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Universal critical amplitudes in finite-size scaling

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It is argued that there is no nonuniversal, system-dependent, multiplicative metric factor in the finite-size scaling relation for the singular part of the free energy near a bulk critical point. New universality properties for various critical-point amplitudes follow: one universal ratio is $L/\xi_{||}(T_c)$ in finite-size transfer-matrix calculations.

I. INTRODUCTION

The universality of critical exponents and of certain critical amplitude ratios[1–4] is a central concept in the modern theory of critical phenomena. For an ordinary continuous transition or critical point of, for instance, a ferromagnet, renormalization-group (RG) theory predictions may be summarized in asymptotic scaling relations. For example, for the singular part of the reduced bulk *free-energy density* of a simple ferromagnet one has, as $t \equiv (T-T_c)/T_c \to 0$ and $h = H/k_BT \to 0$,

$$f_\infty^{(s)} \equiv F^{(s)}/Vk_BT \approx A_1 \, |\, t \,|^{2-\alpha} W^{\pm}(A_2 h \,|\, t \,|^{-\Delta}) \,, \tag{1.1}$$

where $\pm$ refers to $t \gtrless 0$. The exponents α and $\Delta \equiv \beta+\gamma$, and the *scaling functions* $W^{\pm}$, are the same for all systems in a given *universality class*.[1–3] Within the universality class, lattice structure, coupling constants, etc. may vary, but all such variation is summarized in the values of the nonuniversal *metric factors*, A_1 and A_2. In recent years there has been an increasing interest in the properties of systems with one or more finite dimensions when bulk parameters are close to their critical values (i.e., $h \simeq 0$, $T \simeq T_c$). A phenomenological account of the behavior is given by finite-size scaling theory[5,6] which finds diverse applications[7] in Monte Carlo, transfer-matrix,[8] and other numerical calculations, as well as in analyzing experimental data. In this paper we consider the question of scaling-function universality and the role of the nonuniversal metric factors in finite-size scaling.

For simplicity, we restrict attention mainly to *cubes*, of dimensions $L \times L \times \cdots \times L = L^d$, and *cylinders*, of dimensions $L^{d-1} \times \infty$; with *periodic* boundary conditions. The corresponding asymptotic finite-size scaling relation may then, quite generally, be written as

$$f^{(s)} \approx L^{-d} Y(C_1 t L^{1/\nu}, C_2 h L^{\Delta/\nu}) \,. \tag{1.2}$$

However, we argue (in Sec. II) that for spatial dimensionalities, d, less than the upper critical dimension $d_>$, the metric factors C_1 and C_2 are the *only* nonuniversal, *system-dependent* parameters entering: In other words, the scaling function $Y(x,y)$ is universal (for cubes or cylinders, respectively), but no further nonuniversal prefactor C_0 is required. In Sec. III we discuss various implications of this conclusion and its generalization to other boundary conditions. More specifically, we address, in Sec. IV, the universality of the ratio $L/\xi_{||}(T_c)$ in transfer-matrix calculations. A calculation of this ratio for the honeycomb-lattice Ising model illustrates and confirms the general conclusions. Finally, we present the generalization appropriate to other shapes of the finite system.

II. FINITE-SIZE SCALING FOR THE FREE ENERGY

In order to establish the common universality of, let us say, the fcc and the sc lattice Ising models, one first demonstrates that they can both be represented by the same more general, all encompassing model, say G [e.g., by the same class of Ginzburg-Landau-Wilson Hamiltonians in the context of, for instance, $\epsilon = (4-d)$-expansion RG calculations[1,9,10]], but with different "initial" parameters. Next, one needs to establish that the critical points of both models lie on critical manifolds in the domain of attraction of the same RG fixed point. Now a *bulk* RG transformation in the vicinity of a given fixed point can, in general, be represented in terms of a complete set of nonlinear scaling fields of the general model G.[11,4,3] For ordinary continuous transitions, as in simple ferromagnets, there are two relevant scaling fields: temperature-like, g_t, and fieldlike, g_h, with eigenexponents $\lambda_t = 1/\nu$ and $\lambda_h = \Delta/\nu$, respectively.[1,2] Upon approaching criticality, one may normally neglect corrections to scaling arising from the irrelevant scaling fields and from the nonlinearity of g_t and g_h; one then has $g_t \approx c_1 t$ and $g_h \approx c_2 h$, where c_1 and c_2 depend on the particular system for which t and h represent the reduced temperature and magnetic field. Hence, one can then represent[2,3,11] the asymptotic *bulk* RG transformation, with spatial rescaling factor $b \gg 1$, in the form

