Analytical results on the periodically driven damped pendulum. Application to sliding charge-density waves and Josephson junctions

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The differential equation $\epsilon\ddot{\phi}+\dot{\phi}-\frac{1}{2}\alpha\sin(2\phi)=I+\sum_{n=-\infty}^{\infty}A_n\delta(t-t_n)$ describing the periodically driven damped pendulum is analyzed in the strong damping limit $\epsilon \ll 1$, using first-order perturbation theory. The equation may represent the motion of a sliding charge-density wave (CDW) in ac plus dc electric fields, and the resistively shunted Josephson junction driven by dc and microwave currents. When the torque I exceeds a critical value the pendulum rotates with a frequency ω. For infinite damping, or zero mass ($\epsilon=0$), the equation can be transformed to the Schrödinger equation of the Kronig-Penney model. When A_n is random the pendulum exhibits chaotic motion. In the regular case $A_n=A$ the frequency ω is a smooth function of the parameters, so there are no phase-locked subharmonic plateaus in the $\omega(I)$ curve, or the I-V characteristics for the CDW or Josephson-junction systems. For small nonzero ϵ the return map expressing the phase $\phi(t_{n+1})$ as a function of the phase $\phi(t_n)$ is a one-dimensional circle map. Applying known analytical results for the circle map one finds narrow subharmonic plateaus at all rational frequencies, in agreement with experiments on CDW systems.

I. INTRODUCTION

The differential equation for the damped, driven pendulum

$$\epsilon\ddot{\phi}+\dot{\phi}-F(\phi)=I+V(t)\ , \tag{1.1}$$

where F and V are periodic functions, is of interest in a variety of problems in condensed-matter physics. For instance, the equation may describe the resistively shunted Josephson junction driven by a dc current I and a microwave current $V(t)=A\sin(\omega_{\text{ext}}t)$.[1,2] In this case 2ϕ is the phase across the junction and $F(\phi)\sim\sin(2\phi)$. The damping $1/\epsilon$ is given by $\epsilon=R(2eCI_c/h)^{1/2}$, where R is the resistance, C the capacitance, and I_c the critical current. The voltage across the junction is determined by the Josephson equation $V=2RI_c\dot{\phi}$ and the average voltage is thus proportional to the frequency with which the pendulum rotates. When the current I (the torque on the pendulum) is small the frequency is zero, i.e., the pendulum remains near its downward position and there is no voltage across the junction. When I exceeds a critical value the pendulum rotates with a characteristic frequency $\omega>0$, and there will be a nonzero average voltage across the junction,

$$V=2RI_c\langle\dot{\phi}\rangle=2RI_c\omega\ ,$$

where $\langle\dot{\phi}\rangle$ is the average phase increase per time unit.

The equation also describes the motion of a single-domain charge-density-wave system (CDW) in a dc electric field I plus an ac electric field $V(t)$. Grüner *et al.*[3,4] and Monceau *et al.*[5] have analyzed experiments on $NbSe_3$ in terms of the equation, and it was suggested that the strong damping limit $\epsilon\sim 0$ applies. In CDW systems $\phi(t)$ is the position of the rigid CDW relative to a fixed "impurity" with potential $F(\phi)$, which could simply be a contact potential. The nonrotating solutions correspond to pinned CDW states. The rotating solutions describe the sliding of the CDW when I exceeds the critical depinning field I_c, and the current carried by the CDW is given by the simple expression

$$I_{\text{CDW}}\sim\langle\dot{\phi}\rangle=\omega\ .$$

Hence, the roles of fields and currents are the opposite for the CDW systems and the Josephson junction: in the Josephson junction a voltage $V\sim\omega$ is induced by the driving current I; in the CDW a current $I\sim\omega$ is induced by the depinning electric field.

The equation (1.1) has been studied theoretically by numerous authors. Numerical simulations have revealed "subharmonic steps" in which the frequency $\omega=\langle\dot{\phi}\rangle$ "locks-in" at rational fractions of the driving frequency, $\omega=(p/q)\omega_{\text{ext}}$.[6] This explains the occurrence of steps in the I-V characteristics which have been observed in the Josephson junction in a microwave field, in particular by Belykh *et al.*,[7] and in the CDW system $NbSe_3$ by Grüner *et al.*,[8] although more or less competing models for CDW's have been suggested. In addition, numerical simulations have revealed "chaotic" solutions where the frequency spectrum has a broad background.[9,2] This noisy behavior had previously been observed experimentally[10] in Josephson junctions.

