-1} [(K - 1)n + 1],$$

and for convenience we define the generating function as

$$\tilde{g}(x) = \sum_{N=0}^{\infty} \frac{x^N}{N!} \tilde{g}_N = (1 - kx)^{-1/k}. \quad (21)$$

(iii) Finally,  $g(N, s)$  is the number of clusters in which  $n_0$  site follows the site occupied by the  $s$ th particle. Counting the boundary sites one obtains

$$g(N, s) = g_s \prod_{h=0}^{N-1-s} [(s + h - 1)(K - 1) + 1],$$

and the generating function,

$$g(x, s) = \sum_{N=s}^{\infty} \frac{x^N}{(N - s)!} g(N, s) = g_s x^s (1 - kx)^{1-s-1/k}. \quad (22)$$

We can now obtain the number of clusters that fall into two components of size  $n + 1$  and  $N - n - 1$  when one cuts the bond created by the addition of the  $s$ th particle (for definiteness the component of size  $N - n - 1$  is linked to the root). This number is

$$G(N, n, s) = \tilde{g}_{n+1} g(N - n, s) \frac{(N - s)!}{(N - n - s)! n!}, \quad (23)$$

since, after the  $s$ th particle is added, there remains  $n$  particles for one branch and  $N - n - s$  particles for the other branch, which are to be added in all possible orders. Averaging over the  $(N - 1)g_N$  configurations we obtain

$$\langle R^2 \rangle = \frac{1}{N^2 g_N} \sum_{s=2}^N F_s(N), \quad (24)$$

with

$$F_s(N) = \sum_{n=0}^{N-s} (n + 1)(N - n - 1) G(N, n, s).$$

The generating function

$$F_s(x) = \sum_{N=2}^{\infty} F_s(N) \frac{x^N}{N!}$$

has the property

$$\begin{aligned}
x^s \frac{d^s F_s(x)}{dx^s} &= \sum_{N=2}^{\infty} \sum_{n=0}^{N-s} (n+1)(N-n-1) \tilde{g}_{n+1} g(N-n,s) \frac{x^N}{n!(N-s-n)!} \\
&= \sum_{N=2}^{\infty} \sum_{n=0}^{N-2} (n+1)(N-n-1) \tilde{g}_{n+1} g(N-n,s) \frac{x^N}{n!(N-s-n)!} \\
&= \left[ \sum_{n_1=1}^{\infty} n_1 \frac{x^{n_1-1}}{(n_1-1)!} \tilde{g}_{n_1} \right] \left[ \sum_{n_2=1}^{\infty} n_2 \frac{x^{n_2+1}}{(n_2+1-s)!} g(n_2+1,s) \right] \\
&= \left[ \frac{d}{dx} \left[ x \frac{d\tilde{g}}{dx} \right] \right] \left[ x^2 \frac{d}{dx} \left[ \frac{g(x,s)}{x} \right] \right], \tag{25}
\end{aligned}$$

where the second line uses the fact that  $s \geq 2$  and  $g(N-n,s)=0$  for  $n > N-s$ . Using the expressions (20), (21), and (22), we obtain

$$\frac{d^s}{dx^s} F_s(x) = g_s(s-1+x)(1+x)(1-kx)^{-s-2-2/k}, \tag{26}$$

and by considering the coefficient of  $x^{N-s}$  one obtains after some algebra,

$$\begin{aligned}
F_s(N) &= \frac{g_N}{(s+2/k)(s+1+2/k)} \left[ \frac{N(N-1)}{k^2} - (N+1+2/k)(N+2/k) \right. \\
&\quad \left. + s \left[ (N+2/k + N/k + 1)(N+2/k) - \frac{2N-1}{k} \right] - s^2 \left[ \frac{N}{k} + \frac{1}{k^2} \right] \right]. \tag{27}
\end{aligned}$$

