Response of a Higgs kink to a static external force

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The acceleration of a ϕ^4 kink in a weak constant external field $|\chi| << 1$ is studied analytically and numerically. Similarly as for a sine-Gordon kink a non-Newtonian acceleration $\dot{v}(t) \sim \chi t^2$ is observed for $t << T=\sqrt{2}\pi$. However, for very weak fields $|\chi| << 1/T$ the acceleration in the time average over a full period T is Newtonian.

Fernandez, Gambaudo, Gauthier, and Reinisch,[1-3] have obtained the result that an initially static 2π antikink in the sine-Gordon equation (SGE) with a constant external force χ,

$$\phi_{xx}-\phi_{tt}=\sin\phi-\chi \quad , \tag{1}$$

for small times $t \lesssim 1$ does not experience the *a priori* expected constant ("Newtonian") acceleration $\dot{v}(t)=\pi\chi/4$, but $\dot{v}(t) \sim t^2$. However, when a constant is added to the initial soliton, the acceleration becomes proportional to this constant.[3,4] In both cases, the behavior has been ascribed to the interaction between the phonon waves excited by the force χ about the kink profile, and the kink itself.

In this Brief Report the nonrelativistic motion of a kink, initially at rest,

$$\phi(x,0)=\tanh(x/\sqrt{2})+u(0), \quad \phi_t(x,0)=0 \quad , \tag{2}$$

is studied in the Higgs scalar equation (HE) with an x- and t-independent external force χ,

$$\phi_{xx}-\phi_{tt}=\phi^3-\phi-\chi \quad , \tag{3}$$

by an analytical approach and numerical experiments.

Similarly as in Ref. 1, the solution to Eqs. (2) and (3) is supposed to be the sum of an exact kink and an x-independent perturbation function $u(t)$, i.e.,

$$\phi(x,t) \approx \tanh\frac{z}{\sqrt{2}}+u(t) \quad , \tag{4a}$$

where

$$z(x,t)=\gamma[x-s(t)], \quad \gamma=(1-v^2)^{-1/2} \quad , \tag{4b}$$

and

$$s(t)=\int_0^t v(t')\,dt', \quad v(0)=0 \quad . \tag{4c}$$

That $u(t)$ may be considered to depend approximately only on t, has been justified in Ref. 2 for the SGE. Taking the ansatz (4a) seriously, the position of the soliton at time t is determined by the maximum of the derivative $\phi_x(x,t)$, i.e., by $x=s(t)$, where $s(t)$ is given by Eq. (4c) and $v(t)$ is the kink's velocity. Then the ansatz (4a) leads to a simple expression for the acceleration $\dot{v}(t)$ which for small times, in the approximation $\gamma^3(1+v^2)\approx$ const, can be elementarily integrated two times to obtain $v(t)$ and $s(t)$. This procedure we will follow here.

Inserting Eqs. (4a)–(4c) into Eq. (3), one obtains for $x=s(t)$ or $z=0$,

$$-u_{tt}+\gamma^3(1+v^2)\dot{v}/\sqrt{2}=u^3-u-\chi \quad , \tag{5a}$$

while Eq. (3) in $x=\pm\infty$ reduces to

$$-u_{tt}=2u \pm 3u^2+u^3-\chi \quad . \tag{5b}$$

Equation (5b) shows that the perturbation function u depends also on x when terms of order u^2 are taken into account. Here we consider Eqs. (5a) and (5b) only in the first order of u, i.e., for $|u| << 1$ (and $|\chi| << 1$). Then, due to the supposed x independence of u, both equations may be subtracted from each other yielding

$$\gamma^3(1+v^2)\dot{v} \approx -3\sqrt{2}u \quad . \tag{6a}$$

The general solution to the first-order approximation of Eq. (5b)

$$-u_{tt} \approx 2u-\chi \quad , \tag{6b}$$

reads

$$u(t) \approx \frac{u_t(0)}{\sqrt{2}}\sin(\sqrt{2}t)+[u(0)-\chi/2]\cos(\sqrt{2}t)+\chi/2 \quad . \tag{6c}$$

In this first-order approximation $u(t)$ oscillates with a period $T=\sqrt{2}\pi$ and amplitude $|u(0)-\chi/2|$ [for $u_t(0)=0$] about the equilibrium $u=\chi/2$. Similarly as for the SGE,[1] for larger values of $|\chi|$ and $u(0)$ the perturbation function is no longer of the form (6c). (The numerical solution for the HE shows that for $\chi=0.3$ [and $u(0)=u_t(0)=0$] the ansatz (4) is no longer satisfied for $t \gtrsim 2$.) In spite of this the following approach for $\dot{v}(t)$ is valid also for larger values of $|\chi|$, for small times, as long as $|u(t)| << 1$.

