Exact exponents for infinitely many new multicritical points

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Andrews, Baxter, and Forrester have recently solved two infinite sequences of models that are generalizations of the Ising and hard-square models and exhibit multicriticality. The nature of these new two-dimensional multicritical points is examined in order to determine what universality classes they may represent. Appropriate order parameters are defined and their critical exponents reported. One infinite sequence of multicritical points are continuous melting transitions of $p\times 1$ commensurate ordered phases for all integers $p\geq 2$. The other sequence appears to consist of "generic" multicritical points terminating lines of n-phase coexistence, again for all integers $n\geq 2$. The multicritical exponents in this latter sequence coincide with those found by Friedan, Qiu, and Shenker on the basis of assuming conformal invariance and unitarity.

I. INTRODUCTION

Only a small number of statistical mechanical models exhibiting critical phenomena have been exactly solved. They are the two-dimensional Ising model,[1,2] the six-vertex model,[3] the eight-vertex model,[4,5] and, more recently, Baxter's hard-square lattice gas (which includes the hard-hexagon model as a special case).[5–9] The Ising model has the simplest critical point, at which *two* coexisting phases become indistinguishable; the order parameter is simply the difference between the two coexisting phases and vanishes with a critical exponent $\beta=\frac{1}{8}$. *Three* coexisting phases become simultaneously indistinguishable at either of the critical points exhibited by Baxter's hard-square model,[5–9] thus let us call them threefold multicritical or tricritical points. The order parameters are differences between the coexisting phases: when there are n coexisting phases then $(n-1)$ different order parameters can be constructed. Therefore, two different order parameters vanish at a tricritical point.

The three coexisting $\sqrt{3}\times\sqrt{3}$ commensurate phases in the hard-hexagon model are simply spatial translations of one another.[5,6] This special symmetry under cyclic permutation of the phases (Z_3 symmetry) dictates that the two order parameters differ only by a multiplicative factor and vanish with the same exponent, $\beta=\frac{1}{9}$, at the critical point. Thus the hard-hexagon—model critical point is a special tricritical point that terminates a line of three-phase coexistence with the three phases being related by a Z_3 symmetry. The other tricritical point exhibited by Baxter's exactly solved hard-square model terminates a line of three-phase coexistence where the three phases are *not* related by any special symmetry.[7–9] The three phases involved here are a fluid phase and two $\sqrt{2}\times\sqrt{2}$ commensurate solid phases.[7–9] The leading order parameter is the difference between the two solid phases and vanishes with a tricritical exponent of $\beta_1=\frac{3}{32}$.[8] The second order parameter is the density difference between the fluid and solid phases and has exponent $\beta_2=\frac{1}{4}$.[8] The lack of any special symmetry relating the two order parameters suggests that this latter tricritical point represents the universality class of "generic" tricriticality, unlike the hard-hexagon model with its special symmetry.

Andrews, Baxter, and Forrester[10] have recently generalized the Ising model with its two-phase coexistence, and the hard-square model, with its three-phase coexistence, to an infinite sequence of exactly solved models, with n-phase coexistence for all integers n. These models are defined on a square lattice with the degree of freedom at each site of the lattice being a spin or "height" variable that may take on only a finite number of discrete states. A more detailed description of these "restricted solid-on-solid" (RSOS) models is given in Sec. II of this paper (see also Ref. 10). Andrews *et al.* number the models in their sequence by the integers $r\geq 4$; $r=4$ is simply the spin-$\frac{1}{2}$ Ising model, $r=5$ is the hard-square lattice gas, and the models with $r>5$ are new.[10] As r increases, so does the number of different states allowed to each local spin or height variable and, consequently, the number of parameters necessary to fully specify the interactions in the model also increases. The full phase diagram of the general RSOS model is thus a multidimensional parameter space. Andrews *et al.*[10] have succeeded in solving each model only on two-dimensional manifolds within this larger space. For each model, $r=4,5,\ldots,$ they have found two distinct manifolds of exact solution.[10] Each such manifold is divided into two phases by a line of critical points and may be parametrized by the variables t and v, with the critical line being $t=0$. [The parameter t used here to measure deviation from criticality is defined as $t=-p$, where p is the parameter used in Ref. 10 in, for example, their Eq. (1.5.4).] The order parameters and critical behavior are independent of the parameter v,[10] which, roughly speaking, measures the spatial anisotropy of the interactions in the model. For simplicity, this parameter v will be ignored for the remainder of this paper; it may be assumed to be fixed at some value for each manifold of exact solution.

Ignoring v, the manifolds of exact solution are simply lines parametrized by t, which runs on the interval

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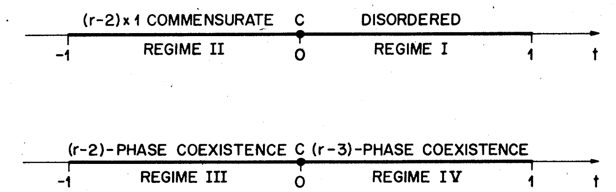

FIG. 1. Exact solution manifolds of the RSOS models. Each of the lines is divided into two regimes by a critical point C at $t=0$.

