Excitation spectrum and superfluid density of ^{3}He-A at $T=0$

T. Dombre and R. Combescot
Groupe de Physique des Solides de l'Ecole Normale Supérieure, Laboratoire associé au Centre National de la Recherche Scientifique, 24 Rue Lhomond, F-75231 Paris Cedex 05, France
(Received 13 March 1984)

The excitation spectrum of ^{3}He-A at $T=0$ is studied and it shows a discretization. This is linked to the localization of the excitations which are trapped in the existing texture. The excitation density of states is calculated and is found to be nonzero at zero energy. A corresponding departure of the superfluid density from its ideal value is obtained.

I. INTRODUCTION

In a recent paper[1] (hereafter referred to as I), we have studied the expression for the superfluid current in ^{3}He-A at $T=0$. This has been motivated by the present lack of a consistent set of hydrodynamic equations for this superfluid.[2] Actually there are hydrodynamic equations agreeing with the symmetry of ^{3}He-A, namely, those of a Bose liquid with particles carrying an intrinsic angular momentum, however, they contradict our present microscopic knowledge on ^{3}He-A at $T=0$.

At $T=0$ singularities arise in microscopic theory because there are two nodes for the gap on the Fermi surface. This makes it impossible to use the standard gradient expansion of Gor'kov's equations. In I, we have been able to overcome this problem by solving exactly these equations. This has been done with the only approximation that the order parameter is slowly varying as in any texture. As a result we kept only gradients of the order parameter and dropped higher-order terms. The validity of this approximation will be discussed in detail in the next section.

In this paper we use our solution to pursue the study of the statics of the system (a brief summary of our results has been already published[3]). We investigate the excitation spectrum and find that it is continuous with respect to two parameters and discrete for the third one. This is in contrast with the free case where this spectrum depends continuously on $\vec{k}$. Corresponding to this discrete behavior of the spectrum, we find that the excitations correspond to localized states. This localization arises because low-energy excitations are trapped, for a given $\vec{k}$, in potential wells created by the studied texture. We obtain the density of states $N(\omega)$ for these excitations and find it to be nonzero at zero energy. Finally we have obtained directly, through a gauge transformation, the superfluid density tensor. Its component parallel to the anisotropy axis $\hat{l}$ differs from the total density by an amount proportional to $N(0)$ which strongly suggests a nonzero normal density in ^{3}He-A at $T=0$. All our results agree quite well qualitatively with the physical picture proposed by Volovik and Mineev,[4] namely, trapping of the excitations by the texture and corresponding existence of a normal density. However, they are quantitatively markedly different.

II. THE EXCITATION SPECTRUM

Let us first recall the formulation of the problem: If one neglects in Gor'kov's equations terms of order δ/E_F compared to the dominant ones, they can be written as (see I for conventions)

$$\begin{bmatrix} i\omega_n-\xi_k & \Delta_k(\vec{r}) \\ -\Delta_k^*(\vec{r}) & -i\omega_n-\xi_k \end{bmatrix} g+i\vec{v}_k\cdot\vec{\nabla}g=1 \ . \tag{1}$$

Because of the nodes of the gap, difficulties arise in solving these equations in the presence of textures of the $\hat{l}$ vector; indeed, for wave vectors $\vec{k}$ pointing near $\pm\hat{l}$, the usual gradient expansion is no longer useful and one must look for a nonperturbative solution of Eq. (1). We expect that in order to grasp the main physical effects issuing from this unique situation, we need to retain only the first variations of the gap; in other words, to put into Eq. (1) a linearized version of the texture around the origin $\vec{r}=\vec{0}$:

$$\Delta_k(\vec{r})=\Delta_k^0+\vec{r}\cdot\vec{\nabla}\Delta_k \ . \tag{2}$$

This will later be shown to be a consistent procedure.