To derive Eq. (6a) it is not necessary that $u(t)$ is independent of x. A weaker (sufficient) condition is that the derivative $u_x(x,t)$ is symmetric with respect to $x=s(t)$ and vanishes faster than $1/x$ for $|x| \rightarrow \infty$. (The latter is satisfied for $t<\infty$.) Then the (finite) values of u (and correspondingly u_{tt}) in $x=\pm\infty$ and $x=s(t)$ are related to each other by

$$u(x=+\infty,t)+u(x=-\infty,t)=2u[x=s(t),t] \quad , \tag{7}$$

[and Eqs. (6a)–(6c) are meant for $x=s(t)$].

Integrating Eq. (6a) with Eq. (6c) twice for $u_t(0)=s(0)=v(0)=0$ [and $\gamma^3(1+v^2)\approx 1$] one obtains for the position of the kink in leading order of t,

$$s(t) \approx -3u(0)t^2/\sqrt{2} \quad \text{for } u(0)\neq 0 \quad , \tag{8a}$$

and

$$s(t) \approx -\sqrt{2}\chi t^4/8 \quad \text{for } u(0)=0 \quad . \tag{8b}$$

These results show an analogous time dependence as the corresponding results for a sine-Gordon kink.[1–4]

To check formulas (8a) and (8b), Eqs. (2) and (3) are integrated numerically (by the method of characteristics), for seven sets of the parameters χ and $u(0)$. In Table I the symbol $s_m(t)$ denotes the point of steepest slope in the numerical solution $\phi(x,t)$ at time t. This definition agrees with that used above to derive the expressions (8a) and (8b) from the ansatz (4a). Since the perturbation, in general, causes small asymmetries in the shape of the kink, the "true" expression for the position of the soliton is hard to find. Therefore we consider in the numerical solution also another definition for the position of the kink,[1–3] namely,

$$s_I(t)=\int_{-\infty}^{+\infty} x\phi_x(x,t)\,dx \Big/ \int_{-\infty}^{+\infty} \phi_x(x,t)\,dx \ . \tag{9}$$

In the examined time interval $0 \leqslant t \leqslant 1$ both definitions, $s_m(t)$ and $s_I(t)$, lead to good agreement with each other and also with $s(t)$, given by Eqs. (8a) and (8b), respectively (cf., Table I). [This is different from the situation in the SGE (1) where $s_m(t)$ and $s_I(t)$, even for small $|\chi|$, exhibit considerable differences.] Also the prediction [Eq. (8a)] that for different signs of $u(0)$ the kink moves into opposite directions is verified by the numerical results. For $\chi=0.2$ and $u(0)=-0.2$ differences of about 15–25% appear for $t \gtrsim 0.8$ between $s(t)$ and the numerical results $s_m(t)$ and $s_I(t)$. It is interesting that this is corrected by the next order term

$$3\sqrt{2}(u(0)-\chi/2)t^4/12 \approx -0.11t^4$$

in Eq. (6c), although higher orders in u have been neglected. [Obviously, this fourth-order term in t becomes less important when χ and $u(0)$ have equal signs.]

For very small values $|\chi|$, $|u(0)| \ll 1/T$ [such that $\gamma(v)$ and v may be considered as constant during a full period T] Eqs. (6a) and (6c) can be integrated for larger times $t>1$, too. In this case one obtains in the average over a full period T, as long as $|v| \ll 1$, the acceleration

$$\bar{\dot{v}}_T \approx -3\chi/\sqrt{2} \ , \tag{10a}$$

while the position of the kink at time $t=T=\sqrt{2}\pi$ is given by

$$s(T) \approx -3\chi T^2/(2\sqrt{2}) \approx -20.9\chi \ , \tag{10b}$$

independently from $u(0)$ [and $u_t(0)$]. The numeral solution of Eqs. (2) and (3) yields $s_m(T)=-0.209$, -0.206, and -0.205 for the three sets $(\chi,u(0))=(0.01,0)$, $(0.01,0.03)$, and $(0.01,-0.03)$, and $s_I(T)=-0.209$ for all three sets, in excellent agreement with Eq. (10b). For arbitrary times (but $|v| \ll 1$) Eq. (6a) gives [with $u_t(0)=0$],

$$s(t) \approx -6\{[u(0)-\chi/2][1-\cos(\sqrt{2}t)]/2+\chi t^2/4\}/\sqrt{2} \ . \tag{10c}$$

For the above three sets of χ and $u(0)$ one obtains by a numerical integration of Eqs. (2) and (3) for $t=T/2=\pi/\sqrt{2}$ the results $s_m(T/2) \approx s_I(T/2)=-0.031$, -0.158, and $+0.097$, respectively, in complete agreement with $s(T/2)$ obtained by Eq. (10c).