The average square radius is given to leading order by the dominant term in Eq. (24). For  $K > 1$ ,

$$\sum_{s=2}^N F_s(N) \sim g_N \left[ \sum_{s=2}^N \frac{s}{(s+2/k)(s+1+2/k)} \right] N^2(1+1/k),$$

and finally,

$$\langle R^2 \rangle \sim \left[ \frac{K}{K-1} \right] \ln N, \tag{28}$$

and for  $K=1$ ,

$$\langle R^2 \rangle \sim N/6. \tag{29}$$

If we remember that the bonds of the tree are locally embedded on a hypercubic lattice, so that the Euclidean distance between the root and a site on level  $m$  is  $m^{1/2}$ , we see that the square of the radius of gyration [Eq. (28)] is equal to the mean level number [Eq. (12)]. For  $K=1$ , the aggregate reduces to a linear polymer and the result (29) is the expected gyration radius of such a polymer in mean-field theory.<sup>12</sup>

#### IV. DISCUSSION OF ISOTROPIC DIFFUSION

##### A. General remarks

A universal feature of the aggregates studied above is that their size is of order  $\ln N$  [or  $(\ln N)^{1/2}$  if we consider Euclidean distance]. If the tree sites were all occupied up to a maximum level  $M$  one would have  $M \sim \ln N / \ln K$ : The actual aggregates are less dense since most particles lie around level  $m_N \sim [K/(K-1)] \ln N$ . A surprising re-

sult is that this size increase is not accompanied by a smooth decrease of the density in the interior of the cluster, but by a shrinking of the "core" of the cluster and the creation of a surface layer.

The  $\ln N$  behavior is to be contrasted with the behavior of lattice animals (branched polymers) on a Cayley tree, which have an average square radius of order  $N$ .<sup>5,6,12</sup> As pointed out in the Introduction, the physical reason is that a given cluster shape has very different weights in both cases. In the Eden process, compact clusters can be created in many different ways, so their weight is large, while a configuration with a very long "arm" of occupied sites can be produced in only one way, and its contribution to  $\langle R^2 \rangle$  is quite small. In the animal problem such configurations actually dominate  $\langle R^2 \rangle$ .

##### B. Fractal dimensionality

A more precise statement of "compactness" involves the fractal dimension  $D$  defined in the Introduction. A  $\ln N$  dependence of the cluster size corresponds to an infinite  $D$ , in agreement with general arguments of Witten and Sander,<sup>2</sup> which state that there exists no critical dimensionality above which  $D$  becomes constant for the diffusion model as well as for the Eden model.

The effective dimension of the tree is itself infinite, and therefore nontrivial fractal properties may appear only in properties that depend on the difference  $d-D$ . This is the case for the average density at a distance  $r$  from the cluster seed:

$$\rho(r) \sim r^{d-D}. \tag{30}$$

On a tree, the number of sites on level  $m$  varies exponentially (as  $K^m$ ), but if the number of occupied sites is of the form

$$A_N^m \sim m^{-\alpha} K^m \quad (31)$$

for some range of levels, one may have

$$\rho(m) \sim m^{-\alpha} \quad (32)$$

in that range.

The theories of Ball *et al.*<sup>8</sup> and of Muthukumar<sup>10</sup> both predict that

$$d - D \rightarrow 1, \quad d \rightarrow \infty,$$

and thus one might hope to observe a power-law behavior of type (32) for the density of diffusion-controlled aggregates on the tree.

The explicit expression we have obtained shows no such power-law behavior, suggesting that  $\lim(d - D) = 0$  for the aggregates studied here. This is in agreement with the general expectation for Eden growth, and to our knowledge it is the first case where it can be rigorously shown that Eden clusters are compact. However, this result seems to contradict the conclusion of Ball *et al.*<sup>8</sup> for diffusion-limited aggregation (DLA), and raises the question of the high-dimensionality limit of DLA. The spherically averaged model of Ball *et al.*<sup>8</sup> does not take into account the large fluctuations which are inherent in the geometry of a fractal object, and replaces a few occupied regions in an empty space by a low uniform density. It is not clear whether or not this approximation becomes valid for high dimensions or overestimates the screening of the interior by the outer regions.