Most of these phenomena can be understood, at least in principle, by considering the underlying return map of the equation (1.1). Suppose we watch the system with "stroboscopic light" at times $t_1, t_2, \ldots, t_n, \ldots$ with $\tau_n = n\tau$, and τ the periodicity of $V(t)$. Since ϕ and $\dot{\phi}$ contains all the information of the state of the system, the value of ϕ after $n+1$ cycles of the rf field, $\phi_{n+1} = \phi(t_{n+1})$ is a (generally unknown) function of the values of ϕ and $\dot{\phi}$ after n cycles:

$$\phi_{n+1} = h'(\phi_n, \dot{\phi}_n) .$$

However, it has been shown numerically[11] that for sufficiently large damping the derivative $\dot{\phi}_n$ becomes a unique function of ϕ_n and the return map collapses to a one-dimensional (1D) map after an initial transient period:

$$\phi_{n+1} = h'(\phi_n, \dot{\phi}_n(\phi_n)) = h(\phi_n) , \tag{1.2}$$

where

$$h(\phi_n) = \phi_n + \Omega + g(\phi_n) ,$$

and g is periodic, $g(\phi) = g(\phi + \pi)$. The one dimensionality of the return map is trivial for $\epsilon = 0$ where the differential equation is of first order. Once the transformation to the circle map has been established one can apply theoretical results from the circle map; this permits a rather detailed understanding of the subharmonic steps and the transition to chaos. At the transition point $h(\phi_n)$ loses its analyticity.[11]

Renne and Polder[12] and Waldram and Wu[13] have analyzed the equation in the case $\epsilon = 0$ with $F(\phi) = I_c \sin(2\phi)$. Renne and Polder found that the equation can be transformed into a Schrödinger equation for a particle in a periodic potential. The wave vector k of the Bloch solutions corresponds to the frequency ω. The gap functions corresponding to the pth gap give the periodic solution $\phi(t)$ at the pth "Shapiro step," where $\omega = p\omega_{\text{ext}}$. Since k is varying smoothly as a function of the parameters (between the band gaps) there can be no subharmonic plateaus where the frequency ω locks-in. Waldram and Wu constructed the return map for the differential equation with the sinusoidal potential. The return map takes the trivial form

$$\theta_{n+1} = \theta_n + \omega , \tag{1.3}$$

where θ_n is a function of ϕ_n and $|\phi_n - \theta_n| < \pi$, so ω is the frequency of the pendulum, which again is a smooth function of I, I_c, etc.

The purpose of the present paper is to generalize and extend previous work to the case $\epsilon \ll 1$, and to "random" applied forces $V(t)$. To our knowledge, no analytical results for $\epsilon > 0$ exist prior to the present work. Let us briefly outline the paper. In Sec. II we consider the case $\epsilon = 0$:

$$\dot{\phi} - F(\phi) = I + \sum_{n=-\infty}^{\infty} A_n \delta(t - t_n) . \tag{1.4}$$

The advantage of using the δ-function potential is that we can obtain *explicit* analytic solutions, probably without loss of generality. Complete solutions can be outlined for any distribution A_n, not only for the periodic arrangement $A_n = A$. The solutions are most elegantly derived through a transformation to the Schrödinger equation for the Kronig-Penney model, where A_n is the strength of the potential at site n. In the periodic case we confirm for general $F(\phi)$ that there can be no subharmonic steps; in the random case the problem can be related to a 1D localization model and the phase $\phi(t)$ (not surprisingly) exhibits chaotic behavior.

Our most interesting results, derived in Sec. III, are for the strongly damped case $0 < \epsilon \ll 1$. Again we calculate the return map $\phi_{n+1}(\phi_n)$ for any distribution A_n. The return map is indeed one dimensional, confirming previous numerical results. For a periodic force, $A_n = A$, the return map takes the form

$$\theta_{n+1} = \theta_n + \Omega + \epsilon g(\theta_n) , \tag{1.5}$$

where g is a complicated periodic function and $|\theta_n - \phi_n| < \pi$. A map of the form (1.5) is called a circle map. *The parameter ϵ giving the strength of the nonlinear term of the circle map is thus essentially the coefficient of the second-order term of the differential equation (1.1).* The phase diagram in Ω-ϵ space is shown in Fig. 1. For nonzero ϵ the frequency locks-in at any rational value $(p/q)\omega_{\text{ext}}$. The locked portions of the diagram do not fill up the whole parameter space,[14] i.e., there is a nonzero probability that the motion is quasiperiodic (incommensurate).