Oscillations of $\dot{v}(t)$, related with those of $u(t)$, are predicted for all values of $|\chi|$ and of $|u(0)|$ [in the following we consider $u(0)=0$] for which the given approach holds. The difference between the situations for $|\chi| \ll 1/T$ and $|\chi| \ll 1$ is the following. For very small values $|\chi| \ll 1/T$ the velocity $|v| \ll 1$ remains small at least during a full period T. Therefore the amplitudes $(\sim\chi/2)$ of the oscillations of $\dot{v}(t)$ diminish slowly. In this nonrelativistic case the motion of the kink is Newtonian in the average over a full period as shown above, although it follows Eq. (8b) for $t \ll T$. With increasing $|\chi|$ (but $|\chi| \ll 1$, for example, $\chi \approx 0.1 \sim 0.2$) the velocity $v(t)$ tends faster to the relativistic value $|v|=1$, and the acceleration $\dot{v}(t)$ to $\dot{v}=0$. Then the amplitude in the oscillation of $\dot{v}(t)$ in the second half of the first period $(T/2 \lesssim t \lesssim T)$ is already considerably smaller than in the first half $(0 \lesssim t \lesssim T/2)$, and the non-Newtonian behavior for $t \ll T$ is no longer compensated in the average over the whole period $0 \leqslant t \leqslant T$. The latter can be understood as a consequence of the fact that the kink has left already during the first period the nonrelativistic, i.e., the Newtonian region.

In conclusion, the Higgs and sine-Gordon kinks behave similarly in a constant external field. It is emphasized here explicitly that also for the SGE, for $|\chi| \ll 1/T=(2\pi)^{-1}$, small oscillations of $\dot{v}(t)$ with a period $T=2\pi$ about the Newtonian acceleration are predicted. The result given in Ref. 3 (cf., Eq. (56) there) that "for very weak amplitude χ . . . the kink dynamics is 'classical'—i.e., Newtonian" has to be understood in the average over a full period. [The amplitude in the oscillation of $\dot{v}(t)$ is of the same order as the Newtonian acceleration, i.e., proportional to χ.]

The nonrelativistic motion of the kinks for both Eqs. (1) and (3) may also be discussed by means of the field

TABLE I. For $0 \leqslant t \leqslant 1$ the position $s(t)$ of the Higgs kink, given by Eq. (8a) for $u(0) \neq 0$ and by Eq. (8b) for $u(0)=0$, agrees well with the results $s_m(t)$ (point of steepest slope) and $s_I(t)$ (center of mass) of the numerical experiments. All numbers given here for s, s_m, and s_I must be multiplied by 10^{-3}. (For $t=0$ the kink [Eq. (2)] is at rest in $x=0$, i.e., $s(0)=s_m(0)=s_I(0)=0$.)

	t		0.2			0.4			0.6			0.8			1.0	
χ	$u(0)$	s	s_m	s_I	s	s_m	s_I	s	s_m	s_I	s	s_m	s_I	s	s_m	s_I
0.2	0	−0.06	>-0.1	−0.06	−0.9	−0.8	−0.9	−4.6	−4.5	−4.5	−14.5	−13.8	−13.9	−35.4	−33.2	−33.1
0.4	0	−0.11	≈ -0.1	−0.11	−1.8	−1.8	−1.8	−9.2	−8.9	−8.9	−29.0	−27.4	−27.8	−70.7	−66.1	−66.2
0.2	0.1	−8	−8	−8	−34	−33	−34	−76	−76	−77	−136	−135	−136	−212	−208	−213
0.2	0.2	−17	−17	−17	−68	−67	−67	−153	−148	−150	−272	−252	−261	−424	−372	−399
0.2	0.4	−34	−34	−34	−136	−134	−135	−305	−289	−303	−543	−470	−533	−849	−653	−818
0.4	0.2	−17	−17	−17	−68	−68	−68	−152	−152	−154	−272	−266	−276	−424	−402	−434
0.2	−0.2	+17	+17	+17	+68	+65	+65	+153	+139	+141	+272	+229	+233	+424	+311	+331

momentum

$$P = -\int_{-\infty}^{+\infty} \phi_x \phi_t \, dx \quad ,$$

supposing that its time derivative dP/dt is constant in time and using the ansatz (4), or the corresponding one for the SGE (1). For the HE one finds with $dP/dt = -2\chi$ and $u(0) = u_t(0) = 0$,