$$f_\infty^{(s)}(t,h) \approx b^{-d} \bar{f}_\infty^{(s)}(c_1 t b^{1/\nu}, c_2 h b^{\Delta/\nu}; 0,0,\ldots) \,, \tag{2.1}$$

where $\bar{f}_\infty(g_t, g_h; 0,0,\ldots)$ denotes the free energy of the model G, with all irrelevant scaling fields set to zero. Note that we have here implicitly excluded the occurrence of any marginal or *dangerous irrelevant variables*.[12,13] Thus our conclusion is restricted to $d < d_>$, as are our subsequent considerations, since only then may one expect that the irrelevant thermodynamic variables are not dangerous.

Now, on the basis of field-theoretic calculations for finite systems with periodic boundary conditions, it has been argued recently by Brézin[14] that for dimensions L, greatly exceeding all fixed microscopic lengths a_0 (e.g., lattice spacings), the renormalization-group transformation and the scaling fields are affected by L only through corrections to the leading scaling behavior. If we accept this hypothesis (which is discussed briefly below) the conclusion (2.1) may be generalized to yield the asymptotic behavior of the singular part of the free energy of a finite system in the form

$$f^{(s)}(t,h;L)\approx b^{-d}\bar{f}^{(s)}(c_1tb^{1/\nu},c_2hb^{\Delta/\nu};0,0,\ldots;L/b)\,, \tag{2.2}$$

where the dependence of $\bar{f}^{(s)}$ on L may be regarded as entering through appropriate infrared cutoffs on momentum variables.

Now one may choose $b=L/l_0$ where l_0 is some fixed, arbitrary, system-independent reference length satisfying only $l_0\gg a_0$. Then for $L\gg l_0$, the relation (2.2) reduces to

$$L^d f^{(s)}(t,h;L)\approx l_0^d\bar{f}^{(s)}[(c_1tL^{1/\nu})l_0^{-1/\nu},(c_2hL^{\Delta/\nu})l_0^{-\Delta/\nu};0,\ldots;l_0]\,. \tag{2.3}$$

But the left-hand side (for $L\gg a_0$) cannot depend on the choice of l_0. Consequently, it must be a definite, and hence universal function only of the two combinations $c_1tL^{1/\nu}$ and $c_2hL^{\Delta/\nu}$. Hence we obtain the assertion (1.2) (with C_1/c_1 and C_2/c_2 universal constants). Likewise, by choosing $b=B\,|\,c_1t\,|^{-\nu}$ with a system-independent proportionality constant B, one obtains (1.1) from (2.1) [or from (2.2) with $L\to\infty$] with $A_1/|\,c_1\,|^{2-\alpha}$ and $A_2/c_2\,|\,c_1\,|^{-\Delta}$ universal constants. Note that both these thermodynamic metric factors *and* the finite-size metric factors, C_1 and C_2, depend only on the two system parameters c_1 and c_2. We have thus obtained a finite-size form of the so-called "two-scale factor universality" hypothesis,[15–18] or "hyperuniversality" hypothesis,[19,20] which, in its bulk form may be stated as[19]

$$\lim_{t\to 0+} f^{(s)}(t,h=0)[\xi_\infty(t)]^d=Q_1=\text{universal}\,, \tag{2.4}$$

where, in (1.1), we have adopted the normalization $W^+(0)=1$, while

$$\xi_\infty(t)\approx c_0t^{-\nu},\quad \text{as } t\to 0+\,, \tag{2.5}$$

is the zero-field bulk correlation length (measured, for example, by the second moment of the two-point correlation function in the infinite system). Alternatively, one may say that $A_1c_0^d\ (\equiv Q_1)$ is universal (see also Appendix A).

Note that the condition $L\gg l_0$ (i.e., $b\gg 1$) enters into the derivation so that corrections due to the nonlinearities of the scaling fields, g_t and g_h, may be neglected, and in order to justify setting the irrelevant variables to zero. The conditions $L\gg a_0$ and $l_0\gg a_0$ likewise enable us to formally neglect finite-size corrections in $f^{(s)}$ which depend explicitly on a_0/L. However, Brézin[14] has not really established that such dependences are not dangerous (even for $d<d_>$). Thus it is worthwhile to consider the problem from a somewhat different viewpoint. Accordingly, in the next section, we present an alternative, phenomenological derivation of (1.2) which is, hopefully, more transparent physically and which one may expect will also be valid for nonperiodic boundary conditions.