On the other hand, the tree model neglects the screening effect altogether: An incoming particle is not prevented from entering the interior of the cluster since different branches do not interact. It was not obvious from the start that statistical effects alone could not give rise to fractal features: After all, there is no general proof that Eden clusters are compact, and screening plays no role for lattice animals which have a fractal dimension  $D=4$  on a tree. The conclusion of our study is that this is not the case, and screening is indeed essential to give a nonzero  $d - D$ . A better model should take it into account, for instance, by a perturbative approach, but we have not found a simple and tractable way to allow for interbranch diffusion in the tree model.

### C. Interface structure

The fact that the density profile has a sharp variation at a smaller distance from the root than  $m_N$  indicates that, in the intermediate region between the core and the surface, a delicate balance is struck between the growing number of available sites and the decreasing occupation density. A huge reservoir of particles exists at the surface, and small perturbations may have a large effect by allowing a few particles to "leak" into the interior and alter the density profile radically. This suggests that modifications of the model may lead to a different situation and maybe even to a power-law behavior of the form (32).

The width of the particle distribution function on the tree is related to the average cluster size  $m_N$  by [see Eq. (16)]

$$W \sim (\ln N)^{1/2} \sim (m_N)^{1/2}. \quad (33)$$

This is very different from the result  $W \sim \text{const}$  for compact objects suggested by an argument presented by Witten<sup>9</sup> for DLA in the smoothed-density approximation, and this is also different from  $W \sim R$ , as found numerically by Peters *et al.*,<sup>13</sup> for Eden clusters of radius  $R$  in dimensions  $d=2$  and 3. This indicates that the surface region has an interesting structure which should be investigated in more detail.

## V. GROWTH IN THE PRESENCE OF BIASED DIFFUSION

### A. Formulation

Having obtained an analytic solution for isotropic diffusion, it is natural to investigate the case of biased diffusion. Physically, this is a very crude model for the effect of long-range interactions, such as Van der Waals forces or electrostatic forces, which are likely to be present in many realistic situations. A more elaborate model would involve diffusion probabilities depending on the distance between the particles and the cluster, but here we just want to explore the basic features using the simplest possible assumptions.

It was shown in Sec. II that if a particle has probability  $a$  ( $< \frac{1}{2}$ ) to jump towards the root of the tree, it has probability  $a/(1-a)$  to eventually reach level  $m-1$ , starting from level  $m$ . Along lines similar to those followed for isotropic diffusion, one then finds that for  $a < \frac{1}{2}$ , the probability  $P_m$  that a new particle starting on level  $M$  is finally added to the cluster on level  $m$  is

$$P_m = Cq^{M-m} b_N^m. \quad (34)$$

$C$  is a normalization factor, and  $q = Ka/(1-a)$ ,  $a_N^m$ , and  $b_N^m = Ka_N^{m-1} - a_N^m$  are now, respectively, the number of occupied sites and of empty boundary sites on level  $m$  for a given sample. This gives the growth equations

$$a_{N+1}^m - a_N^m = 1 \quad \text{with probability } P_m = \frac{q^{-m} b_N^m}{\sum_1^M q^{-m} b_N^m},$$

$$= 0 \quad \text{with probability } (1 - P_m). \quad (35)$$

$M$  denotes the highest level with an empty boundary site.

The denominator in  $P_m$  explicitly depends on the  $a_N^m$ , in contrast with the isotropic case, so the equation is nonlinear and cannot be expressed only in terms of the average occupation numbers  $A_N^m$ . We are unable to solve the problem analytically and have to rely on a numerical study.