$$\gamma v(t) = -3\chi t/\sqrt{2} + (3\chi/2)\sin(\sqrt{2}t) \quad , \tag{11}$$

which corresponds for $|v| << 1$ (and small t) approximately with the above result (8b). For the SGE one can reproduce Eq. (4) of Ref. 1 with $dP/dt = -2\pi\chi$ (up to a negative sign, since in Ref. 1 a 2π antikink is considered). This indicates for both equations that, while dP/dt is a conserved quantity, the time derivative dP_{kink}/dt of the kink's momentum

$$P_{\text{kink}} = -\int_{-\infty}^{+\infty} \phi_x (\phi - u)_t dx$$

is not conserved. The difference is due to the excitation of the vacuum.

Since the similar topological structure of ϕ^4 kinks and 2π kinks leads one to expect analogous results also for their acceleration by an external force, we would like to emphasize the slight differences. In the SGE (1), the ansatz (4) leads to two different values of the acceleration (for $v = 0$ they differ by a factor $4/\pi$) according to whether one considers the maximum of $\phi_x(x,t)$ or the field momentum $P(t)$. This explains why the numerical results $s_m(t)$ and $s_l(t)$ differ considerably for a 2π kink, but agree well with each other for a ϕ^4 kink. This agreement for a ϕ^4 kink indicates that if for small velocities and small times the ϕ^4 kink becomes even slightly deformed, the derivative u_x of the perturbation function should remain approximately symmetric with respect to $x = s(t)$. Secondly, the almost perfect correspondence between the numerical results and the theoretical values $s(t)$ suggests that the formulas (8a) and (8b) for a ϕ^4 kink in leading order of t are already correct. For a 2π kink a corrective factor $8/\pi^2$ was obtained by considering the interaction between the kink and the phonons of wave number $k \neq 0$.[3] This interaction leads to the nonadiabaticity in the motion of a 2π kink. The corrected result was recently also obtained by perturbative methods based on the inverse scattering method.[5]

It seems that the debate about Newtonian or non-Newtonian behavior of 2π kinks has still not been finished. Eboli and Marques[6] claim that the answer depends on whether radiation is taken into account or not. (Here, radiation is meant in the sense of the vacuum excitation, and not of dissipating energy as it is usually understood for quasistable soliton solutions.) However, the oscillations of the vacuum arise since the boundary conditions for the soliton are changed by the external force. Therefore, one does not have the freedom to take radiation into account or not. The only situation where the vacuum does not oscillate [for $0 < |\chi| << 1$, $|u(0)| << 1$, and $|u_t(0)| << 1$] corresponds (in first order of u) to the initial conditions $u(0) = \chi$ (SGE) or $u(0) = \chi/2$ (HE) and $u_t(0) = 0$. In this case the kink experiences a constant acceleration, equal to the Newtonian value.[4] Recently, Dash[7] has made some comments in favor of a Newtonian behavior. He requires "When the external field is not time dependent the exact solution $u_\infty = \text{const}$, independent of t, is to be considered . . . ." As pointed out above, a solution $u_\infty = \text{const}$ (corresponding to $u = \text{const}$ in our notation) does not exist for $\chi \neq 0$ and $u(0) = u_t(0) = 0$, i.e., an initially exact (unshifted) kink. Furthermore, when differentiating Eq. (1) of his comments, Dash would also arrive at the non-Newtonian acceleration

$$\gamma^3 \dot{v}(t) = \pi\chi[t^2/8 + O(t^3)] \quad ,$$

which is the contribution from the real part $\chi\cos(\Omega t) \sim \chi$ (for $t << 1$) of the periodic external force. The imaginary part $i\chi\sin(\Omega t)$ contributes only to higher orders in t since $\sin(\Omega t) \sim \Omega t$ for $t << 1$.

Finally, the following should be emphasized. That the acceleration of initially exact unshifted kinks for small times is different from the Newtonian value does not mean that in sine-Gordon or ϕ^4 theory Newtonian dynamics is not valid. Just to the contrary, the field momentum $P(t)$ in the nonrelativistic limit satisfies Newton's law $dP/dt = -2\pi\chi$ (SGE) or $dP/dt = -2\chi$ (HE). Velocity and acceleration, however, are directly related to the (bare) kink's momentum P_{kink} and its derivative. In all cases where u_{tt} does not vanish identically dP_{kink}/dt and dP/dt are different from each other, and therefore the first cannot satisfy Newton's law. In consequence, the acceleration of the kink is different from the Newtonian value.

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