III. PHENOMENOLOGICAL APPROACH: UNIVERSAL AMPLITUDES

In a bulk system the analytic "background" of the free-energy density, $f_\infty^{(0)}(t,h)$, can be identified unambiguously. Then, for a finite system the singular part of the finite-size free-energy density may be *defined* by

$$f^{(s)}(t,h;L)=f(t,h;L)-f_\infty^{(0)}(t,h)\,. \tag{3.1}$$

The relevance of this definition to microscopically based RG calculations will be discussed briefly below. Now the original finite-size scaling prescription[5–7,14] asserts that all lengths diverging at bulk criticality should scale with the bulk correlation length $\xi_\infty(t)\approx c_0t^{-\nu}$ [see (2.5)]. This provides for the natural generalization of the bulk scaling relation (1.1) to

$$f^{(s)}(t,h;L)\approx A_1\,|\,t\,|^{2-\alpha}\widetilde{W}^{\pm}(A_2h\,|\,t\,|^{-\Delta};L/\xi_\infty)\,, \tag{3.2}$$

where $\widetilde{W}^{\pm}(w,x)$ is universal. This universality embodies, of course, the fact that the near-critical fluctuations in a large system should not be sensitive to detailed structure with characteristic dimensions shorter than $a_0\ll L$ so that ξ_∞ must set all length scales.

However, even though the argument L/ξ_∞ itself entails no further nonuniversal factor, an additional metric factor c_0 does enter the finite-size scaling relation through the variation of $\xi_\infty(t)$; but this *same* metric factor must, equally, enter into the scaling of the net two-point correlation function in a bulk system through

$$G(\vec{R};t,h)\approx D_1R^{2-d-\eta}X^{\pm}(\vec{R}/\xi_\infty;\ D_2h\,|\,t\,|^{-\Delta}), \tag{3.3}$$

where $X^{\pm}(x,y)$ is universal. Granted this relation, one may invoke the fluctuation-susceptibility relation to link the metric factors D_1 and D_2 to A_1 and A_2 (see Appendix A). Furthermore, for $d<d_>$, the corresponding scaling form should *also* hold for the total correlation function, $\Gamma(\vec{R};t,h)=\langle s_{\vec{0}}s_{\vec{R}}\rangle$, which, when $|\,\vec{R}\,|\to\infty$, relates to the squared magnetization.[21] As shown in Appendix A, this closes the circle and yields

$$A_1=Q_1/c_0^d\,, \tag{3.4}$$

where Q_1 is universal. This is just the hyperuniversality relation[15–20] already alluded to in Sec. II. The hyperscaling relations $d\nu=2-\alpha$, etc.[21] also follow, and the metric factors D_1 and D_2 are seen to be universally related to A_1 and A_2 [see (A8)]. Finally, if (3.4) is substituted into the finite-size scaling relation (3.2) and this is rescaled, in the standard way, in terms of L instead of t, we recapture the basic result (1.2) with, now, $C_1/A_1^{1/d\nu}$ and $C_2/A_2A_1^{\Delta/d\nu}$ universal constants.

Since the analytic background $f_\infty^{(0)}(t,h)$ cannot contribute to the singular behavior of field derivatives one obtains from (1.2) scaling expressions for the finite-size magnetization,

$$m=-\frac{\partial f}{\partial h}\approx C_2L^{-\beta/\nu}Y^{(1)}(C_1tL^{1/\nu},C_2hL^{\Delta/\nu})\,, \tag{3.5}$$

for the susceptibility,

$$\chi=\frac{\partial m}{\partial h}\approx C_2^2 L^{\gamma/\nu} Y^{(2)}(C_1 t L^{1/\nu}, C_2 h L^{\Delta/\nu}) , \tag{3.6}$$

and for the fourth derivative (or nonlinear susceptibility),

$$\chi^{(4)}=-\frac{\partial^4 f}{\partial h^4}\approx C_2^4 L^{(\gamma+2\Delta)/\nu} Y^{(4)}(C_1 t L^{1/\nu}, C_2 h L^{\Delta/\nu}) , \tag{3.7}$$

and so on.

