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# Laughlin states in higher Landau levels

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We propose a procedure for determining the pair-correlation function and the energy for the higher-Landau-level generalization of the incompressible quantum fluid states proposed by Laughlin [Phys. Rev. Lett. 50, 1395 (1983)]. On the basis of explicit calculations for Landau-level filling factors $\nu=\frac{1}{3}$ and $\nu=\frac{1}{5}$, we conclude that the fractional quantum Hall effect is not restricted to the lowest-orbital Landau level.

For a two-dimensional electron gas in a strong magnetic field the Hall conductivity, $\sigma_H$, is given by[1,2]

$$\sigma_H=\frac{e^2\nu}{\hbar} \quad , \tag{1}$$

when the chemical potential has a discontinuity which, as a function of magnetic field, is pinned at a certain Landau-level filling factor, $\nu$. [$\nu\equiv 2\pi a_L^2\rho$, where $\rho$ is the electron density and $a_L=(\hbar c/eB)^{-1/2}$ is the magnetic length.] Discontinuities occurring at integral values of $\nu$ are associated with the splitting of the energy spectrum into Landau levels and are reflected in the integral quantum Hall effect.[3] Similarly, the observation of a fractional quantum Hall effect[4] shows that at some set of rational values of $\nu$, the electron gas has especially stable states. An explicit form for these states has been proposed by Laughlin[5] for $\nu=1/M$, where $M$ is an odd integer, and it is generally expected that the states corresponding to the other stable filling factors seen in experiments are generalizations of these, possibly ones to states involving both spins.[6-8] To our knowledge, the fractional quantum Hall effect has, to date, been observed only in the lowest-orbital Landau level[9] and theoretical work has been directed toward that case. However, as we see below, there exist natural generalizations of Laughlin's state in higher Landau levels. Moreover it is possible to evaluate the pair-correlation functions and hence the energies of these states, and we find them to be lower in energy than the corresponding charge-density-wave (CDW) states. The elementary excitations of these states are found to have a localized fractional charge as in the lowest Landau level, and the excitation energies can be estimated. On this basis we conclude that for sufficiently low temperatures, strong fields, and high electron mobilities, the fractional quantum Hall effect should occur in higher-orbital Landau levels.

We use a symmetric gauge vector potential

$$\vec{A}=B(-y/2,\ x/2,\ 0) \quad , \tag{2}$$

take $a_L$ as the unit of length and $\hbar\omega_c\equiv\hbar(eB/mc)$ as the unit of energy. Then the single-particle Hamiltonian (kinetic energy) operator may be written as

$$\hbar=a^\dagger a+\tfrac{1}{2} \quad , \tag{3a}$$

where

$$a\equiv\frac{1}{\sqrt{2}}(\alpha+\beta) \quad , \tag{3b}$$

$$\alpha\equiv\frac{x+iy}{2}\equiv\frac{Z}{2} \quad , \tag{3c}$$

and

$$\beta\equiv\frac{\partial}{\partial x}+i\frac{\partial}{\partial y} \quad . \tag{3d}$$

The normalized eigenfunctions of $\hbar$ may be written as

$$\psi_{n,m}(x,y)=(\pi n!m!)^{-1/2}(a^\dagger)^n(b^\dagger)^m\exp(-|z|^2/4) \quad , \tag{4a}$$

where

$$b\equiv\frac{1}{\sqrt{2}}(\alpha^\dagger-\beta^\dagger) \quad . \tag{4b}$$

(Note that $[a,a^\dagger]=[b,b^\dagger]=1$ and $[a,b]=[a,b^\dagger]=0$.) $H_0\psi_{n,m}=(n+\frac{1}{2})\psi_{n,m}$, so $n$ labels a Landau level and $m$ labels the eigenstates within a Landau level. The unnormalized many-body states invented by Laughlin for $\nu=1/M$ are

$$\Phi_M^0[Z]=\prod_{j<k}(Z_j-Z_k)^M\exp\left(-\tfrac{1}{4}\sum_l|Z_l|^2\right) \quad . \tag{5}$$

Noting that $\psi_{n=0,m}\propto Z^m$, it is easily seen that this state is entirely contained with the lowest Landau level. The corresponding state in the $n$th Landau level is[10]

$$\Phi_M^n[Z]=\prod_j(a_j^\dagger)^n\Phi_M^0[Z] \quad . \tag{6}$$

For $\psi_M^0[z]$, Laughlin[5] pointed out that the pair-correlation function $g_M^0(r)$ is identical to that of a two-dimensional one-component plasma (2DOCP) with plasma parameter $\Gamma=2M$ and ion-disk radius $a=a_L/\sqrt{2M}$. Below we use Eq. (6) to construct $g_M^n(r)$ from $g_M^0(r)$.