At the origin, $\hat{l}$ is supposed to be along $\hat{z}$. The singularity we are interested in arises because $\hat{l}\times\text{curl}\hat{l}=-(\hat{l}\cdot\vec{\nabla})\hat{l}\neq\vec{0}$. We take $\hat{y}$ along $\hat{l}\times\text{curl}\hat{l}$ at $\vec{r}=\vec{0}$ and we set

$$q=|\hat{l}\times\text{curl}\hat{l}|_{\vec{r}=\vec{0}}=-\partial_z l_y>0 \ . \tag{3}$$

If $\hat{l}\times\text{curl}\hat{l}=\vec{0}$, the singularity is higher order and becomes unimportant. It may be useful to think of the example of a texture where $\hat{l}$ varies in the y-z plane along the z direction. In this case $\partial_z l_y$ is the only nonvanishing derivative of $\hat{l}$ at $\vec{r}=\vec{0}$. To solve Eq. (1) we are only interested in $\vec{r}$ parallel to $\hat{k}$. We set

$$\vec{r}=\rho\hat{k}, \quad \alpha=\hat{k}\cdot\vec{\nabla}\Delta_k \ , \tag{4}$$

where α can always be taken real positive for fixed $\hat{k}$ (the only sacrifice is a simple phase change in Δ_k^0 depending on $\hat{k}$). With these conventions, Eq. (2) reads

$$\Delta_k(\rho)=\Delta_k^0+\alpha\rho=\alpha(\rho+\rho_0)+i\,\text{Im}\Delta_k^0 \ , \tag{5}$$

where $\rho_0=\mathrm{Re}(\Delta_k^0)/\alpha$. In the case of our simple texture we have merely $\alpha=(q\delta)\hat{k}_z^2$, $\rho_0=(\delta/\alpha)\hat{k}_y$, and $\mathrm{Im}\Delta_k^0=-\hat{k}_x\delta$. Our exact result for g^{11} is [see Eq. (32) in I]

$$g^{11}(\omega_n,\hat{k},\xi,\rho)=\frac{-i}{v_F\det G}\left[G^{22}(\rho)\int_{+\infty}^{\rho}du\, e^{i(u-\rho)\xi/v_F}G^{11}(u)-G^{12}(\rho)\int_{-\infty}^{\rho}du\, e^{i(u-\rho)\xi/v_F}G^{21}(u)\right], \tag{6}$$

where

$$\begin{aligned}
&\det G=-\frac{2\sqrt{2\pi}}{\lambda\Gamma(\mu+1)},\quad \lambda=\frac{\omega_n+i\mathrm{Im}\Delta_0}{(2\alpha v_F)^{1/2}},\\
&\mu=|\lambda|^2,\quad x=\left[\frac{2\alpha}{v_F}\right]^{1/2}(u+\rho_0),\\
&G^{11}=U_{\mu+1/2}(x)-(1/\lambda)U_{\mu-1/2}(x),\\
&iG^{12}=U_{\mu+1/2}(x)+(1/\lambda)U_{\mu-1/2}(x),\\
&-iG^{21}=U_{\mu+1/2}(-x)+(1/\lambda)U_{\mu-1/2}(-x),\\
&G^{22}=U_{\mu+1/2}(-x)-(1/\lambda)U_{\mu-1/2}(-x),
\end{aligned} \tag{7}$$

and U is the parabolic cylinder function.[5]

This seems to be a rather messy expression. But it is easy to study the analytical behavior in the complex plane of $\omega=i\omega_n$. As a function of ω we have

$$\lambda=i\frac{\mathrm{Im}\Delta_0-\omega}{(2\alpha v_F)^{1/2}},\quad \mu=\frac{(\mathrm{Im}\Delta_0)^2-\omega^2}{2\alpha v_F}. \tag{8}$$

Since the parabolic cylinder functions are analytical in μ, the only singularities arise for $\lambda=0$ and when $\Gamma(\mu+1)$ diverges, which occurs for $\mu=-p$ where p is a positive integer. Therefore we obtain a set of poles on the real ω axis:

$$\begin{aligned}
&\omega=\mathrm{Im}\Delta_0,\\
&\omega_p=\pm[(\mathrm{Im}\Delta_0)^2+2p\alpha v_F]^{1/2},\quad p\geq 1
\end{aligned} \tag{9}$$

which is the excitation spectrum. We see that we obtain, for fixed $\hat{k}$, discrete energy levels which correspond physically to localized states. This is discussed further below.