One can equally identify amplitudes in the scaling forms for the singular parts of the entropy, specific heat, and higher temperature derivatives of f, in terms of powers of C_1. Thereby, one obtains new universal critical-point ratios, for *finite-size* ($L<\infty$) quantities evaluated *at bulk criticality* ($T=T_c$, with $H=0$). In particular, the coefficients of proportionality U_j in asymptotic relations such as

$$f_c^{(s)}(L)\approx U_0 L^{-d} , \tag{3.8}$$

for the singular part of the critical free energy, and

$$\chi_c^{(4)}(L)/\chi_c^2(L)\approx U_1 L^d , \tag{3.9}$$

etc., should be universal. [Note, however, that $m_c^2 L^d/\chi_c$ vanishes identically because symmetry cannot be broken in a finite system so that $Y^{(1)}(x,0)\equiv 0$.] Quantities like $\chi_c^{(4)}/\chi_c^2$ can be estimated straightforwardly from Monte Carlo or transfer-matrix data.

Although the derivation just presented for (1.2) and (3.5)—(3.9) may be appealing heuristically, it is certainly open to challenge. Accordingly, we enlarge on a few further aspects here. First, the basic finite-size ansatz (3.2) remains applicable for general boundary conditions[5–7] although, naturally, the scaling functions must depend on the boundary conditions[22,23] and further scaled arguments for surface fields, etc. may enter when more realistic surface parameters are introduced. Various analytical checks, particularly of surface properties, which vary as $1/L$ relative to bulk properties, confirm the validity of (3.2) for free boundary conditions, for antiperiodic boundary conditions, etc., imposed on planar Ising models, ideal Bose fluids, etc.[5–7,22,23] However, for nonperiodic boundary conditions the absence of less rapidly decaying powers of L in (3.8) may be questioned. Indeed, powers such as $(a_0/L)^2$, might well be anticipated, but if they do occur, they should perhaps be regarded only as part of a "nonsingular" finite-size background; thus the corresponding terms might appear in (3.5)—(3.7) and (3.9) merely as higher-order, nondivergent corrections. However, analyticity in $1/L$ can hardly be used as a criterion in defining a finite-size background contribution for the free energy!

Similar problems arise if one approaches the issue from a renormalization-group standpoint. Although block-spin and other renormalization groups may be defined for finite systems, the character to be attributed to behavior near a bulk fixed point when L/a_0 is finite but large is not obvious. More specifically, it is not clear, in general, if the only additive contributions to the free energy in the renormalization process should be those entering in the bulk limit, as Brézin seems to find for periodic boundary conditions (at least up to exponentially small terms), or if important additive terms depending on L should enter, as would seem plausible for free boundary conditions. Likewise, it is not obvious that the simple multiplicative renormalization $L \Longrightarrow L/b$, invoked in (2.2), is always adequate, even in leading order.[24–26] To assert that this suffices is equivalent to supposing that $1/L$ acts as a distinct linear scaling field which does not mix with other thermodynamic fields such as t and h; while plausible, this, again, is not entirely convincing at criticality even though corrections due to surfaces can be separated clearly from bulk contributions away from criticality.[5–7,26–28] In summary, a deeper analysis of finite-size scaling within a renormalization-group context is needed to fully cement our confidence in "finite-size hyperuniversality" and might even lead to nontrivial modifications of the conclusion (1.2) in certain circumstances.

IV. CRITICAL-POINT CORRELATION RATIOS

If $\Lambda_0(T,H;L)$ is the largest eigenvalue of a transfer matrix which builds up an $L^{d-1}\times\infty$ cylinder from $L^{d-1}\times a$ slices, the free-energy density is

$$f(T,H;L)=f_0\equiv-(1/aL^{d-1})\ln\Lambda_0(T,H;L) . \tag{4.1}$$

Furthermore, if Λ_j with $\Lambda_0>\Lambda_1\geq\Lambda_2\geq\cdots$ denotes the subdominant eigenvalues of the transfer matrix, the *longitudinal correlation length* of the system is

$$\xi_{\parallel}(t,h;L)\equiv\xi_1=a[\ln(\Lambda_0/\Lambda_1)]^{-1} . \tag{4.2}$$

It is then natural to postulate the finite-size scaling relation[5,6,14]

$$\xi_{\parallel}(t,h;L)/\xi_\infty(t)\approx K^{\pm}(A_2 h\,|\,t\,|^{-\Delta};L/\xi_\infty) , \tag{4.3}$$

where $K^{\pm}(w,x)$ is universal. This clearly represents a generalization of (3.2), together with the hyperuniversality relation (3.4). By repeating the steps which led previously to (1.2), we obtain the analogous result, namely

$$\xi_{\parallel}(t,h;L)\approx LS(C_1 t L^{1/\nu}, C_2 h L^{\Delta/\nu}) , \tag{4.4}$$

where C_1 and C_2 are the *same* metric factors as in (1.2). Thus we see that the critical-point ratio $L/\xi_{\parallel,c}$ should be a universal number.