Hence, narrow subharmonic steps are expected in damped systems with $\epsilon \ll 1$. Grüner *et al.*[8] have discovered very recently a multitude of steps in the CDW system $NbSe_3$. This indicates that $NbSe_3$ is not in the overdamped regime $\epsilon = 0$ as was previously assumed. We suggest that experiments on CDW systems and Josephson junctions be analyzed in terms of the present theory.

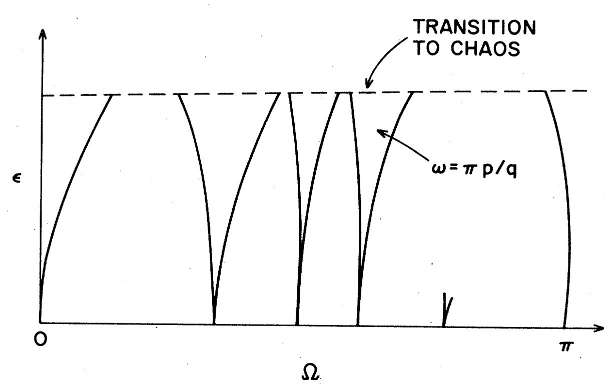

FIG. 1. Phase diagram in Ω-ϵ space. For $\epsilon > 0$ there is mode locking at every single rational frequency, $\omega - \pi p/q$. The parameter Ω is a complicated analytic function of the parameters of the differential equation.

II. THE OVERDAMPED CASE $\epsilon=0$

Let us first consider the overdamped case $\epsilon=0$:

$$\dot{\phi}-F(\phi)=I+\sum_{n=-\infty}^{\infty} A_n\delta(t-t_n) . \tag{2.1}$$

At $t_{n-1}<t<t_n$ where $\dot{\phi}=F(\phi)+I$, one obtains

$$\phi(t)=f(t-\tilde{t}_n) ,\qquad f^{-1}(x)=\int_{\tilde{x}}^{x}\frac{d\phi}{F(\phi)+I} , \tag{2.2}$$

where the choice of $\tilde{x}$ is arbitrary, and $\tilde{t}_n$ is an integration constant, $\tilde{x}=\phi(\tilde{t}_n)$. In the case $F(\phi)=\frac{1}{2}\alpha\sin(2\phi)$,

$$f(x)=\tan^{-1}\left[\frac{Z(x)}{I}-\frac{\alpha}{2I}\right] , \tag{2.3}$$

where

$$Z(x)=\begin{cases}\Omega\tan(\Omega x), & q=I^2-\frac{\alpha^2}{4}=\Omega^2>0\\ -\Omega\coth(\Omega x), & q=-\Omega^2<0,\quad |Z|>\Omega\\ -\Omega\tanh(\Omega x), & q=-\Omega^2<0,\quad |Z|<\Omega .\end{cases} \tag{2.4}$$

The tan solution describes a rotating motion for I sufficiently large; the other solutions are oscillating modes with zero rotation frequency.

We now match the solutions in two neighbor intervals:

$$\phi(t_n+0)-\phi(t_n-0)=A_n$$

and thus

$$f(t_n-\tilde{t}_{n+1})-f(t_n-\tilde{t}_n)=A_n . \tag{2.5}$$

Introducing

$$\tilde{\tau}_{n+1}=t_n-\tilde{t}_{n+1},\quad \tau_n=t_n-t_{n-1} , \tag{2.6}$$

one obtains from (2.3), (2.5), and (2.6)

$$\phi_{n+1}\equiv\phi(t_n+0)=\tan^{-1}\left[\frac{Z(\tilde{\tau}_n+\tau_n)}{I}-\frac{\alpha}{2I}\right]+A_n \tag{2.7}$$

and

$$Z(\tilde{\tau}_{n+1})=I\tan\phi_{n+1}+\frac{\alpha}{2} =\frac{\alpha}{2}+\frac{I\tan A_n+Z(\tilde{\tau}_n+\tau_n)-\alpha/2}{1-(\tan A_n/I)[Z(\tilde{\tau}_n+\tau_n)-\alpha/2]} , \tag{2.8}$$

where [by Eq. (2.4)]

$$Z(\tilde{\tau}_n+\tau_n)=\begin{cases}\dfrac{Z(\tilde{\tau}_n)+\Omega\tan(\Omega\tau_n)}{1-[\tan(\Omega\tau_n)/\Omega]Z(\tau_n)} & \text{if } q=\Omega^2\\ \dfrac{Z(\tilde{\tau}_n)-\Omega\tanh(\Omega\tau_n)}{1-[\tanh(\Omega\tau_n)/\Omega]Z(\tau_n)} & \text{if } q=-\Omega^2 .\end{cases} \tag{2.9}$$