To go further we must calculate the residues of g^{11} corresponding to the different poles given by Eq. (9). They will be noted $R^p(\hat{k},\xi,\rho)$ and for evaluating these quantities we shall distinguish between the zeroth level ω_0 and the higher ones ω_p with $p\geq 1$. From Eqs. (6) and (7) one has

$$\begin{aligned}
g^{11}=\frac{i}{v_F}\frac{\lambda\Gamma(\mu+1)}{2(2\pi)^{1/2}}\Bigg[&\left[U_+(-y)-\frac{1}{\lambda}U_-(-y)\right]\int_{+\infty}^{\rho}du\, e^{i(u-\rho)\xi/v_F}\left[u_+(x)-\frac{1}{\lambda}U_-(x)\right]\\
&-\left[U_+(y)+\frac{1}{\lambda}U_-(y)\right]\int_{-\infty}^{\rho}du\, e^{i(u-\rho)\xi/v_F}\left[U_+(-x)+\frac{1}{\lambda}U_-(-x)\right]\Bigg],
\end{aligned} \tag{10}$$

where $y=(2\alpha/v_F)^{1/2}(\rho+\rho_0)$ and $U_\pm(y)$ stands for $U_{\mu\pm1/2}(y)$.

For R^0 we find

$$R^0(\hat{k},\xi,\rho)=-\frac{1}{2}\left[\frac{\alpha}{\pi v_F}\right]^{1/2}\left[U_{-1/2}(-y)\int_{+\infty}^{\rho}e^{i(u-\rho)\xi/v_F}U_{-1/2}(x)du-U_{-1/2}(y)\int_{-\infty}^{\rho}e^{i(u-\rho)\xi/v_F}U_{-1/2}(-x)du\right], \tag{11}$$

and for R^p,

$$\begin{aligned}
R^p(\hat{k},\xi,\rho)=\frac{1}{4}&\left[\frac{\alpha}{\pi v_F}\right]^{1/2}\frac{\omega_p-\mathrm{Im}\Delta_0}{\omega_p}\frac{(-1)^p}{(p-1)!}\\
&\times\Big[[U_{-p+1/2}(-y)-\lambda_p^{-1}U_{-p-1/2}(-y)]\int_{+\infty}^{\rho}e^{i(u-\rho)\xi/v_F}du\,[U_{-p+1/2}(x)-\lambda_p^{-1}U_{-p-1/2}(x)]\\
&-[U_{-p+1/2}(y)+\lambda_p^{-1}U_{-p-1/2}(y)]\int_{-\infty}^{\rho}e^{i(u-\rho)\xi/v_F}du[U_{-p+1/2}(-x)+\lambda_p^{-1}U_{-p-1/2}(-x)]\Big],
\end{aligned} \tag{12}$$

where $\lambda_p^{-1}=i(2\alpha v_F)^{1/2}/(\omega_p-\mathrm{Im}\Delta_0)$.

But for $p\in\mathbf{N}$, the functions $U_{-p-1/2}$ are given by

$$U_{-p-1/2}(x)=2^{-p/2}e^{-x^2/4}H_p(x/\sqrt{2})=(-1)^pU_{-p-1/2}(-x), \tag{13}$$

where H_p is the pth Hermite polynomial.