The universality of the ratio $L/\xi_{\parallel,c}$ for systems with *periodic* boundary conditions has already been noticed, mainly on the basis of numerical data, for quite a large class of two-dimensional models.[8,24,29–33] However, it has previously been regarded as something of a puzzle. Our argument does not, of course, determine the *value* of the ratio $L/\xi_{\parallel,c}$ although this happens to equal $\pi\eta$ in all the spatially *isotropic* models studied so far. (However, the generality of this particular result is *not* obvious,[26] even in two-dimensional systems.) Our present considerations (see also below) reveal the origin of this universality property for $d<d_>$. However, one should recall again that the scaling functions,[22,23] and thus the *values* of the universal quantities including the U_j of Sec. III, will normally depend on the boundary conditions.

At criticality ($T=T_c$, $H=0$) infinitely many of the eigenvalues $\Lambda_j(L)$ approach $\Lambda_0(L)$ when $L\to\infty$. The corresponding effective correlation lengths

$$\xi_j(t,h;L)\equiv a[\ln(\Lambda_0/\Lambda_j)]^{-1}, \tag{4.5}$$

thus diverge at T_c. (Normally both ξ_1 and ξ_2 are included in this diverging set.) Then the standard scaling hypothesis,[5,6] asserting that all diverging lengths become proportional, implies that the ratios $L/\xi_{j,c}$ approach finite constants when $L\to\infty$. Indeed, Nightingale and Blöte have concluded[26] that $L/\xi_{2,c}$ asymptotically approaches a *universal* constant, in several models (see Ref. 26 for details). This behavior of ξ_2 is related, in systems with a magnetic symmetry ($H\Longleftrightarrow -H$), to the fact that ξ_2 describes the decay of energy-energy correlations (represented by four-point expectations with the points grouped in two well-separated pairs). Consequently, ξ_2 has a normal bulk limit so that scaling relations of the type (4.3) can be expected to apply.

An alternative route to the conclusions about the ratios $L/\xi_{j,c}$ can be formulated in terms of the "free-energy levels"[34,35]

$$f_j(T,H;L)\equiv -(1/aL^{d-1})\ln\Lambda_j(T,H;L). \tag{4.6}$$

It is generally expected[34–36] that f_0 and certain of the higher levels f_j, including both f_1 and f_2, constitute branches of one analytic function of H, at fixed T, or of T, at fixed H. Thus it is natural to *speculate* that, at least for the lowest few among the appropriate levels f_j, it is valid to generalize (1.2) and (3.1), to conclude

$$f_j-f_\infty^{(0)}\approx L^{-d}Y_j(C_1tL^{1/\nu},C_2hL^{\Delta/\nu}), \tag{4.7}$$

with universal scaling functions $Y_j(x,y)$. For those levels for which this conjecture holds, we conclude, using (4.5) and (4.6), that

$$L/\xi_{j,c}\approx Y_j(0,0)-Y_0(0,0), \tag{4.8}$$

so that again the limiting value of $L/\xi_{j,c}$ is universal.

It is important to stress that in all our considerations the linear dimension L is an actual *geometrical length* and not measured just by, say, a number of lattice layers, etc. This is important because anisotropy is a *marginal* RG field.[37] Thus a simple distortion of spatial coordinates is *not* a harmless operation. In this connection it is interesting that most of the exact analysis for Ising models on a triangular lattice and the corresponding numerical calculations for other models have been performed with the lattice distorted from the standard threefold symmetry, the sites being mapped rather onto a square lattice. In order to employ our finite-size universality relations, one must reexpress results so derived in terms of the length L of the original, isotropic lattice.