$-1<t<1$, and are shown in Fig. 1. In this representation the phase diagrams are essentially identical for each RSOS model in the infinite sequence $r=4,5,\ldots$. Each exact solution line is divided into two "regimes" by its critical point (C in Fig. 1) at $t=0$. One line consists of regime I for $0<t<1$, which is simply a disordered phase, and regime II for $-1<t<0$, which is a line of $p=(r-2)$-phase coexistence.[10] The critical point here is therefore an order-disorder transition, with increasing t acting much like increasing temperature. The other line of exact solution consists of regime III for $-1<t<0$, which is a line of $(r-2)$-phase coexistence, and regime IV for $0<t<1$, which is a line of $(r-3)$-phase coexistence.[10] Thus for $r>4$ the critical point here separates two ordered phases and is of a very different nature than that separating regimes I and II.

The purpose of this paper is to examine these critical points in some more detail, in order to determine what universality classes they may represent. The coexisting phases in regime II all have ordering that is spatially modulated along one diagonal of the square lattice, with a period of precisely $p=(r-2)$ next-nearest-neighbor spacings; the ordering is spatially uniform along the other diagonal.[10] The p phases differ only in how this nonuniform ordering is placed on the lattice; they are simply translations of one another. The ordering present in each phase of regime II therefore breaks the translational invariance of the underlying lattice such that the unit cell of each ordered phase is $p=(r-2)$ times as long as that of the disordered phase in one direction, but of the same width. Such commensurate ordered phases are denoted $p\times1$.[11] It has been argued[11] that the continuous melting of a $p\times1$ commensurate phase, such as we have at the critical point separating regimes I and II, should be in the same universality class as a multicritical point of the p-state clock model, whose Hamiltonian is invariant under cyclic permutations of the p states allowed to the local "spin" variables.[11] The critical exponents exhibited by the exact solution[10] are consistent with this expectation for those cases, namely $p=2$, 3, and 4, in which the exponents of the clock model are known, as shown in Sec. III of this paper. For $p>4$, however, the clock models are generally expected to disorder either via a first-order transition or via a pair of transitions with an intermediate phase.[12] The exact multicritical points found[10] do not exhibit either of these expected behaviors and so must correspond to special points in the space of general p-state clock models, presumably to points at which the order-disorder transition crosses over from a single first-order transition to a pair of continuous transitions with an intermediate "massless" or "floating" phase with algebraically decaying spin-spin correlations.[12] This is discussed in more detail in Sec. III below.

The $n=(r-2)$ phases that coexist in regime III of the exact solution line[10] are not all simply related to one another as in regime II. In general, at a point of n-phase coexistence, as in regimes II and III, $(n-1)$-independent order parameters may be constructed; they are simply differences between the n phases. In regime III these order parameters may be defined (see Sec. IV below) such that each has its own distinct critical exponent at the multicritical point separating regimes III and IV. In regime II, on the other hand, the translational symmetry that relates the coexisting phases dictates that the order-parameter exponents are pairwise degenerate, leaving fewer distinct critical exponents. The existence of a distinct exponent for each order parameter at the regime III to IV critical point indicates that this is a generic n-fold multicritical point, with no special symmetry present relating the n coexisting phases, even in the scaling limit. Thus it appears that Andrews *et al.*[10] have found examples of generic n-fold multicritical points for all integers $n=(r-2)\geq 2$. They have solved each model only on a line approaching the multicritical point, so their exact solution does not necessarily exhibit *all* of the multicritical exponents. The exponents they do report[10] are

$$2-\alpha=r/2 \tag{1.1}$$

for the free energy on both sides (regimes III and IV) of the multicritical point and

$$\beta_1=\frac{3}{8(r-1)} \tag{1.2}$$

for the leading order parameter in regime III. Other order parameters can be constructed for $r>4$, as is shown in Sec. IV below, and the corresponding exponents in regime III are

$$\beta_k=\frac{(k+1)^2-1}{8(r-1)}\ , \tag{1.3}$$

for integers $k=2,\ldots,(r-3)$.

Friedan, Qiu, and Shenker have recently found that assuming conformal invariance and unitarity severely limits the possible values of multicritical exponents in two-dimensional systems.[13] They find an infinite sequence of possible multicritical points under these assumptions, each one with only a finite set of rational critical exponents.[13] For each such multicritical point, which they label in sequence by the integers $m\geq 3$,[13] there is a critical exponent that is consistent with the free-energy singularity (1.1) in regimes III and IV of the exact solution of Andrews *et al.*[10] for $r=m+1$. Furthermore, the corresponding $(m-2)=(r-3)$ leading order-parameter exponents allowed by the assumptions of Friedan *et al.*[13] also coincide precisely with the exponents (1.2) and (1.3) found in regime III of the exact solution. Thus Andrews *et al.*[10] have found an exact multicritical point for each set of critical exponents of Friedan *et al.*[13] The infinite sequence of possible multicritical points of Friedan *et al.*[13]

therefore appears to represent the universality class of generic $(m-1)$-fold multicritical points.