This simplifies the above expressions into

$$R^0(\hat{k},\xi,\rho)=\frac{1}{2}\left[\frac{\alpha}{\pi v_F}\right]^{1/2}\left[U_{-1/2}(y)\int_{-\infty}^{+\infty}du\, e^{i(u-\rho)\xi/v_F}U_{-1/2}(x)\right], \tag{14}$$

$$R^p(\hat{k},\xi,\rho)=\frac{1}{4}\left[\frac{\alpha}{\pi v_F}\right]^{1/2}\frac{\omega_p-\mathrm{Im}\Delta_0}{\omega_p}\frac{1}{(p-1)!}[U_{-p+1/2}(y)+\lambda_p^{-1}U_{-p-1/2}(y)]$$
$$\times\int_{-\infty}^{+\infty}du\, e^{i(u-\rho)\xi/v_F}[U_{-p+1/2}(x)-\lambda_p^{-1}U_{-p-1/2}(x)]\ . \tag{15}$$

Using[6]

$$\int_{-\infty}^{+\infty}e^{-y^2/2}e^{ixy}H_p(y)dy=\sqrt{2\pi}(i)^p e^{-x^2/2}H_p(x)\ , \tag{16}$$

integration over the variable u is easily done and leads to

$$R^0(\hat{k},\xi,\rho)=\frac{1}{\sqrt{2}}\exp\left[-\frac{\alpha}{2v_F}(\rho+\rho_0)^2-\frac{\xi^2}{2\alpha v_F}-i(\rho+\rho_0)\frac{\xi}{v_F}\right] \tag{17}$$

and

$$R^p(\hat{k},\xi,\rho)=\frac{(i/2)^{p-1}}{(p-1)!}\frac{\omega_p-\mathrm{Im}\Delta_0}{\omega_p}\frac{R^0}{2}\left[H_{p-1}\left[\frac{y}{\sqrt{2}}\right]+\frac{\lambda_p^{-1}}{\sqrt{2}}H_p\left[\frac{y}{\sqrt{2}}\right]\right]\left[H_{p-1}\left[\frac{\xi}{(\alpha v_F)^{1/2}}\right]-i\frac{\lambda_p^{-1}}{\sqrt{2}}H_p\left[\frac{\xi}{(\alpha v_F)^{1/2}}\right]\right]. \tag{18}$$

We see that, as a function of ρ, R^0 is centered around $\rho=-\rho_0$ and has a spatial extension of order $\tilde{L}=(v_F/\alpha)^{1/2}$. This is much smaller than the length scale $L=\delta/\alpha$ over which the order parameter changes since the ratio between these two lengths is $(v_F/\delta L)^{1/2}=(\xi_0/L)^{1/2}$, where $\xi_0=v_F/\delta$ is the coherence length. On the other hand, $\xi_0/\tilde{L}=(\xi_0/L)^{1/2}$, and we have $\xi_0\ll\tilde{L}\ll L$. In the case of our simple texture (and $\hat{k}_z\sim 1$), $L=q^{-1}$ and $\tilde{L}=(\xi_0/q)^{1/2}$. The preceding inequalities show that our basic approximation of linearizing the order parameter is valid since the spatial extension $\tilde{L}$ of the excitation is much smaller than the length scale L for order-parameter changes.

In the same way, the excitation ω_p has a spatial extension $\sqrt{2p}\tilde{L}$ around $-\rho_0$. Our approximation holds up to $p\lesssim(L/\tilde{L})^2=L/\xi_0$, where it begins to fail. But at this stage $\omega_p\sim\delta$ and we are not interested in excitations of such high energies. If we needed it, we could solve the problem in this range by a quasiclassical approximation.

The localization of the excitation is easy to understand. The local energy of an excitation $\vec{k}$ located at ρ is

$$[\xi_k^2+(\mathrm{Im}\Delta_0)^2+\alpha^2(\rho+\rho_0)^2]^{1/2}\ .$$

We see that an excitation with a fixed energy which is sufficiently low cannot escape far from the point $\rho=-\rho_0$ and is trapped around this point.