The pair-correlation function is defined by

$$g_M^n(|Z_1-Z_2|)=\frac{N(N-1)}{\rho^2\langle\Phi_M^n|\Phi_M^n\rangle}\int dZ_3\cdots\int dZ_N|\Phi_M^n[Z]|^2 \quad . \tag{7}$$

But using Eq. (4a) and the commutation relations among the $a$ and $b$ operators we see that for all $m$ and $m'$

$$\int dZ_k[(a^\dagger)^n Z_k^{m'}\exp(-|Z_k|^2/4)]^*[(a^\dagger)^n Z_k^m\exp(-|Z_k|^2/4)]=n!\int dZ_k(Z_k^{m'})^*Z_k^m\exp(-|Z_k|^2/2) \quad . \tag{8}$$

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It follows that $\langle \Phi_M^n | \Phi_M^n \rangle = (n!)^N \langle \Phi_M^0 | \Phi_M^0 \rangle$. The same identity can be used for the $N-2$ coordinates which are integrated over in Eq. (7) with the result that if $g_M^0(|Z_1 - Z_2|)$ is written (without loss of generality) in the form

$$g_M^0(|Z_1 - Z_2|) = \sum_{S_1', S_1} \sum_{S_2', S_2} A(S_1' S_1; S_2' S_2) Z_1^{*S_1'} Z_2^{*S_2'} Z_1^{S_1} Z_2^{S_2} \exp(-|Z_1|^2/2) \exp(-|Z_2|^2/2) \;, \tag{9}$$

then the corresponding function in the $n$th Landau level is given by

$$\begin{aligned} g_M^n(|Z_1 - Z_2|) = \sum_{S_1', S_1} \sum_{S_2', S_2} & A(S_1' S_1; S_2' S_2) (n!)^{-2} [(a^\dagger)^n Z_1^{S_1'} \exp(-|Z_1|^2/4)]^* \\ & \times [(a^\dagger)^n Z_2^{S_2'} \exp(-|Z_2|^2/4)]^* [(a^\dagger)^n Z_1^{S_1} \exp(-|Z_1|^2/4)] [(a^\dagger)^n Z_2^{S_2} \exp(-|Z_2|^2/4)] \;, \end{aligned} \tag{10}$$

$g_M^0(r)$ can be accurately determined by Monte Carlo calculations for the 2DOCP.[11,12] Our procedure for determining $g_M^n(r)$ is to fit these results to an appropriate analytic form, expand this expression in the manner of Eq. (9) and then use Eq. (10). We have done this in several different ways, one of which is discussed in detail in the following paragraph.

Girvin[13] has suggested an analytic form for $g_M^0(r)$ which is motivated by the observation that only odd powers of $r^2$ appear in the Taylor series expansion of $\exp(r^2/4) g(r)$.

$$g_M^0(r) = 1 - \exp(-r^2/2) - 2 \sum_{l=1}^{\infty}{}' \frac{C_l}{4^l l!} r^{2l} \exp(-r^2/4) \;. \tag{11}$$

[The prime on the sum in Eq. (12) indicates that only odd values of $l$ appear.] Of the forms we have tried this one allowed us to fit the Monte Carlo results for $M=3$ and $M=5$ reported in Ref. 12 with the fewest number of parameters. Each term in Eq. (11) can be expressed in the form of Eq. (9). To obtain a simple expression for $g_M^n(r)$ we then evaluate the right-hand side of Eq. (10) at $Z_2 \equiv 0$, $Z_1^* = Z_1 = r$, and use the identity

$$(a^\dagger)^n Z^M \exp(-|Z|^2/4) \big|_{Z=Z^*=r} = 2^{-n/2} \exp(-r^2/4) \times F_{n,M}(r) \;, \tag{12}$$

where

$$F_{n,m}(r) = 2^i r^{s-i} (-)^i i! L_i^{s-i}(r^2/2) \;, \tag{13}$$

$s$ is the greater of $m$ and $n$, $i$ the lesser of $m$ and $n$, and $L_n^\alpha(x)$ a generalized Laguerre polynominal. [Note that $F_{n,m}(r=0) = 0$ unless $m=n$.] The result is[14]

$$\begin{aligned} g_M^n(r) = {} & 1 - \exp(-r^2/2) [L_n^0(r^2/2)]^2 \\ & - 2 \exp(-r^2/2) \sum_{l=1}^{\infty}{}' \frac{C_l}{4^l l!} \sum_{s=0}^{\infty} \frac{1}{4^s s!} [F_{n,s+l-n}(r)]^2 \sum_{i,j}{}' (-)^j \begin{pmatrix} s \\ i \end{pmatrix} \begin{pmatrix} l \\ j \end{pmatrix} \begin{pmatrix} s+l-i-j \\ n-(i+j)/2 \end{pmatrix} \begin{pmatrix} i+j \\ (i+j)/2 \end{pmatrix} \;. \end{aligned} \tag{14}$$