II. THE MODELS

The models solved by Andrews *et al.* are presented as restricted solid-on-solid (RSOS) models on a square lattice.[10] At each site, i, of the lattice is an integer "height," l_i, which is restricted to the interval

$$1 \le l_i \le (r-1) , \tag{2.1}$$

with the integer $r \ge 4$. The heights at nearest-neighbor sites of the lattice, l_i and l_j, must differ by unity:

$$| l_i - l_j | = 1 . \tag{2.2}$$

The Boltzmann weight of a given configuration is simply the product of weights assigned to each 4-site square plaquette of the lattice.[10] These plaquette weights W are determined by the heights at the four sites, l_1,l_2,l_3,l_4, counting clockwise from the northwest (NW) corner [see Fig. 2(a)], and have the symmetries

$$\begin{aligned} W(l_1,l_2,l_3,l_4) &= W(l_3,l_2,l_1,l_4) \\ &= W(l_1,l_4,l_3,l_2) \\ &= W(r-l_1,r-l_2,r-l_3,r-l_4) . \end{aligned} \tag{2.3}$$

The restriction (2.2) imposes a sublattice structure on the system:[10] the full square lattice may be divided into two interpenetrating sublattices of next-nearest-neighbor sites, let us call them odd and even. If we demand, as a further restriction on the model, that the height l_i at one site on the odd sublattice be an odd integer, then all the other heights on the odd sublattice must also be odd and all the heights on the even sublattice must be even.[10]

The general RSOS model as specified in the preceding paragraph still has many free parameters, namely the weights $W(l_1,l_2,l_3,l_4)$. Some of this freedom has no physical significance; thus changing the weights via

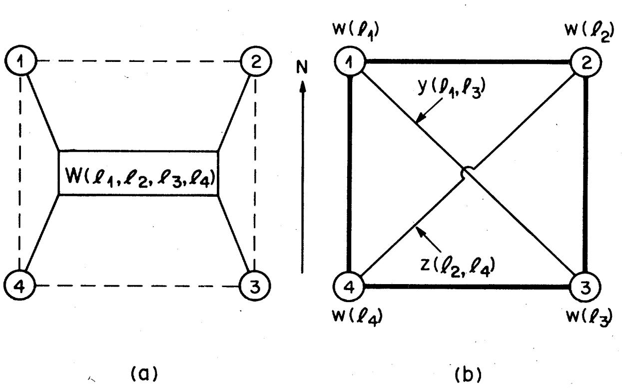

FIG. 2. Interactions in each four-site square plaquette of the RSOS model. (a) Andrews *et al.* presented the model using general four-site interactions W, coupling the four sites, 1, 2, 3, and 4. (b) The model may also be presented as one with only one-site interactions, represented by the $w(l)$, nearest-neighbor interaction of a very restrictive nature, given by (2.2) and represented by the thick lines, and diagonal interactions represented by y and z.

$$W(l_1,l_2,l_3,l_4) \to W'(l_1,l_2,l_3,l_4) = \frac{CW(l_1,l_2,l_3,l_4)X(l_1)X(l_3)}{X(l_2)X(l_4)} , \tag{2.4}$$

where $X(l)=X(r-l)$ and C and the $\{X(l)\}$ are otherwise arbitrary positive real numbers, does not alter *relative* weights of configurations with periodic boundary conditions. The remaining number of physically relevant parameters in the model is $(3r-8)/2$ for even r and $(3r-9)/2$ for odd r. Andrews *et al.*[10] have solved the model only on two-dimensional manifolds within the full $[(3r-8)/2]$- or $[(3r-9)/2]$-dimensional space of physically distinct models with weights satisfying (2.3). They parametrize the weights on their exact solution manifold using elliptic θ functions. The precise formulas are given in Ref. 10, but do not appear to be very illuminating. The weights are all analytic functions of t for $-1<t<1$.

The presence of plaquette or four-site interactions in this model appears to inhibit understanding, but by utilizing the freedom of (2.4) the plaquette weights can be transformed and factorized as

$$W'(l_1,l_2,l_3,l_4)=w(l_1)w(l_2)w(l_3)w(l_4)y(l_1,l_3)z(l_2,l_4) , \tag{2.5}$$

where the one-site weights satisfy

$$w(l)=w(r-l) , \tag{2.6}$$

and the weights for the next-nearest-neighbor interactions satisfy

$$\begin{aligned} &y(l,l')=z(l,l')=1 \quad \text{for } l \ne l' , \\ &y(l,l) \equiv y_l = y_{r-l}, \quad z(l,l) \equiv z_l = z_{r-l} , \end{aligned} \tag{2.7}$$

for all l. Thus the model may be presented in, perhaps, more familiar terms as one with single-site "interactions," represented by the weights $\{w(l)\}$, nearest-neighbor interactions of a very restrictive nature (2.2), represented by the thick lines in Fig. 2(b), and next-nearest-neighbor interactions, represented by the weights $\{y(l_1,l_3),z(l_2,l_4)\}$, see Fig. 2(b).