This equation gives a relation between $Z(\tilde{\tau}_{n+1})$ and $Z(\tilde{\tau}_n)$, and hence between $\phi_{n+1}=\phi(t_n+0)$ and ϕ_n, and Eq. (2.9) thus defines the return map for the equation. Using the notation $Z(\tilde{\tau}_n)=Z_n$ one reduces Eqs. (2.8) and (2.9) to

$$Z_{n+1}=\frac{\alpha}{2}+\frac{P_n+Q_nZ_n}{R_n+S_nZ_n} , \tag{2.10}$$

where P_n, Q_n, R_n, and S_n are known numbers. The substitution

$$Z_n=\frac{\alpha}{2}+s_nu_n , \tag{2.11}$$

with the proper s_n, reduces Eq. (2.10) to

$$u_{n+1}=w_n\frac{u_n-r_n}{1+r_nu_n} \tag{2.12a}$$

or

$$u_{n+1}=w_n\frac{u_n-r_n}{1-r_nu_n} . \tag{2.12b}$$

These equations may of course be studied "as is." However, more insight into the structure of the solutions can be achieved by transforming the recurrence relations (2.12) to a linear Schrödinger equation (see Azbel and Soven, Refs. 15 and 16). Consider, for instance, Eq. (2.12a). The equivalent Schrödinger equation is that of the Kronig-Penney model:

$$\psi''+\left[k^2-\sum_{\nu} v_\nu\delta(x-x_\nu)\right]\psi=0 . \tag{2.13}$$

The connection between the two problems is through the transfer matrix relating the wave function in the interval $(x_{n-1}<x<x_n)$ to that in the interval $(x_{n+1}<x<x_n)$.

The currentless wave function of the Schrödinger equation in the interval $x_{n-1}<x<x_n$ (with energy $E=0$) is

$$\psi=\exp(G_n/2)\cos[k(x-x_{n-1})-\Phi_n/2] . \tag{2.14}$$

The transfer matrix $\underset{\sim}{\Theta}_n$ defined by the equation

$$\begin{bmatrix}\exp\left[\frac{G_{n+1}}{2}-i\frac{\Phi_{n+1}}{2}\right]\\ \exp\left[\frac{G_{n+1}}{2}+i\frac{\Phi_{n+1}}{2}\right]\end{bmatrix}=\underset{\sim}{\Theta}_n\begin{bmatrix}\exp\left[\frac{G_n}{2}-i\frac{\Phi_n}{2}\right]\\ \exp\left[\frac{G_n}{2}+i\frac{\Phi_n}{2}\right]\end{bmatrix} \tag{2.15}$$

is given by

$$\underset{\sim}{\Theta}_n=\begin{bmatrix}\sec(h_n)\exp(-ika_n-ih_n) & i\tan(h_n)\exp(ika_n)\\ -i\tan(h_h)\exp(-ika_n) & \sec(h_n)\exp(ika_n+ih_n)\end{bmatrix} \tag{2.16}$$

with

$$h_n=\tan^{-1}(v_n/2k),\quad a_n=x_n-x_{n-1} . \tag{2.17}$$

The recursion relation can be reduced to (2.12a). The relations between the parameters in (2.15) and (2.16) and those in Eq (2.12a) are

$$u_n=\tan\left[\frac{\Phi_n-h_n}{2}-\frac{\pi}{4}\right],$$
$$w_n=\tan^2\left[\frac{\pi}{4}-\frac{h_n}{2}\right], \tag{2.18}$$
$$r_n=\tan\left[ka_n-\frac{h_n+h_{n-1}}{2}\right].$$

Of course, one could forget about the Schrödinger equation and consider Eq. (2.18) as a corollary of Eq. (2.15) and (2.16). Reversing (2.18) one finds

$$h_n=\frac{\pi}{2}-2\tan^{-1}\sqrt{w_n},$$
$$ka_n=\frac{h_n+h_{n-1}}{2}+\tan^{-1}(r_n), \tag{2.19}$$

and $G_0=0$, so that if u_0 is given then

$$\phi_0=h_0+\pi/2+\tan^{-1}u_0 .$$

Thus Eqs. (2.15)–(2.19) reduce the solutions of Eq. (2.1) to the phase change of the $E=0$ wave function of the Hamiltonian (2.13).