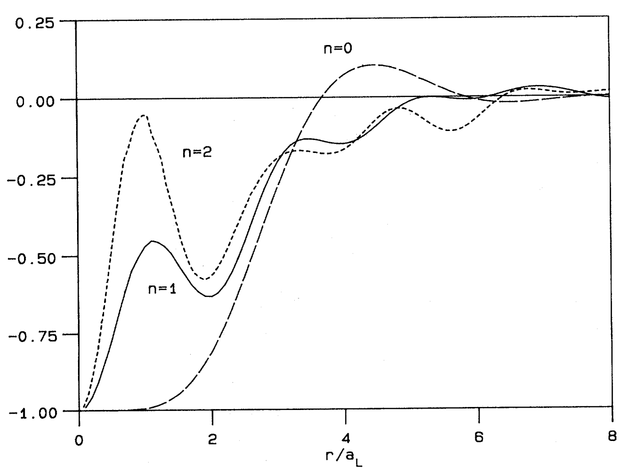


FIG. 1. Pair-correlation functions, $g^n(r)$, for the Laughlin states at $\nu = \frac{1}{3}$ in the $n=0$, $n=1$, and $n=2$ Landau levels. The long-dashed line is for $n=0$, the solid line for $n=1$, and the short-dashed line for $n=2$.

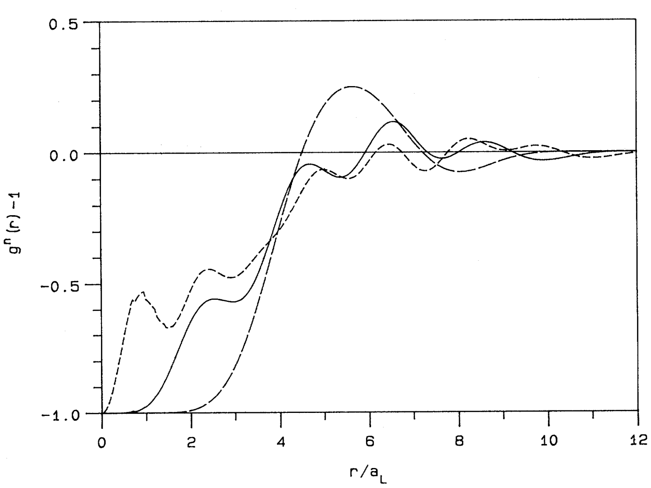


FIG. 2. As in Fig. 1 but for the $\nu = \frac{1}{5}$ Laughlin states.

TABLE I. Energy per electron in units of $e^2/a_L$ for Laughlin and CDW states at $n=0$, 1, and 2 at $\nu=1/M$ for $M=3$ and 5. The CDW states are of the hexagonal type.

| | $M=3$ | | $M=5$ | |
|---|---|---|---|---|
| $n$ | CDW | Laughlin | CDW | Laughlin |
| 0 | −0.388 | −0.410 | −0.322 | −0.328 |
| 1 | −0.284 | −0.325 | −0.252 | −0.294 |
| 2 | −0.252 | −0.263 | −0.209 | −0.247 |

(The prime on the sum over $i$ and $j$ indicates that it is restricted to values for which $i+j$ is even.) The first two terms on the right-hand side of Eq. (14) give the pair correlation function for a full Landau level. Pair-correlation functions for $n=0$, 1, and 2 obtained using this procedure are illustrated in Fig. 1 for $M=3$ and in Fig. 2 for $M=5$. Note that $g_M^n(r=0)=0$ is guaranteed for all $n$ by the antisymmetry of the many-body wave function $\Phi_M^n[Z]$. It is also worth noting that the charge-neutrality sum rule,

$$\int_0^\infty dr\, r(1-g_M^n(r)=M \quad , \tag{15}$$

must be, and is, preserved by our procedure.[15]

The energy per electron for these Laughlin states is related to the pair-correlation function by[16]

$$E_M^n=-\frac{e^2}{2M}\int_0^\infty dr[1-g_M^n(r)] \quad . \tag{16}$$

In Table I we have compared the energies at $M=3$ and $M=5$ with the energies of the corresponding CDW states[14] for several values of $n$. It is quite clear that Laughlin's states remain lower in energy, at least in some cases, in the higher Landau levels. For example, for $M=3$, the density is ideally suited for the CDW state if $n=0$ but not if $n=1$.[14] As a result the energy difference between Laughlin and CDW states is actually larger in the higher Landau level.