III. DENSITY OF STATES AND NORMAL DENSITY

The density of states at the origin $N(\omega)$ is a sum of terms $N_p(\omega)$ corresponding to each successive level

$$N_p(\omega)=N_0\int\frac{d\Omega}{4\pi}d\xi R^p(\hat{k},\xi,\rho=0)\delta(\omega-\omega_p)\ , \tag{19}$$

where $N_0=mk_F/\pi^2$ is the density of states at the Fermi level in the normal liquid. Since ξ does not appear in the expression of ω_p, we can write directly

$$N_p(\omega)=N_0\int\frac{d\Omega}{4\pi}\tilde{R}_p(\hat{k})\delta(\omega-\omega_p)$$

with

$$\tilde{R}_p(\hat{k})=\int_{-\infty}^{+\infty}d\xi R^p(\hat{k},\xi,\rho=0)\ . \tag{20}$$

The ξ-integrated residue $\tilde{R}^p$ is obtained by using again formula (16). We obtain

$$\tilde{R}^0(\hat{k})=(\pi\alpha v_F)^{1/2}\exp\left[-\alpha\frac{\rho_0^2}{v_F}\right]$$

$$\tilde{R}^p(\hat{k})=\tilde{R}^0\frac{(\omega_p-\mathrm{Im}\Delta_0)}{\omega_p}\frac{2^{-p}}{(p-1)!}\times\left[H_{p-1}^2\left[\left[\frac{\alpha}{v_F}\right]^{1/2}\rho_0\right]-\frac{\lambda_p^{-2}}{2}H_p^2\left[\left[\frac{\alpha}{v_F}\right]^{1/2}\rho_0\right]\right]. \tag{21}$$

We now note the following. We are interested in the density of states for low energies $\omega\ll\delta$. This means low ω_p and from Eq. (9) this implies $|\mathrm{Im}\Delta_0|$, $(p\alpha v_F)^{1/2}\ll\delta$. Therefore, since $\exp(-x^2/2)H_p(x)$ (the wave function of the harmonic oscillator) is maximum for $x\sim\sqrt{2p}$ and decreases exponentially after that, we have a non-negligible contribution in Eq. (17) only if $|\mathrm{Re}\Delta_0|=\alpha|\rho_0|\sim(p\alpha v_F)^{1/2}\ll\delta$. Therefore the main contribution to the density of states comes as expected from the vicinity of the nodes of the gap. This allows us to set $\hat{k}_z=\pm 1$ in the integral (20), which implies $\alpha=q\delta$, and to replace $\int d\Omega/4\pi$ by $\int d\hat{k}_x d\hat{k}_y/2\pi$ (a factor of 2 coming from the two directions $\pm\hat{l}$). We can also change the x and y axes in the integration so that $\mathrm{Re}\Delta_0=\hat{k}_x\delta$, $\mathrm{Im}\Delta_0=\hat{k}_y\delta$.

For $N_0(\omega)$ we obtain

$$N_0(\omega)=N_0(\pi q v_F \delta)^{1/2}\frac{1}{2\pi}\times\int d\hat{k}_x d\hat{k}_y \delta(\omega-\hat{k}_y\delta)e^{-\hat{k}_x^2\delta^2/qv_F}\,,$$
$$=\frac{N_0}{2}\frac{qv_F}{\delta}\,. \quad (22)$$

For $N_p(\omega)$ we use the orthogonality property of the Hermite polynomials

$$\int_{-\infty}^{+\infty} e^{-x^2}H_p^2(x)dx=\sqrt{\pi}2^p p!\,,$$

and we obtain

$$N_p(\omega)=\frac{N_0}{2}qv_F\int_{-\infty}^{+\infty} d\hat{k}_y\delta[\omega\pm(\delta^2\hat{k}_y^2+2pqv_F\delta)^{1/2}]\,,$$
$$=N_0\frac{qv_F}{\delta}\frac{|\omega|}{(\omega^2-2pqv_F\delta)^{1/2}}\,. \quad (23)$$

This gives the following structure for $N(\omega)$ in the range $\omega \ll \delta$

$$N(\omega)=\frac{N_0}{2}\frac{qv_F}{\delta}+N_0\frac{qv_F}{\delta}\sum_{p=1}^{+\infty}\frac{|\omega|}{(\omega^2-2pqv_F\delta)^{1/2}}\,. \quad (24)$$