In the case of strongly biased diffusion ( $a > \frac{1}{2}$ ), a simplification occurs: Every new particle has probability 1 to join the cluster, so  $q = K$  and the denominator in  $P_m$  is

$$\sum_{m=2}^L q^{-m} b_N^m = \sum_{m=2}^L (a_N^{m-1}/K^{m-1} - a_N^m/K^m) = K^{-1}.$$

The recurrence relations become linear again:

$$a_{N+1}^m = \left[ 1 - \frac{1}{K^{m-1}} \right] a_N^m + \frac{1}{K^{m-2}} a_N^{m-1}, \quad (36)$$

and using the same type of methods as above it can be shown that

$$\bar{m} \sim \ln N / \ln K, \quad N \rightarrow \infty$$

and that the particle distribution is Gaussian around  $\bar{m}$ . The cluster is therefore as compact as possible, as one would expect intuitively.

**B. Numerical results**

There is no difficulty of principle to study Eq. (35) by a Monte Carlo method, but the generation of samples takes more computer time than for isotropic diffusion, and it is relatively difficult to study very large clusters.

The results obtained for  $K=2$  are displayed in Fig. 3.

(i) For an attractive bias ( $a > \frac{1}{3}$ ) the mean level number behaves as  $\ln N$ , as for the isotropic case. This is expected intuitively since the probability factor  $q^{-m}$  favors small values of  $m$ , and any "hole" in the interior of the cluster is likely to be filled rapidly, so the clusters can only be more compact than for Eden growth.

(ii) For a repulsive bias ( $a < \frac{1}{3}$ ) the situation is radically altered and the data are consistent with a linear growth:

$$m_N \sim C(a)N, \quad (37)$$

although we did not study clusters which were sufficiently large to reach the asymptotic regime. The probability factor  $q^{-m}$  now favors large values of  $m$ , and if a branch is longer than average it is likely to grow more rapidly. This mechanism leads to very elongated structures, but it is clear from Fig. 3 that there is a large crossover region for  $a$  close to  $\frac{1}{3}$ .

The particle distribution and the density profile are shown in Fig. 4 for  $N = 10^5$  particles. For one given sample very large fluctuations in the distribution are observed, i.e., nearly 2 orders of magnitude for  $N = 5 \times 10^5$ . The distribution averaged over 50 samples, for  $N = 10^5$ , is relatively smooth and quite different from the corresponding distribution for isotropic diffusion. It is very asymmetric, with a long tail for large values of the level number. The very flat maximum observed for  $m \leq \langle m \rangle$  may be due to the fact that the asymptotic regime has not been reached yet.

The density profile does not look striking, except that it has a sharp boundary at a *smaller* level than for isotropic diffusion: The core of the cluster shrinks because particles are drawn out of it towards the surface, but this effect is not strong enough to give rise to a power-law decay of the density.

**VI. CONCLUSION**

The model we have studied confirms the expectation that in high dimensions the clusters grown by diffusion-

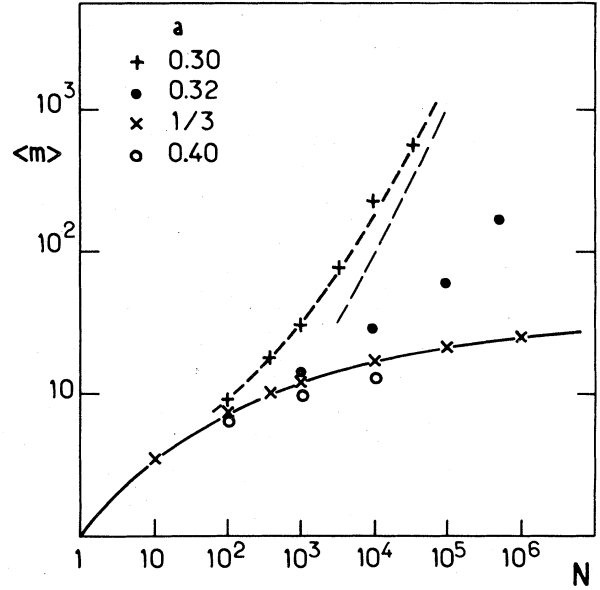


FIG. 3. Mean level number as a function of number of particles for biased diffusion. Isotropic diffusion corresponds to  $a = \frac{1}{3}$ . The dashed straight line indicates an asymptotic behavior  $\langle m \rangle \sim N/100$ .

controlled aggregation are very different from lattice animals, and that their fractal dimension does not tend to a finite limit. It suggests that Witten-Sander aggregation may be viewed as a perturbation of Eden growth, the two processes being identical on a Cayley tree, and that a field theory of dynamic growth should begin with a formulation for the Eden mechanism.