For r even, the model may also be translated into an Ising model by changing to spin variables

$$s_i=(r-2l_i)/4 . \tag{2.8}$$

This produces spin-$[(r-2)/4]$ Ising spins on the odd sublattice and spin-$[(r-4)/4]$ spins on the even sublattice. For $r=4$ the spin-0 spins on the even sublattice have only one possible state ($s=0$, $l=2$) and may be ignored. The spin-$\frac{1}{2}$ spins on the odd sublattice then interact only with their closest neighbors on that sublattice. These diagonal interactions are specified by the two parameters $y_1=y_3$ and $z_1=z_3$. Thus in this, the simplest case, $r=4$, the RSOS model is equivalent to an anisotropic spin-$\frac{1}{2}$ Ising model in zero field with only nearest-neighbor interactions on a square lattice, as solved by Onsager.[1] For $r=6$ we have spin-$\frac{1}{2}$ spins on the even sublattice and spin-1 spins on the odd sublattice. The nearest-neighbor restriction (2.2) is $|s_i-s_j|=\frac{1}{2}$ in terms of these spins; for $r=6$ the only configurations forbidden are those with antiparallel nearest-neighbor spins. This restriction is the only interaction between sublattices. Thus we have an anisotropic spin-$\frac{1}{2}$ Ising model on the even sublattice, with two coupling parameters, one for each diagonal. On the odd sublattice we have a spin-1 Ising or Blume-Capel[14] model, with again a coupling parameter for each diagonal and also a one-site coupling to the magnitude of the spin. This gives a fairly physical representation of the five parameters in the $r=6$ model. For larger r the model becomes more and more complicated and this representation as an Ising model probably becomes less useful.

For r odd, Andrews *et al.*[10] propose a lattice-gas representation of the model, changing to the occupation variables

$$\sigma_i=\begin{cases}(l_i-2)/2 & \text{for } l_i \text{ even },\\ (r-l_i-2)/2 & \text{for } l_i \text{ odd }.\end{cases} \tag{2.9}$$

This produces a lattice gas with maximum occupation $(r-3)/2$. The generalized hard-hexagon (or hard-square) model as solved by Baxter[6] is the case $r=5$ and is normally presented in this lattice-gas representation.[5–9] Again, the model becomes rapidly more complicated as r increases and it is not clear if this lattice-gas representation is useful for $r>5$.

III. REGIMES I AND II; $p\times 1$ COMMENSURATE MELTING

In regime I the restricted SOS model is disordered for all r.[10] This means that the probability

$$P_a(i)\equiv\langle\delta_{a,l_i}\rangle\,, \tag{3.1}$$

of finding the height $l_i=a$ at a given site i in the interior of the system is independent of the boundary conditions in the thermodynamic limit.[10] In regime II, on the other hand, there are $p=(r-2)$ distinct ground states and as many corresponding ordered phases, so the local ordering can be affected by far boundary conditions. The ground states in regime II have a spatially nonuniform structure; they are invariant under global translations along the SW to NE diagonal of one next-nearest-neighbor spacing, but along the other (NW to SE) diagonal they are invariant only under translations by an integer multiple of $p=(r-2)$ next-nearest-neighbor spacings.[10] A ground state for $r=6$ is illustrated in Fig. 3. As one moves along a row of the square lattice the height variables, l_i, in each ground state move up and down within the allowed limits, first increasing in unit steps to $(r-1)$, then decreasing to 1, then increasing to $(r-1)$, etc.[10] A unit cell of this ground state can be chosen to be $(2r-4)$ adjacent sites in a row, numbered $n=0,1,2,\ldots,(2r-5)$ (west to east, see Fig. 3), with the odd sites on the odd sublattice. The $(r-2)$ ground states and their corresponding phases may then be numbered by the odd integer $1\le j\le(2r-5)$ so that in each ground state $l_j=1$. The pattern of order in each phase is the same; they differ from each other only in how that order is placed on the underlying lattice. Thus the probabilities $\{P_a^{(j)}(n)\}$ in the different phases j are related via[10]

$$P_a^{(j)}(n)=P_a^{(j')}(n') \quad \text{if } j-n=j'-n' \;[\text{mod}(2r-4)]\,. \tag{3.2}$$

1	2	3	4	5	4	3	2	1	2	3	4	5
2	3	4	5	4	3	2	1	2	3	4	5	4
3	4	5	4	3	2	1	2	3	4	5	4	3
4	5	4	3	2	1	2	3	4	5	4	3	2
5	4	3	2	1	2	3	4	5	4	3	2	1
4	3	2	1	2	3	4	5	4	3	2	1	2
3	2	1	2	3	4	5	4	3	2	1	2	3
2	1	2	3	4	5	4	3	2	1	2	3	4

FIG. 3. Ground state in regime II of the $r=6$ RSOS model. The height variable l_i is given at each site of the square lattice. The long horizontal boxes denote unit cells of this 4×1 commensurate ordered phase. The unit cell of the disordered phase in regime I contains two sites, one on each sublattice.