The structure of this density of states is quite similar to the one created by Landau levels for an electron gas in a magnetic field. For $(qv_F\delta)^{1/2} \ll \omega \ll \delta$, many levels contribute to $N(\omega)$ which can be approximated by (we set P equal to the integral part of $\omega^2/2qv_F\delta$):

$$\frac{\delta}{qv_F}\frac{N(\omega)}{N_0}\simeq\sum_{p=1}^{P}\frac{\omega}{(\omega^2-2pqv_F\delta)^{1/2}}$$
$$\simeq\int_0^{\omega^2/2qv_F\delta}dp\frac{\omega}{(\omega^2-2pqv_F\delta)^{1/2}}=\frac{\omega^2}{qv_F\delta}\,. \quad (25)$$

Therefore in this range, $N(\omega)$ reduces as expected to the ordinary density of states $N_0(\omega/\delta)^2$ for the low-lying excitations in the homogeneous system.

A main feature of $N(\omega)$ is the existence of a finite density of states for $\omega=0$:

$$N(0)=\frac{N_0}{2}\frac{v_F}{\delta}|(\hat{l}\cdot\vec{\nabla})\hat{l}|\,. \quad (26)$$

Therefore we expect a normal density at $T=0$ linked to $N(0)$ and a corresponding correction to the superfluid density ρ^s in order to ensure Galilean invariance. However, we can only calculate ρ^s within our framework. The rest of this section is devoted to this question.

The best way to obtain the $\vec{v}_s$ contribution to the current is to perform a gauge transformation in order to get rid of $\vec{v}_s$.[7] This is done as follows: If $\psi_\alpha(\vec{r},t)$ is the old field operator a new one $\phi_\alpha(\vec{r},t)$ is introduced by the relation

$$\psi_\alpha(\vec{r},t)=\exp\left[im\frac{\vec{v}_s\cdot\vec{r}}{\hbar}\right]\phi_\alpha(\vec{r},t)\,. \quad (27)$$

After the transformation the gap dependence on $\vec{v}_s$ has disappeared. The new Hamiltonian has the same expression as before except one supplementary term if we keep only the linear order in $\vec{v}_s$:

$$\int d^3r\vec{v}_s\frac{1}{2i}(\phi_\alpha^\dagger\vec{\nabla}\phi_\alpha-\vec{\nabla}\phi_\alpha^\dagger\phi_\alpha)\equiv\int d^3r\vec{v}_s\cdot\vec{g}'\,. \quad (28)$$

Gor'kov's equations read now as in (1) but it is easy to see that, because of the additional term, $i\omega_n$ is replaced by $i\omega_n-\vec{k}\cdot\vec{v}_s$; that is,

$$\omega_n\rightarrow\omega_n+i\vec{k}\cdot\vec{v}_s\,. \quad (29)$$

Furthermore the density current $\vec{g}$ is given by

$$\vec{g}=\vec{g}'+\rho\vec{v}_s\,.$$

Therefore by calculating the density current from our solution of the Gor'kov's equations with (29), we shall obtain directly the deviations of ρ^s from ρ at $T=0$.

We must calculate

$$\vec{g}'=N_0k_BT\sum_n\int d\xi\frac{d\Omega}{4\pi}\vec{k}g^{11} \quad (30)$$

with g^{11} given by Eqs. (6) and (29). In the limit of zero temperature, the sum over Matsubara frequencies goes into an integral

$$\vec{g}'=\frac{N_0}{2\pi}\int d\omega'\int d\xi\left[\frac{d\Omega}{4\pi}\right]\vec{k}g^{11}\,. \quad (31)$$

In order to integrate over ω' the function g^{11} of $\omega'+i\vec{k}\cdot\vec{v}_s$, we shift the integration path as shown in the figure, replacing the path $\mathscr{C}$ by $\mathscr{C}_1+\mathscr{C}_2+\mathscr{C}_3$. (See Fig. 1.)

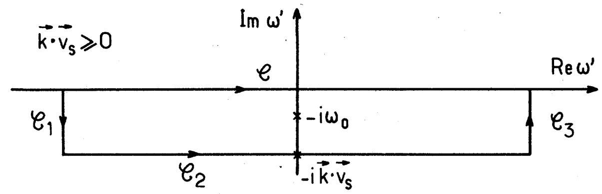

FIG. 1. Change in the path of integration for the calculation of ρ_s.