It may also be that simple modifications would lead to a different behavior, and, in particular, the results for biased diffusion suggest that a more elaborate model, with the diffusion bias a function of distance, may lead to a different law for cluster size. This would be in analogy

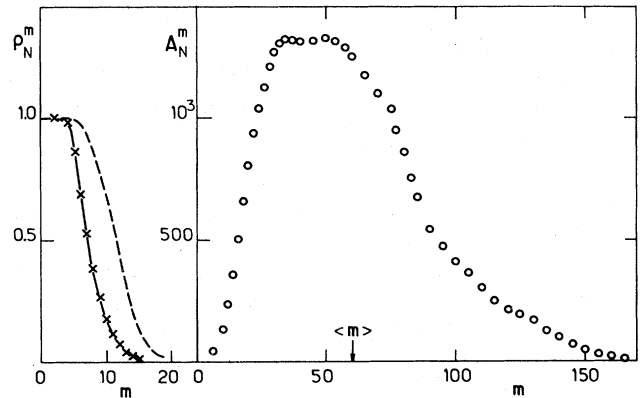


FIG. 4. Density profile and particle distribution averaged over 50 samples of  $N = 10^5$  particles for repulsive biased diffusion ( $a = 0.32$ ). The dashed line gives the density for isotropic diffusion. Note the different scales for  $\rho_N$  and  $A_N$ .



with the effect of long-range forces on critical exponents. In the same context, Monte Carlo studies on aggregation in the presence of long-range forces, both attractive and repulsive, seem worthwhile. It would be very interesting to know whether the fractal dimensionality of the aggregates changes, this being a step towards a classification of universality classes for growth models.

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\*Permanent address: University of Guelph, Department of Physics, Guelph, Canada N1G 2W1.

- <sup>1</sup>T. A. Witten and L. M. Sander, *Phys. Rev. Lett.* **47**, 1400 (1981).
- <sup>2</sup>T. A. Witten and L. M. Sander, *Phys. Rev. B* **27**, 5686 (1983).
- <sup>3</sup>P. Meakin, *Phys. Rev. A* **27**, 604 (1983); **27**, 1495 (1983).
- <sup>4</sup>H. P. Peters, D. Stauffer, H. P. Hölters, and K. Loewenich, *Z. Phys. B* **34**, 399 (1979).
- <sup>5</sup>D. A. Klarner, *Can. J. Math.* **10**, 581 (1967); T. C. Lubensky and J. Isaacson, *Phys. Rev. A* **20**, 2130 (1979).
- <sup>6</sup>M. E. Fisher and J. W. Essam, *J. Math. Phys.* **2**, 609 (1961).
- <sup>7</sup>H. Gould, F. Family, and H. E. Stanley, *Phys. Rev. Lett.* **50**, 686 (1983).
- <sup>8</sup>R. Ball, M. Nauenberg, and T. Witten, Institute of Theoretical Physics, University of California, Santa Barbara, Report No. 83-37 (unpublished).
- <sup>9</sup>T. A. Witten, Institute of Theoretical Physics, University of California, Santa Barbara Report No. 82-162.
- <sup>10</sup>M. Muthukumar, *Phys. Rev. Lett.* **50**, 839 (1983).
- <sup>11</sup>P. Meakin and T. A. Witten, *Phys. Rev. A* **28**, 2985 (1983).
- <sup>12</sup>B. H. Zimm and W. H. Stockmeyer, *J. Chem. Phys.* **17**, 1301 (1949); G. Dobson and M. Gordon, *ibid.* **41**, 2389 (1964).
- <sup>13</sup>H. P. Peters, D. Stauffer, H. P. Hölters, and K. Loewenich, *Z. Phys. B* **34**, 399 (1979).