In the case of a random distribution of A_n's, (2.13) is a Hamiltonian for electrons in a one-dimensional random on-site potential. Such Hamiltonians have been studied in the context of localization in one-dimensional systems. The solutions to (2.15) are always chaotic (see Azbel and Rubinstein, Ref. 17). Translating this result to our original problem we conclude that, not surprisingly, the pendulum perturbed by a random periodic driving force will exhibit chaotic motion.

In the periodic case $A_n=A$ for all n one finds, since $\underline{\Theta}_n=\underline{\Theta}$,

$$\begin{bmatrix}\exp\left[\dfrac{G_{n+1}-i\Phi_{n+1}}{2}\right]\\ \exp\left[\dfrac{G_{n+1}+i\Phi_{n+1}}{2}\right]\end{bmatrix}=\underline{\Theta}^n\begin{bmatrix}\exp\left[-\dfrac{i\Phi_0}{2}\right]\\ \exp\left[\dfrac{i\Phi_0}{2}\right]\end{bmatrix}, \tag{2.20}$$

$$\underline{\Theta}^n=\frac{1}{D}\begin{bmatrix}\Theta_{12} & \lambda_1-\Theta_{11}\\ \lambda_2-\Theta_{22} & \Theta_{21}\end{bmatrix}\begin{bmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{bmatrix}\begin{bmatrix}\Theta_{21} & \Theta_{11}-\lambda_1\\ \Theta_{22}-\lambda_2 & \Theta_{12}\end{bmatrix} \tag{2.21}$$

where λ_1 and λ_2 are the eigenvalues of $\underline{\Theta}$ and $D=\lambda_1\Theta_{22}+\lambda_2\Theta_{11}-2$. In the pth "energy gap" where $|\cos(ka+h)|>|\cos(h)|$, λ_1 and $\lambda_2=1/\lambda_1$ are real and $\phi_n\to$const when $n\to\infty$, so

$$\Phi_{n+1}=\Phi_n+2\pi p . \tag{2.22}$$

The relation between the original phase ϕ_n and Φ_n is given by combining (2.18), (2.3), and (2.11):

$$u_n=\tan\left[\frac{\Phi_n-h}{2}-\frac{\pi}{4}\right]$$
$$=\frac{I}{s}\tan(\phi_n) \tag{2.23}$$

so $|\Phi_n/2-\phi_n|<\pi$ and the frequency of the original problem becomes

$$\omega=\lim_{n\to\infty}\frac{\phi_n-\phi_0}{n}=\frac{\Phi_n-\Phi_0}{2n}=\pi p . \tag{2.24}$$

In average, the pendulum performs p rotations between two successive kicks. These solutions give the main Shapiro steps in the Josephson-junction problem. The width of the pth Shapiro step when I is varied is a complicated function of p which is given by the condition

$$|\cos(ka+h)|>|\cos(h)|$$

for $ka\sim p\pi$, with ka and h given by Eqs. (2.19) and (2.9)–(2.12).

In the allowed bands where

$$\lambda_1=\exp(i\omega),\ \ \lambda_2=\exp(-i\omega),$$
$$\cos\omega=\sec(h)\cos(ka+h), \tag{2.25}$$

the phase Φ_n is given by

$$\Phi_n=2\tan^{-1}\left[\frac{A_1\cos(n\omega)+B_1\sin(n\omega)}{A_2\cos(n\omega)+B_2\sin(n\omega)}\right]$$
$$=2\tan^{-1}[Q\tan(n\omega+a_1)-a_2] \tag{2.26}$$

or

$$\tan^{-1}\left[\frac{1}{Q}\tan\left[\frac{\phi_n}{2}+a_2\right]\right]=n\omega+a_1 , \tag{2.27}$$

where A_1, B_1, A_2, B_2, Q, a_1 and a_2 are known numbers. Defining

$$\theta_n=\tan^{-1}\left[\frac{1}{Q}\tan\left[\frac{\phi_n}{2}+a_2\right]\right],$$

Eq. (2.27) takes the form

$$\theta_{n+1}=\theta_n+\omega , \tag{2.28}$$

which is a trivial "circle map" mapping one point θ_n on the circle onto another point θ_{n+1} on the circle.