The modulated ordering in these phases is commensurate with the lattice, with the unit cell of the ordered phases being p times as long as the unit cell of the disordered phase (which consists of two sites) in one direction, but of the same width. Such commensurate ordered phases are denoted $p\times 1$.[11]

The multicritical transition from regime II to regime I of the exact solution surface is the melting of a $p\times 1$ commensurate phase, with $p=r-2$. It is a continuous phase transition; the free energy is singular with exponent[10]

$$2-\alpha=(p+2)/p \tag{3.3}$$

in both regimes. The $\{P_a^{(j)}(n)\}$ in regime II are given by Eq. (3.3.18b) of Ref. 10 and their expansions about the critical point $(t=0)$ are

$$P_a^{(j)}(n)=\frac{4}{r}\sin\left[\frac{\pi a}{r}\right]\times\sum_{k=0}^{r-3}\left[|t|^{k(r-2-k)/2(r-2)^2}\sin\frac{(k+1)\pi a}{r}\times\cos\frac{k\pi(j-n)}{r-2}\left[1+O(|t|^{2/(r-2)})\right]\right]. \tag{3.4}$$

The spatial Fourier transforms of these local probabilities,

$$\hat{P}_a^{(j)}(k)\equiv(r-2)^{-1}\sum_{n=0}^{(2r-5)}P_a^{(j)}(n)\exp[ikn\pi/(r-2)]\,, \tag{3.5}$$

vanish, for integer $1\le k\le(r-3)$, in each phase j for $t\to 0$ as

$$\hat{P}_a^{(j)}(k)\approx\frac{2}{r}|t|^{\beta_k}\sin\left[\frac{\pi a}{r}\right]\times\sin\left[\frac{(k+1)\pi a}{r}\right]\exp\left[\frac{ikj\pi}{r-2}\right], \tag{3.6}$$

with critical exponent

$$\beta_k = \frac{k(r-2-k)}{2(r-2)^2} . \tag{3.7}$$

These $\hat{P}_a^{(j)}(k)$ take on different values in each of the different ordered phases and therefore represent order parameters for regime II. Note that the symmetry of the ordered phases dictates that the order-parameter exponents satisfy $\beta_k = \beta_{r-2-k}$.

The continuous melting transition of a $p\times 1$ commensurate phase, such as is present in this exact solution, is expected to be in the same universality class as the disordering transition of a p-state chiral clock model.[11] Such a clock model consists of p-state local-spin variables, n_i, that may take on the values $n_i = 0, 1, \ldots, (p-1)$. The clock-model Hamiltonian, with, for example, just nearest-neighbor interactions, reflects the Z_p spin symmetry by being invariant under the global transformation

$$n_i \rightarrow n_i' = n_i + 1(\mathrm{mod}\, p) \tag{3.8}$$

that cyclically permutes the spin states. This Z_p symmetry corresponds to the invariance of the restricted SOS model under translations along the NW to SE diagonal by one next-nearest-neighbor spacing. In the ordered phase this Z_p symmetry is broken; in the restricted SOS model a spatial modulation of the local order breaks this symmetry, while in the clock model one of the p spin states is selected out in a "ferromagnetically" ordered phase. The order parameters of the clock models are

$$M_k = \langle \exp(2\pi i k n_i / p) \rangle \tag{3.9}$$

and correspond directly to those of the restricted SOS model (3.5).

For $p=2$, $r=4$, the RSOS model reduces to a spin-$\frac{1}{2}$ Ising model on the odd sublattice, as discussed in the preceding section. In regimes I and II the couplings are ferromagnetic along the SW to NE diagonal and antiferromagnetic along the other diagonal. This produces a 2×1 antiferromagnetically ordered phase in regime II. The two-state clock model is also precisely an Ising model so both it and the $r=4$ RSOS model have the same critical behavior with $\alpha=0$, $\beta=\frac{1}{8}$.

For $p>3$ a p-state clock-model Hamiltonian may or may not be invariant under the transformation

$$n_i \rightarrow n_i' = p - n_i \tag{3.10}$$

which reverses the ordering of the spin states (thus interchanging clockwise and counterclockwise). If this (3.10) is a symmetry of the system, then it is a *symmetric* clock model, while couplings that break this symmetry are dubbed "chiral."[11] For $p=3$ the symmetric clock model is the three-state Potts model, at whose critical point chiral symmetry breaking is known to be relevant.[11,15] Thus the continuous melting of a 3×1 commensurate phase, such as we have in regime II of the $r=5$ RSOS model, will be in the three-state Potts-model universality class only if the effective chiral symmetry breaking vanishes at the critical point. Otherwise the transition will be in a universality class governed by a fixed point with chiral symmetry breaking.[11] In the disordered phase the spatial correlation function will decay in an oscillatory fashion at long distances, with the period of the oscillations incommensurate with the lattice, if chiral symmetry breaking is present.[11] In regime I of the $r=5$ RSOS model (the generalized hard-hexagon model) such incommensurate oscillations are indeed seen.[16] However, in the scaling limit, $t\rightarrow 0$, these oscillations vanish, so they are not present in the critical scaling function for the correlation function. Thus the leading chiral symmetry-breaking operator, which is relevant, must vanish at the regime I to II critical point for $r=5$, a feature of the model which is not at all apparent from the symmetries and nature of the basic interactions (see Ref. 9 for more discussion of this). This vanishing of chiral symmetry breaking allows the critical point to be in the three-state Potts-model universality class, with exponents $\alpha=\frac{1}{3}$ and $\beta=\frac{1}{9}$.