A similar example arises in the calculation of $L/\xi_{\|,c}$ for the *honeycomb*-lattice Ising model, which we will describe briefly. (Further details are given in Appendix B.) Recall that for the standard square lattice one finds $L/\xi_{\|,c}\to\frac{1}{4}\pi=\pi\eta$. Now Fig. 1 illustrates a honeycomb lattice with an even number of layers, N, and periodic boundary conditions; if a is the spacing between adjacent columns along a row, the length is

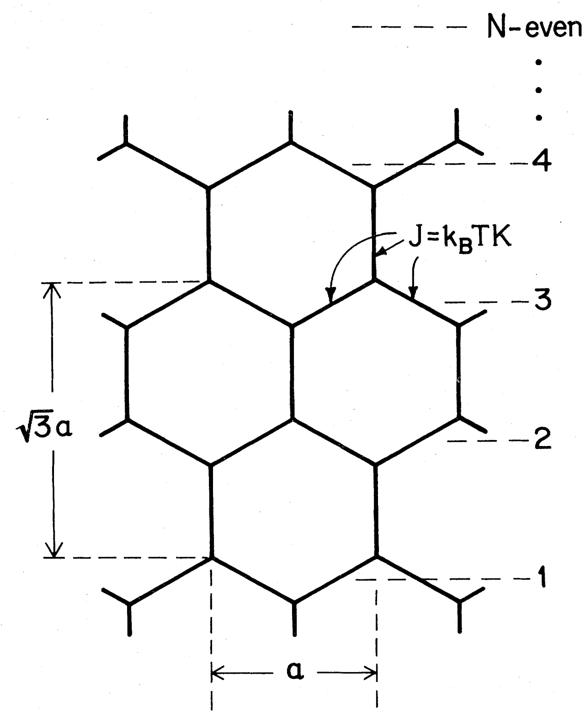

FIG. 1. Section of a honeycomb lattice of N layers with periodic boundary conditions which is built up by the transfer matrix in the horizontal direction; a denotes the length of an $L\times a$ slice added in one application of the transfer matrix.

$$L=\tfrac{1}{2}N(3^{1/2}a). \tag{4.9}$$

Thus if, as normally considered, the transfer matrix adds on two rows at a time, one has

$$a/\xi_{\|}=\ln(\Lambda_0/\Lambda_1). \tag{4.10}$$

The exact expressions for Λ_0 and Λ_1 are known (see Domb[38] for references to the literature; the results of Husimi and Syôzi[39] are particularly useful). A calculation thence gives

$$\ln(\Lambda_0/\Lambda_1)=\pi/2\sqrt{3}N+O(N^{-3}), \tag{4.11}$$

so that $L/\xi_{\|,c}\approx\frac{1}{4}\pi$ just as for the square lattice. Evidently, lattice-dependent geometrical factors cancel properly.

Finally we mention that in the case of a finite block or cylinder with unequal dimensions of, say, $L_1\times L_2\times\cdots\times L_d$ or $L_1\times L_2\times\cdots\times L_{d-1}\times\infty$, the scaling relation (1.2) generalizes naturally to

$$f^{(s)}\approx L_0^{-d}\tilde{Y}^{\pm}(C_1tL_0^{1/\nu},C_2hL_0^{\Delta/\nu};L_1/L_0,L_2/L_0,\ldots), \tag{4.12}$$

where L_0 is a (diverging) scale length and the shape ratios $l_j=L_j/L_0$ are supposed to approach bounded nonzero constants[13] as $L_0\to\infty$. Then the scaling functions should again be universal and combinations such as $L_0/\xi_{\|,c}$ or $f_c^{(s)}L_0^d$ will be universal functions of the asymptotic shape ratios l_j.

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APPENDIX A

We outline briefly the derivation of (3.4). All the calculations here are with *bulk* quantities. We start with the standard scaling form of the net correlation function, namely,[2]

$$G(\vec{R};t,h)\equiv\langle s_{\vec{0}}s_{\vec{R}}\rangle-\langle s_{\vec{0}}\rangle\langle s_{\vec{R}}\rangle \approx D_1R^{2-d-\eta}X^{\pm}(\vec{R}/\xi_\infty;D_2h\,|t\,|^{-\Delta}) , \quad (A1)$$

where $X^{\pm}(x;y)$ is universal and D_1 and D_2 are nonuniversal metric factors. Note that *no* metric factor is associated with the scaled argument $\vec{R}/\xi_\infty$. The reduced susceptibility χ follows by integrating on $\vec{R}$ which yields

$$\chi_\infty(t,h)\approx D_1\xi_\infty^{2-\eta}\tilde{X}^{\pm}(D_2h\,|t\,|^{-\Delta}) , \quad (A2)$$

where $\tilde{X}^{\pm}(y)$ is likewise found to be universal. Next we appeal to the "locality" property[21] of a standard "linear" renormalization group,[1–3] namely the simple multiplicative renormalization of the order parameter, via