We see that poles inside the total contour $\mathscr{C}-(\mathscr{C}_1+\mathscr{C}_2+\mathscr{C}_3)$ will contribute to the integral. By Cauchy formula we have

$$\frac{1}{2\pi}\int_{\mathscr{C}}d\omega' g^{11}(\hat{k},\xi,\omega')=\frac{1}{2\pi}\int_{\mathscr{C}_1+\mathscr{C}_2+\mathscr{C}_3}d\omega' g^{11}-\mathrm{sgn}(\vec{k}\cdot\vec{v}_s)\sum_{p=0}^{+\infty}R^p(\hat{k},\xi)\Theta(\omega_p(\vec{k}\cdot\vec{v}_s-\omega_p))\,. \quad (32)$$

The integral along $\mathscr{C}_2$ on the right-hand side of Eq. (32) has no more $\vec{v}_s$ dependence and may be discarded. The integrals along $\mathscr{C}_1$ and $\mathscr{C}_3$ vanish when $\mathscr{C}_1$ and $\mathscr{C}_3$ go to infinity because $g_{11}\sim 1/i\omega'$ when $\mathrm{Re}\omega'\rightarrow\pm\infty$. This is a general behavior of the Green's function; it results directly from Eq. (1) and it can be checked on Eq. (6) by making use of Darwin's expansion for the parabolic cylinder functions $U(a,x)$ valid for large $X=(x^2+4a)^{1/2}$.

Finally we obtain for the $\vec{v}^s$ dependence of $\vec{g}'$ the exact expression:

$$\vec{g}'=-N_0\int d\xi\frac{d\Omega}{4\pi}\vec{k}\,\mathrm{sgn}(\vec{k}\cdot\vec{v}^s)\times\sum_{p=0}^{+\infty}R^p(\hat{k},\xi)\Theta(\omega_p(\vec{k}\cdot\vec{v}_s-\omega_p))\ . \quad (33)$$

If we let $\vec{v}^s$ go to 0, the only contribution is from $p=0$ because $\omega_p^2\geq 2qv_F\delta$ for $p\geq 1$. This remains true even if we assume that $m\vec{v}^s$ and curl$\hat{l}$ are of the same order because this means $k_Fv^s\sim qv_F\ll(qv_F\delta)^{1/2}$ if $qv_F\ll\delta$. Now the Θ function implies $|\mathrm{Im}\Delta_0|<|\vec{k}\cdot\vec{v}^s|$ and R^0 gives only contributions for $|\mathrm{Re}\Delta_0|\ll(qv_F\delta)^{1/2}$. Therefore only the vicinity of the nodes of the gap contribute and we have again $\hat{k}_z\simeq\pm 1$. We see that only g'_z is nonzero to leading order. Taking into account the contributions from the two nodes, we obtain

$$\begin{aligned}g'_z&=-N_0k_F\frac{1}{2\pi}\int d\hat{k}_x d\hat{k}_y\tilde{R}^0(\hat{k})\Theta(\omega_0(k_Fv_z^s-\omega_0))\\&=-k_F\int d\omega\,N_0(\omega)\Theta(\omega(k_Fv_z^s-\omega))\\&=-N(0)k_F^2v_z^s\ .\end{aligned}\quad (34)$$

Finally we obtain for the superfluid density ρ_{zz}^s along $\hat{l}$:

$$\rho_{zz}^s=\rho-N(0)k_F^2=\rho\left[1-\frac{3}{2}\frac{v_F|(\hat{l}\cdot\vec{\nabla})\hat{l}|}{\delta}\right]\ , \quad (35)$$

where the departure from ρ is linked to the finite density of states and it is natural to associate this departure to a finite normal density at $T=0$:

$$\rho_{zz}^n=\frac{3}{2}\rho v_F\frac{|(\hat{l}\cdot\vec{\nabla})\hat{l}|}{\delta}\ . \quad (36)$$