The correlation functions in the $r\geq 6$ RSOS models have not been calculated, so no direct evidence on the presence or lack of chiral symmetry breaking in regime I or at the critical point is available yet. For $r=6$ the regime I to II multicritical point is the melting of a 4×1 commensurate phase, so it should be in the same universality class as a four-state clock model. The *symmetric* four-state clock is the Ashkin-Teller model, which has continuously varying critical exponents and is in the same universality class as the eight-vertex model.[17] The critical exponents of the $r=6$ RSOS model, namely $\alpha=\frac{1}{2}$, $\beta_1=\frac{3}{32}$, and $\beta_2=\frac{1}{8}$ are precisely equal to those of a point on the critical line of Ashkin-Teller model,[17] so it appears that these two critical points are in the same universality class. Since chiral symmetry breaking is also relevant at this Ashkin-Teller—model critical point,[15,18] it must vanish at the RSOS-model critical point for it to remain in the Ashkin-Teller universality class. Of course, the transition might be in a chiral universality class[11] that happens to have the same exponents as the Ashkin-Teller model; this cannot be ruled out until the correlation function is calculated.

For $p>4$ the melting of a $p\times 1$ commensurate phase or a p-state clock-model ordered phase is expected to generally be either a first-order transition or a double transition with an intermediate massless or floating phase.[11,12,18] However, there must also be multicritical points where the crossover between these two types of behavior occurs; precisely at such crossover points the melting may occur in a single continuous phase transition. Since the transitions found in the $r>6$ RSOS model by Andrews *et al.*[10] are unique and continuous, it appears that they have found such special cases of $p\times 1$ melting. Exactly how special the exact solution manifold is for $p>4$ is unclear. A better understanding of the situation might be obtained by exploring (presumably by Monte Carlo simulation) the RSOS model in the vicinity of the exact multicritical point, but off of the manifold of exact solution, to see how it fits in the phase diagram in a larger parameter space. Independent of the precise nature of these $p\times 1$ melting transitions, the exact critical exponents for $p>4$ are certainly new and represent new universality classes of multicritical point.

IV. REGIMES III AND IV; GENERIC n-FOLD MULTICRITICALITY

For $r>4$ the restricted SOS model is ordered in both regimes III and IV. The only part of this portion of the exact solution manifold that is disordered is the multicritical point separating regimes III and IV. (For $r=4$ the model reduces to the Ising model and regime IV is the disordered or paramagnetic phase.) The $n=r-2$ phases that coexist in regime III each correspond to a *uniformly ordered* ground state.[10] Here and in the subsequent discussion of regime IV the term "ground state" is used somewhat loosely; the n ground states are not precisely degenerate, but the free energies of the corresponding phases are. Each ground state in regime III has all heights on the odd sublattice equal to l_1, while all heights on the even sublattice are equal to l_2, with $|l_1-l_2|=1$. The $(r-2)$ such ground states and their corresponding phases are labeled by the integer d, which is simply the lesser of l_1 and l_2.[10] The three ground states for the case $r=5$ are illustrated in Fig. 4. In regime IV there are only $(r-3)$ coexisting phases, as is discussed briefly at the end of this section.

The transition from regime III to regime IV is continuous; all the coexisting phases become identical at the multicritical point. The free energy is singular with exponent[10]

$$2-\alpha=r/2\ . \tag{4.1}$$

For r even there is also a logarithmic factor present so the singular part of the free energy behaves as[10]

$$f_s \sim t^{r/2}\ln t \tag{4.2}$$

for $t\to 0$. For $r=4$ this is the familiar Ising-model result. For r odd the singular part of the free energy actually vanishes in regime III, but not in regime IV.[10] Since the phases here are spatially uniform the probabilities $\{P_a(i)\}$ are the same for all sites on a given sublattice. Thus the average of P_a over all sites for the d phase is simply

$$\bar{P}_a^{(d)}=\tfrac{1}{2}[P_a^{(d)}(i)+P_a^{(d)}(j)]\ , \tag{4.3}$$

where the sites i and j are arbitrary sites on the odd and even sublattices, respectively. The exact expression for $\bar{P}_a^{(d)}$ in regime III, as obtained from Eq. (3.3.18c) of Ref. 10, can be expanded about the multicritical point as