$$s_{\vec{r}}=b^{-(d-2+\eta)/2}s'_{\vec{r}/b} . \quad (A3)$$

This shows that the scaling form of the total correlation function, $\Gamma(\vec{R};t,h)\equiv\langle s_{\vec{0}}s_{\vec{R}}\rangle$, is similar to (A1), so we may write

$$\Gamma(\vec{R};t,h)\approx D_1R^{2-d-\eta}Z^{\pm}(\vec{R}/\xi_\infty;D_2h\,|t\,|^{-\Delta}) , \quad (A4)$$

where $Z^{\pm}(x,y)$ is universal and hyperscaling is entailed: see below. In the limit $R\to\infty$ this reduces to

$$m_\infty^2(t,h)\approx D_1\xi_\infty^{2-d-\eta}\tilde{Z}^{\pm}(D_2h\,|t\,|^{-\Delta}) , \quad (A5)$$

where m is the magnetization density.

Now from the original thermodynamic scaling form (1.1) we obtain

$$m_\infty(t,h)\approx A_1A_2\,|t\,|^{\beta}W_1^{\mp}(A_2h\,|t\,|^{-\Delta}) \quad (A6)$$

with $\beta=2-\alpha-\Delta$ and

$$\chi_\infty(t,h)\approx A_1A_2^2\,|t\,|^{-\gamma}W_2^{\mp}(A_2h\,|t\,|^{-\Delta}) , \quad (A7)$$

with $\gamma=2\Delta-2+\alpha$, while $W_1^{\pm}$ and $W_2^{\pm}$ are universal. If the correlation length diverges as $\xi_\infty\approx c_0t^{-\nu}$ when $t\to 0+$ $(h=0)$ we may compare with (A2) and (A5), first, to see that the scaling relation $\gamma=(2-\eta)\nu$ *and* hyperscaling relation $2\beta=(d-2+\eta)\nu$ are implied, and then to obtain the identifications

$$A_1=Q_1c_0^{-d},\ \ A_2=Q_2D_2,\ \ D_1=Q_3A_1^{\psi}A_2^2 , \quad (A8)$$

where Q_1, Q_2, and Q_3 are universal constants and

$$\psi=2(\beta+\gamma)/(2\beta+\gamma)=1+\gamma/d\nu .$$

The first member here represents the hyperuniversality relations (2.4) and (3.4).

APPENDIX B

Husimi and Syôzi[39] obtained expressions for the transfer matrix eigenvalues for the honeycomb lattice described in Fig. 1. The two leading eigenvalues are

$$\Lambda_0=(2\sinh 2K)^N\exp(\gamma_1+\gamma_3+\cdots+\gamma_{N-1}) , \quad (B1)$$

$$\Lambda_1=(2\sinh 2K)^N\exp(\tfrac{1}{2}\gamma_0+\gamma_2+\gamma_4+\cdots+\gamma_{N-2}+\tfrac{1}{2}\gamma_N) , \quad (B2)$$

where $\gamma_j=\gamma(\pi j/N)$ with

$$\gamma(x)=\cosh^{-1}[\cosh 2K\cosh 2K^*-\sin^2x -\cos x(\sinh^2 2K\sinh^2 2K^*-\sin^2x)^{1/2}] . \quad (B3)$$

Here K^* is defined by

$$(\cosh 2K-1)(\cosh 2K^*-1)=1 , \quad (B4)$$

so that one has $K^*=K$ at the critical point and one then obtains

$$\gamma_c(x)=\cosh^{-1}[4-\sin^2x-\cos x\,(9-\sin^2x)^{1/2}] . \quad (B5)$$

This function resembles $\sin\frac{1}{2}x$ on the interval $(0,2\pi)$. We next rewrite the definition (4.2) in the form

$$a/\xi_{\|,c}=-\tfrac{1}{2}\sum_{k=0}^{2N-2}(-1)^k\gamma_c(\pi k/N) . \quad (B6)$$

The sum in (B6) can be evaluated asymptotically by a straightforward extension of the technique described in Ref. 13 for $T<T_c$. For any function of the general form of $\gamma_c(x)$ which is analytic for x in $(0,2\pi)$, is symmetric about $x=\pi$, and has bounded first three derivatives at $x=0$, one finds

$$a/\xi_{\|,c}=\gamma_c'(0)\pi/4N+O(N^{-3}) . \quad (B7)$$

Finally, (4.11) follows because $\gamma_c'(0)=2/\sqrt{3}$.