Combining (2.23) and (2.27), θ_n can be expressed in terms of the original phase ϕ_n:

$$\theta_n=H(\phi_n) \tag{2.29}$$

with $|H(\phi_n)-\phi_n|<\pi$. Hence, the frequency of the pendulum is essentially equal to the frequency or winding number ω of the circle map (2.28) which constitute the 1D return map of the differential equation. The simple return map (2.28) was first found by Waldram and Wu,[13] for $F(\phi)=\sin\phi$.

The band edges give the transitions from locked to unlocked solutions. Since ω is a smooth function of the parameters of the original differential equation there can be no subharmonic frequency-locked steps between the main Shapiro steps. In the following section the result derived here will be extended to the case $0<\epsilon \ll 1$ which is applicable to CDW systems and Josephson junction in rf and microwave fields.

III. THE STRONGLY DAMPED CASE

We now proceed to consider the nontrivial case $\epsilon \ll 1$

$$\epsilon\ddot{\theta}+\dot{\theta}-F(\theta)=\sum_n A_n \delta(t-t_n) \; . \tag{3.1}$$

Note that the function on the right may with arbitrary precision be modified to represent any function $A(t)$. By allowing $A_n=\tau A(n\tau)$, $t_n=n\tau$, and $\tau \to 0$, then

$$\sum_n A_n \delta(t-t_n)=\int A(t')\delta(t-t')dt'=A(t) \; .$$

We suppose $\epsilon \ll 1$ and split the function θ into two functions ϕ and χ

$$\theta=\phi+\chi \; , \tag{3.2}$$

where χ is rapidly varying and always small, while ϕ is slow:

$$|\epsilon\ddot{\phi}| \ll |\dot{\phi}| \; . \tag{3.3}$$

Since χ is small, the winding number, or frequency defined in terms of θ, is the same as the frequency defined in terms of ϕ:

$$\omega=\lim_{t\to\infty}\frac{\phi(t)-\phi(0)}{t}=\lim_{t\to\infty}\frac{\theta(t)-\theta(0)}{t} \; , \tag{3.4}$$

so we need to calculate ϕ only. Retaining terms of order up to ϵ we find by inserting (3.2) into (3.1)

$$\dot{\phi}-F(\phi)+\epsilon\ddot{\chi}+\dot{\chi}=\sum_n A_n\delta(t-t_n) \; . \tag{3.5}$$

The phases at $t=t_n+0$ and $t=t_n-0$ are related by the equations

$$\delta\dot{\theta}_n=\dot{\theta}(t_n+0)-\dot{\theta}(t_n-0)=A_n/\epsilon \; ,$$
$$\delta\theta_n=0 \; , \tag{3.6}$$

which may be combined with (3.2):

$$\delta\phi_n=-\delta\chi_n \; , \tag{3.7}$$

$$\delta\dot{\phi}_n+\delta\dot{\chi}_n=A_n/\epsilon \; . \tag{3.8}$$

Equation (3.5) implies

$$\chi=C\exp(-t/\epsilon) \tag{3.9}$$

so

$$\dot{\chi}_n=-\chi_n/\epsilon$$

and

$$\dot{\phi}_n=F(\phi_n) \; . \tag{3.10}$$

Therefore, finally one obtains from Eqs. (3.7) and (3.8)

$$A_n/\epsilon=\delta\dot{\phi}_n+\delta\dot{\chi}_n=\delta\dot{\phi}-\delta\chi/\epsilon$$
$$=\delta F+\delta\phi/\epsilon=\delta(\epsilon F+\phi)/\epsilon \; . \tag{3.11}$$

Thus, with exponential accuracy

$$\delta(\epsilon F+\phi)=A_n \; , \tag{3.12}$$

which replaces (2.5) for $\epsilon\neq 0$ and

$$\dot{\phi}=F(\phi)=(\alpha/2)\sin(2\phi)+I$$

when $t\neq t_n$, so ϕ is given by the expressions (2.2) and (2.3) as before. Hence, the effects of the $\epsilon\ddot{\theta}$ term can be represented by a change in the matching condition at t_n.