1	2	1	2	1	3	2	3	2	3	3	4	3	4	3
2	1	2	1	2	2	3	2	3	2	4	3	4	3	4
1	2	1	2	1	3	2	3	2	3	3	4	3	4	3
2	1	2	1	2	2	3	2	3	2	4	3	4	3	4
1	2	1	2	1	3	2	3	2	3	3	4	3	4	3
2	1	2	1	2	2	3	2	3	2	4	3	4	3	4
		d=1					d=2					d=3		

FIG. 4. Ground states corresponding to the three ordered phases, $d=1,2,3$, in regime III of the $r=5$ RSOS model.

$$\bar{P}_a^{(d)}=\frac{2\sin(\pi a/r)}{r\sin[\pi d/(r-1)]}\times\sum_{n=1}^{r-2}|t|^{(n^2-1)/8(r-1)}\times\sin\left[\frac{n\pi a}{r}\right]\sin\left[\frac{n\pi d}{r-1}\right][1+O(t)]\ . \tag{4.4}$$

The order parameters in regime III are differences between the various phases $d=1,\ldots,(r-2)$ and may be chosen to be

$$R_a^{(k)}=(r-2)^{-1}\sum_d \bar{P}_a^{(d)}\sin\left[\frac{\pi d}{r-1}\right]\sin\left[\frac{(k+1)\pi d}{r-1}\right]\ , \tag{4.5}$$

for integers $1\le k\le(r-3)$. In regime II the order parameters (3.5) are simple spatial Fourier transforms of the local ordering. Unfortunately, the order parameters here in regimes III and IV do not appear to generally have such familiar interpretations. As $t\to 0$ in regime III the order parameters vanish as

$$R_a^{(k)}\approx\frac{|t|^{\beta_k}}{r}\sin\left[\frac{\pi a}{r}\right]\sin\left[\frac{(k+1)\pi a}{r}\right]\ , \tag{4.6}$$

where the critical exponents are

$$\beta_k=\frac{(k+1)^2-1}{8(r-1)}\ . \tag{4.7}$$

Note that in regime III we have $n=(r-2)$ coexisting phases and consequently $(n-1)$ order parameters. Each order parameter has its own distinct critical exponent at the multicritical point. Thus this appears to be a generic n-fold multicritical point, with no special symmetry present that relates the different order parameters and demands that they have the same exponent. By contrast, at the regime II multicritical point the order-parameter exponents are degenerate pairwise due to the special Z_p symmetry relating the ordered phases.

It is certainly of interest to compare the multicritical exponents found in the exact solution of Andrews, Baxter, and Forrester[10] to those found by Friedan, Qiu, and Shenker[13] based on assuming conformal invariance and unitarity. Friedan *et al.*[13] find an infinite sequence of possible realizations of conformal invariance, which they label with the integers $m=3,4,\ldots,$ each representing a possible multicritical point. (Note this sequence does *not* contain all possible realizations satisfying their assumptions.) For each such realization they find a finite set of rational numbers that must contain all possible critical exponents.[13] In order to compare their exponents to those of the regime III to IV multicritical point we must calculate the critical indices or scaling dimensions for the order parameters (4.6), namely,

$$x_k=\frac{2\beta_k}{2-\alpha}=\frac{(k+1)^2-1}{2r(r-1)}\ . \tag{4.8}$$

The "thermal" critical index is

$$x_t=\frac{2(1-\alpha)}{2-\alpha}=\frac{2(r-2)}{r}=\frac{(2r-3)^2-1}{2r(r-1)} . \tag{4.9}$$

The sequence of $(r-3)$ order-parameter critical indices (4.8) in the RSOS model coincide precisely with those indices in the range $0<x<\frac{1}{2}$ allowed by the analysis of Friedan *et al.*[13] for the realization $m=(r-1)$. The thermal index (4.9) is also allowed in the same realization. Thus the restricted SOS model of Andrews *et al.*[10] appears to provide an explicit realization, namely the regime III to IV transition, of each hypothetical multicritical point in the infinite sequence, $m=3,4,\ldots$, proposed by Friedan *et al.*[13] The regime I to II multicritical exponents (3.3) and (3.6), on the other hand, do not coincide with the exponents of Friedan *et al.*[13] in any simple way. It may be that the regime I to II multicritical points are not generally conformally invariant, or that they represent realizations of conformal invariance that are not in the sequence discussed by Friedan *et al.*[13] Hopefully further investigation will address this question.