In his theory for the fractional Hall effect, Laughlin proposed trial wave functions for states in which a quasiparticle or quasihole is created in $\Phi_M^{(0)}[Z]$.[5,17] These states have a localized excess or deficiency in the charge density containing a total charge $\pm e/M$ and can be raised to higher-Landau levels just as in Eq. (6). The quasiparticle or quasihole creation energy can be approximated by the self-interaction of the excess charge density $\delta\rho^{\pm}(\vec{r})$. For a quasiparticle or quasihole state in the lowest Landau level, with the quasiparticle or quasihole centered at the origin for convenience, $\delta\rho(\vec{r})$ can quite generally be written in the form

$$\delta\rho^{\pm(0)}(\vec{r})=\frac{1}{2\pi m}\sum_{s=0}^{\infty}\frac{C_s^{\pm}r^{2s}}{2^s s!}\exp(-r^2/2) \quad , \tag{17}$$

where $\sum_s c_s^{\pm}=\pm1$. Using arguments similar to those for the pair correlation, the excess charge densities for the corresponding states in higher Landau are

$$\delta\rho^{\pm(n)}(\vec{r})=\frac{1}{2\pi m}\sum_{s=0}^{\infty}\frac{C_s^{\pm}}{2^{s+n}s!n!}[F_{n,s}(r)]^2\exp(-r^2/2) \quad . \tag{18}$$

(Note that the total excess or deficiency of charge is preserved.) For the present purposes it is adequate to approximate $\delta\rho^{\pm(0)}(\vec{r})$ by taking $C_s^{\pm}=\pm1/m$, $s=0,\ldots,m-1$ in accordance with expectations for the size of the quasiparticle or quasihole. In Table II we list the quasiparticle creation energies in units of $e^2/a_L$ obtained from this approximation for $M=3$ and 5 and $n=0$, 1, and 2. For $n=0$ the values are 0.034 for $M=3$ and 0.010 for $M=5$ compared to Laughlin's estimates[18] 0.028 for $M=3$ and 0.007 for $M=5$. For $M=3$ we plot in Fig. 3 the approximate excess charge density of the quasiparticle state for $n=0$, 1, and 2. We see that the localized nature of the excitations is preserved in the higher-Landau-level states and that the dependence of the excitation energy of Landau-level index is approximately the same as that for the total energy.

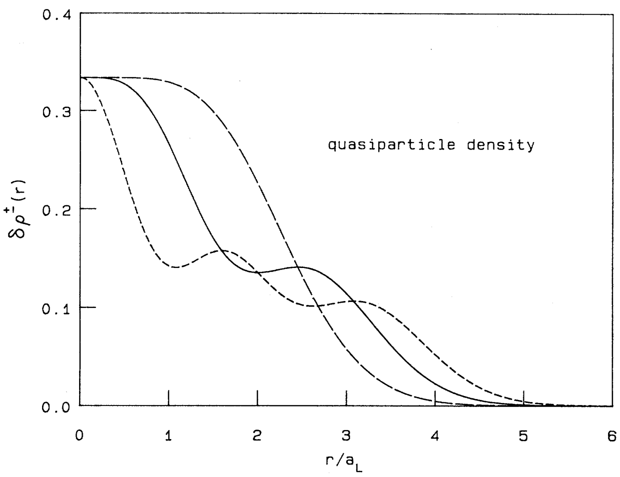


FIG. 3. Quasiparticle density distributions for $n=0$, $n=1$, and $n=2$ Landau levels. These plots are for quasiparticles created in the $\nu=\frac{1}{3}$ Laughlin state as discussed in the text. The long-dashed line is for $n=0$, the solid line is for $n=1$, and the short-dashed line is for $n=2$.

TABLE II. Estimates for quasiparticle or quasihole creation energies in the $n=0$, $n=1$, and $n=2$ Landau levels for $r=1/M$, where $M=3$ and $M=5$. The energies are in units of $e^2/a_L$.

| $n$ | $M=3$ | $M=5$ |
|---|---|---|
| 0 | 0.034 | 0.0098 |
| 1 | 0.028 | 0.0086 |
| 2 | 0.024 | 0.0077 |

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[10]This was originally suggested to the author by R. B. Laughlin.

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[14]Details of this calculation and an account of the associated CDW state calculations will be given elsewhere.

[15]The behavior of other sum rules is discussed in Ref. 14.

[16]We are assuming that $B$ is strong enough to make Landau-level mixing negligible and that all lower Landau levels are fully occupied. Then all states have identical energy contributions from the kinetic energy and from exchange with and among lower Landau levels. We incorporate these into our zero of energy.

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[18]The numbers we give are the average of Laughlin's estimates for the quasiparticle and quasihole creation energies which, in general, need not be identical.