Introducing integration constants τ_n, etc. by Eqs. (2.6), we find

$$f(\tilde{\tau}_{n+1})-f(\tilde{\tau}_n+\tau_n)+\epsilon F(\tilde{\tau}_{n+1})-\epsilon F(\tilde{\tau}_n+\tau_n)=A_n \; . \tag{3.13}$$

In the leading approximation

$$f(\tilde{\tau}_{n+1})=F(\tilde{\tau}_n+\tau_n)+A_n \; ,$$

and to first order in ϵ

$$f(\tilde{\tau}_{n+1})=f(\tilde{\tau}_n+\tau_n)+A_n+\epsilon[F(f(\tilde{\tau}_n+\tau_n)) -F(f(\tilde{\tau}_n+\tau_n+A_n)] \tag{3.14a}$$

For $F(\phi)=(\alpha/2)\sin\phi+I$, (3.14) becomes

$$f(\tilde{\tau}_{n+1})=f(\tilde{\tau}_n+\tau_n)+A_n-\epsilon\alpha\cos[f(\tilde{\tau}_n+\tau)]\sin\frac{A_n}{2} \; . \tag{3.14b}$$

Equation (3.14) expresses $f(\tilde{\tau}_{n+1})$ versus $f(\tilde{\tau}_n)$ [and thus ϕ_{n+1} versus ϕ_n through (2.3) and (2.4)] and defines a one-dimensional return map for the equation. The nontrivial one-dimensionality of the map, which has previously been found numerically[11] has thus been established analytically for $\epsilon \ll 1$ using perturbation theory to first order in ϵ.

How does the result (3.14) modify the simple result (2.28)? Expressing $\tilde{\tau}_{n+1}$ and $\tilde{\tau}_n$ as functions of ϕ_{n+1} and ϕ_n one finds that the return map in the case of $\epsilon=0$,

$$\theta_{n+1}=\theta_n+\omega \; ,$$

or, equivalently,

$$H(\phi_{n+1})=H(\phi_n)+\omega \; ,$$

or

$$\phi_{n+1}=H^{-1}(H(\phi_n)+\omega)=h_0(\phi_n)$$

should be replaced by

$$\phi_{n+1}=h_0(\phi_n)+\epsilon\tilde{g}(\phi_n) \; , \tag{3.15}$$

where $\epsilon\tilde{g}(\phi_n)$ is the last term in Eq. (3.14b) expressed in terms of ϕ_n. Hence

$$\begin{aligned}\theta_{n+1} &= H(\phi_{n+1}(\phi_n)) \\ &= H(h_0(\phi_n)+\epsilon\tilde{g}(\phi_n)) \\ &\simeq H(h_0(\phi_n))+\epsilon H'(h_0(\phi_n))\tilde{g}(\phi_n) \\ &= \theta_n+\omega+\epsilon H'(h_0(\phi_n))\tilde{g}(H^{-1}(\theta_n)) \\ &= \theta_n+\omega+\epsilon g(\theta_n) . \end{aligned} \tag{3.16}$$

where g is a very complicated, but analytical periodic function of θ_n, $g(\theta_n)=g(\theta_n+\pi)$. The original trivial return map has been replaced by the more complicated circle map (3.16). Solutions are generated by iterating the circle map. The function g can be expressed explicitly by tracing the equations which determine the dependence of the functions $\tilde{g}$ and h on the original parameters.

The nonlinear term in the circle map is proportional to the coefficient of the second-order term of the differential equation! This is a main result of the paper. The qualitative features of the maps (3.16) and (2.28) are very different. Whereas there are no subharmonic plateaus for $\epsilon=0$, the circle map with $\epsilon\neq 0$ gives phase locking at *every single rational frequency* $\omega=\pi p/q$,[14,18] as indicated in the figure.

The circle map (3.13) has a transition to chaos[18–20] at a value of ϵ for which θ_{n+1} versus θ_n becomes noninvertible, i.e., the value of ϵ for which

$$\epsilon g'_{\min}(\theta_n)=-1 .$$

The transition is caused by "overlap" of the resonances[18] $\omega=\pi p/q$ and cannot, of course, be treated by the perturbative methods used here. The limit $\epsilon \ll 1$ is far from the transition to chaos.

The function $g(\phi)$ is certainly not sinusoidal but contains higher harmonics with coefficients g_n. To first order in ϵ the width of the subharmonic steps of order q is proportional to the qth harmonic,

$$\Delta(p/q)\sim\epsilon g_q . \tag{3.17}$$

Because of the explicit form of $g(\phi)$ in (3.16), (3.17) tells us what the mode-locked regimes will be like to first order in ϵ. If g had been purely sinusoidal the width of the plateaus of order q would have been proportional to ϵ^q, i.e., extremely narrow. Although the frequency locks-in at every single rational value, the locked portions do not fill up the whole phase diagram.[14] There is room for quasiperiodic (incommensurate) solutions between the commensurate ones for small ϵ.