It is easy to include Fermi-liquid effects in this calculation. One must[7] replace $\vec{v}^s$ by $\vec{v}^s+(mF_1^s/3m^*)\times(\vec{g}/\rho-\vec{v}^s)$ in order to take into account the self-consistent interval field. But $\vec{g}-\rho\vec{v}^s$ is negligible compared to $\vec{v}^s$. The net result is merely to replace m by m^* in N_0 which leads to

$$\rho_{zz}^S=\rho\left[1-\frac{3}{2}\frac{m^*}{m}\frac{v_F|(\hat{l}\cdot\vec{\nabla})\hat{l}|}{\delta}\right]\ . \quad (37)$$

This is what one would expect from a normal liquid effect.

It is worth pointing out that our result Eq. (34) holds only in the regime $v^s/v_c\ll(qv_F/\delta)^{1/2}$ where $v_c=\delta/k_F$ is a typical order for a critical velocity. In the range $(qv_F/\delta)^{1/2}\lesssim v^s/v_c\ll 1$, we obtain

$$\begin{aligned}g'_z&=-k_F\sum_p^{\infty}\int_0^{k_Fv_z^s}d\omega\,N_p(\omega)\\&=-\frac{3}{2}\rho\frac{qv_F}{\delta}\left\{v_z^s+2\sum_{p=1}^{\infty}\left[(v_z^s)^2-2pv_c^2\left[\frac{qv_F}{\delta}\right]\right]^{1/2}\times\mathrm{sgn}v_z^s\right\}\ .\end{aligned}\quad (38)$$

This result displays an interesting nonlinear behavior which reflects directly the structure in the density of states.

We note finally that our result Eq. (35) for ρ^s can easily be generalized by standard methods to finite temperature. The result is merely

$$\rho_{zz}^s=\rho-k_F^2\int_{-\infty}^{+\infty}d\omega\,N(\omega)\left[-\frac{\partial f}{\partial\omega}\right]\ , \quad (39)$$

where $N(\omega)$ is the density of states given by Eq. (24) and $f(\omega)$ the Fermi distribution. At $T=0$, Eq. (39) reduces to Eq. (35) and in the regime $(qv_F\delta)^{1/2}\ll k_BT\ll\delta$, where $N(\omega)$ can be approximated by $N_0(\omega/\delta)^2$, it gives the standard low-temperature result for ρ_{zz}^s. In the intermediate range we remark that $N_p(\omega)$ has the same form as the BCS density of states for an isotropic superfluid with an effective gap $\Delta_p=(2pqv_F\delta)^{1/2}$. Therefore if we introduce the normal density $\rho_{\mathrm{iso}}^n(\Delta,T)$ of such a supefluid we may rewrite Eq. (39) as

$$\rho_{zz}^s=\rho\left[1-\frac{3}{2}\frac{qv_F}{\delta}\right]-\frac{qv_F}{\delta}\sum_{p=1}^{+\infty}\rho_{\mathrm{iso}}^n[(2pqv_F\delta)^{1/2},T]\ . \quad (40)$$

[1]R. Combescot and T. Dombre, Phys. Rev. B **28**, 5140 (1983).
[2]See Ref. 1 for a detailed discussion of this point and references on it.
[3]R. Combescot and T. Dombre, in *Quantum Fluids and Solids—1983 (Sanibel Island, Florida), Proceedings of the Sanibel Symposium on Quantum Fluids and Solids,* edited by E. P. Adams and G. G. Ihas (AIP, New York, 1983).
[4]G. E. Volovik and V. P. Mineev, Zh. Eksp. Teor. Fiz. **81**, 989 (1981) [Sov. Phys.—JETP **54**, 524 (1981)].
[5]*Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1970), p. 686.
[6]*Tables of Integrals, Series, and Products,* I. S. Gradshteyn and I. M. Ryzhik (Academic, New York, 1965), pp. 837,838.
[7]R. Combescot, J. Phys. C **14**, 4765 (1981).