In regime IV of the exact solution manifold of the RSOS models, there are $(r-3)$ coexisting spatially uniform phases. For odd r these are simply all the phases present in regime III except the "middle" one with $d=(r-1)/2$.[10] For even r the two middle phases with $d=(r-2)/2$ and $d=r/2$ are replaced by a new phase which does not correspond to a unique ground state.[10] Order parameters can be constructed for regime IV that correspond directly to the first $(r-4)$ order parameters in regime III. These order parameters, defined in a fashion analogous to (4.5), vanish precisely as given by (4.6), but with different exponents,

$$\beta_k^{\mathrm{IV}}=\frac{(k+1)^2-1}{4(r-2)} . \tag{4.10}$$

A plausible explanation for the presence of different "observed" exponents in regime IV of the $r=5$ RSOS or hard-square model is given in Ref. 9. The same situation appears to be occurring in all the $r\geq 5$ RSOS models. Briefly, if a line of generic $(n-1)$-fold multicritical points is very near regime IV, then the singularities due to this line can modify the exponents observed on regime IV. This will occur if the separation of the multicritical line and regime IV is only due to an irrelevant scaling field with, for example, correction exponent $\theta^{(n)}$. Then the observed exponents in regime IV will be

$$\beta_k^{\mathrm{IV}}=\beta_k^{(n)}+\theta^{(n)}\beta_k^{(n-1)} , \tag{4.11}$$

where $\beta_k^{(n)}$ are the "true" order-parameter exponents (4.7) at the generic n-fold multicritical point, as observed in the corresponding regime III (see Ref. 9 for more details). Solving for $\theta^{(n)}$, we find

$$\theta^{(n)}=\frac{n+2}{n+1} , \tag{4.12}$$

which is a correction-to-scaling exponent seen in some of the exact results for regime III,[10] and also present in the list of exponents allowed by the assumptions of Friedan *et al.*[13]

V. CONCLUSIONS AND QUESTIONS

The recent exact solution of the restricted solid-on-solid (RSOS) models of Andrews, Baxter, and Forrester[10] exhibits two infinite sequences of multicritical points. The regime I to II multicritical point in each model, $r=4,5,\ldots$, is the continuous melting transition of a $p\times 1$ commensurate phase, where $p=r-2$. Appropriate order parameters for these commensurate phases are defined by Eq. (3.5) above. The exact critical exponents (3.7) for each of the $(p-1)$ different order parameters have been extracted from the exact solution. These multicritical points should be in the universality class of the p-state chiral clock models.[11] For $r=4$, $p=2$ this is certainly the case, since the RSOS model is exactly equivalent to the two-state clock or Ising model. For $r=5$, $p=3$ the exact expression for the correlation function[16] shows that chiral symmetry breaking[11] vanishes in the scaling limit at the critical point. Thus the $r=5$ RSOS model critical point is in the universality class of the *symmetric* (nonchiral) three-state clock or Potts model. It will be interesting to see correlation functions for $r\geq 6$, in order to learn whether these multicritical point are in symmetric or chiral universality classes. For $r=6$, $p=4$ the critical exponents coincide precisely with those of the symmetric four-state clock or Ashkin-Teller model, so it appears that chiral symmetry breaking again vanishes at the critical point. This may be a general property of the RSOS models.

The clock models for $p>4$ exhibit a variety of phase transition behavior, including first-order melting, two-stage melting with an intermediate power-law phase, and, presumably, various special multicritical points at which crossover between different types of melting occurs.[12] The RSOS models all exhibit a single continuous melting transition; therefore they must be some special or multicritical case of $p\times 1$ melting for $p>4$. Precisely how these exact solutions fit into the general clock-model phase diagram is not obvious; this question might be answered by simulations of the RSOS models away from the manifold of exact solution.

In regime III of each RSOS model there are $(r-2)$ coexisting phases and consequently $(r-3)$ order parameters; they are given by (4.5) above. At the regime III to IV critical point all of these order parameters vanish, each with its own unique critical exponent. The existence of a distinct critical exponent for each order parameter suggests that the regime III to IV critical point represents a generic $(r-2)$-fold multicritical point, with no special symmetries present to relate the various order parameters. In regime IV the order-parameter exponents are different from regime III, presumably due to a nearby line of multicritical points. Again it might be interesting to understand how the exact solution manifolds and their multicritical points fit into the phase diagram of the general RSOS model; this question could be explored by Monte Carlo simulation. However, even for the simplest new case, the generic tetracritical point, there are six relevant scaling fields, so an understanding of the full phase diagram will not be easily attained.

Friedan, Qiu, and Shenker[13] have recently found an infinite sequence of hypothetical multicritical points by assuming conformal invariance and unitarity. For each such multicritical point they have obtained the full set of critical exponents. The exact exponents of the regime III to IV multicritical points in the RSOS models coincide with a subset of the exponents proposed by Friedan *et al.*[13] Thus the RSOS models appear to provide an explicit realization of each multicritical point in the infinite sequence proposed by Friedan *et al.*[13] Therefore, this infinite sequence of conformally invariant multicritical points should represent generic n-fold multicriticality for all integers n. The regime I to II multicritical points do not appear to fit into this conformal scheme in any simple fashion. This raises the question of whether these multicritical points are conformally invariant and, if so, how can they be fit into a classification scheme like that of Friedan *et al.*[13]

As a final question, the above suggestion implies that the critical exponents of the three-state Potts critical and tricritical points coincide with subsets of the generic tetracritical and pentacritical exponents, respectively (see Ref. 13). What could be the significance of this?

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