[1] K. G. Wilson and J. Kogut, Phys. Rep. 12C, 74 (1974).
[2] M. E. Fisher, Rev. Mod. Phys. 46, 597 (1974).
[3] Shang-keng Ma, Rev. Mod. Phys. 45, 589 (1973).
[4] A. Aharony and M. E. Fisher, Phys. Rev. B 27, 4394 (1983), and references therein.
[5] M. E. Fisher, in *Critical Phenomena,* Proceedings of the Enrico Fermi International School of Physics, edited by M. S. Green (Academic, New York, 1971), Vol. 51.
[6] M. E. Fisher and M. N. Barber, Phys. Rev. Lett. 28, 1516 (1972).
[7] M. N. Barber, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, New York, 1983), Vol. 8.
[8] M. P. Nightingale, J. Appl. Phys. 53, 7927 (1982).

[9]D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena* (McGraw-Hill, New York, 1978).
[10]E. Brézin, J. C. LeGuillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena* edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6.
[11]F. J. Wegner, Phys. Rev. B 5, 4529 (1972).
[12]M. E. Fisher, in *Renormalization Group in Critical Phenomena and Quantum Field Theory, Proceedings of a Conference,* edited by J. D. Gunton and M. S. Green (Temple University Press, Philadelphia, 1974).
[13]V. Privman and M. E. Fisher, J. Stat. Phys. 33, 385 (1983).
[14]E. Brézin, J. Phys. (Paris) 43, 15 (1982).
[15]D. Stauffer, M. Ferer, and M. Wortis, Phys. Rev. Lett. 29, 345 (1972).
[16]M. Ferer, D. Stauffer, and M. Wortis, in *Magnetism and Magnetic Materials—1972 (Denver),* proceedings of the 18th Annual Conference on Magnetism and Magnetic Materials, edited by C. D. Graham, Sr. and J. J. Rhyne (AIP, New York, 1973) p. 836.
[17]M. Ferer, M. A. Moore, and M. Wortis, Phys. Rev. B 8, 5205 (1973).
[18]P. Gerber, J. Math. Phys. A 8, 67 (1975).
[19]P. Gerber and M. E. Fisher (unpublished); M. E. Fisher and D. M. Jasnow, *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, to be published), Vol. 4.
[20]A. Aharony, Phys. Rev. B 9, 2107 (1974).
[21]M. E. Fisher, in *Collective Properties of Physical Systems,* Proceedings of Nobel Symposium No. 24, edited by B. Lundqvist and S. Lundqvist (Academic, London, 1975), p. 16.
[22]M. E. Fisher and A. E. Ferdinand, Phys. Rev. Lett. 19, 169 (1967).
[23]T. W. Capehart and M. E. Fisher, Phys. Rev. B 13, 5021 (1976).
[24]M. Suzuki, Prog. Theor. Phys. 58, 1142 (1977).
[25]M. P. Nightingale and H. W. J. Blöte, Physica 104A, 352 (1980).
[26]M. P. Nightingale and H. W. J. Blöte, J. Phys. A 16, L657 (1983).
[27]V. Privman and M. E. Fisher, J. Phys. A 16, L295 (1983).
[28]H. W. Diehl, J. Appl. Phys. 53, 7914 (1982), and references therein.
[29]J. M. Luck, J. Phys. (Paris) Lett. 42, L275 (1981).
[30]J. L. Pichard and G. Sarma, J. Phys. C 14, L617 (1981).
[31]B. Derrida and L. de Seze, J. Phys. (Paris) 43, 475 (1982).
[32]J. M. Luck, J. Phys. A 15, L169 (1982).
[33]P. A. Rikvold, W. Kinzel, J. D. Gunton, and K. Kaski, Phys. Rev. B 28, 2686 (1983).
[34]C. M. Newman and L. S. Schulman, J. Math. Phys. 18, 23 (1977).
[35]V. Privman and L. S. Schulman, J. Stat. Phys. 29, 205 (1982), and references therein.
[36]M. E. Fisher, Prog. Theor. Phys. Suppl. 69, 14 (1980).
[37]This is evident from the exact solutions of the two-dimensional Ising model; but see also A. D. Bruce, J. Phys. C 7, 2089 (1974).
[38]C. Domb, Adv. Phys. 9, 149,245 (1960).
[39]K. Husimi and I. Syôzi, Prog. Theor. Phys. 5, 177,341 (1950).