It has usually been assumed that the motion of sliding CDW's in $NbSe_3$, etc. is always in the overdamped regime $\epsilon\sim 0$. According to (3.17) this would imply very narrow subharmonic plateaus. Recent experiments by Grüner[8] reveal a multitude of high-order steps in a single-domain CDW sample. Our results imply that this is inconsistent with the completely overdamped limit but could well be understood from the recent theory which is valid for $\epsilon \ll 1$. At high rf amplitudes it seems that the CDW in $NbSe_3$ can be driven into the chaotic regime ($\epsilon\sim 1$) where our theory does not apply and one must resort to numerical methods. Indeed, $NbSe_3$ could very well be an ideal candidate for checking current theories for the transition to chaos in systems for which the return map is a circle map.

In any case, the theory presented here yields explicit results in the case $\epsilon \ll 1$ which are certainly accessible in CDW systems and Josephson systems in not too strong rf and microwave fields. We suggest that such experiments be analyzed in terms of the theory.

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[1] W. C. Stewart, Appl. Phys. Lett. **12**, 277 (1968); D. E. McCumber, J. Appl. Phys. **39**, 3113 (1968).
[2] E. Ben-Jacob, Y. Braiman, R. Shainsky, and Y. Imry, Appl. Phys. Lett. **38**, 822 (1981); Y. Imry, in *Statics and Dynamics of Nonlinear Systems,* edited by G. Benedek, H. Bilz, and R. Zeyher, (Springer, Berlin, 1983), p. 170.
[3] G. Grüner, A. Zawadowski, and P. M. Chaikin, Phys. Rev. Lett. **46**, 511 (1981); J. Bardeen, E. Ben-Jacob, A. Zettl, and G. Grüner, Phys. Rev. Lett. **49**, 493 (1982).
[4] A. Zettl and G. Grüner, Solid State Commun. **46**, 501 (1983); Phys. Rev. B **29**, 755 (1984).
[5] P. Monceau, J. Richard, and M. Renard, Phys. Rev. Lett. **45**, 43 (1980).
[6] See, for instance, W. J. Yeh, D.-R. He, and Y. H. Kao, Phys. Rev. Lett. **52**, 480 (1984); and Y. Imry, in *Statics and Dynamics of Nonlinear Systems,* Ref. 2.
[7] V. N. Belykh, N. F. Pedersen, and O. H. Soerensen, Phys. Rev. B **16**, 4853, 4860 (1977).
[8] G. Grüner (private communications); S. E. Brown, G. Mozurkewich, and G. Grüner, Phys. Rev. Lett. **52**, 2277 (1984).
[9] B. A. Huberman, J. P. Crutchfield, and N. H. Packard, Appl. Phys. Lett. **37**, 751 (1980).
[10] M. T. Levinsen, R. Y. Chiao, M. J. Feldman, and B. A. Tucker, Appl. Phys. Lett. **31**, 776 (1976).
[11] P. Bak, T. Bohr, M. H. Jensen, and P. V. Christiansen, Solid State Commun. **51**, 231 (1984); T. Bohr, P. Bak, and M. H. Jensen, Phys. Rev. A **30**, 1970 (1984).
[12] M. J. Renne and D. Polder, Rev. Phys. Appl. **9**, 25 (1974).
[13] J. R. Waldram and P. H. Wu, J. Low Temp. Phys. **47**, 363 (1982).
[14] M. R. Herman, Lect. Notes Math. **597** 271 (1977).
[15] M. Ya. Azbel and P. Soven, Phys. Rev. Lett. **49**, 751 (1982).
[16] M. Ya. Azbel, Phys. Rev. B **27**, 3901 (1983).
[17] M. Ya. Azbel and M. Rubinstein, Phys. Rev. B **28**, 3793 (1983).
[18] M. H. Jensen, P. Bak, and T. Bohr, Phys. Rev. Lett. **50**, 1637 (1983).
[19] D. Rand, S. Ostlund, J. Sethna, and E. Siggia, Phys. Rev. Lett. **49**, 132 (1982).
[20] M. J. Feigenbaum, L. P. Kadanoff, and S. J. Shenker, Physica (Utrecht) **5D**, 370 (1982); M. J. Feigenbaum and B. Hasslacher, Phys. Rev. Lett. **49**, 605 (1982).