Acoustic-radiation stress in solids. I. Theory

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The Boussinesq radiation stress associated with an acoustic wave propagating in infinite and semi-infinite lossless solids of arbitrary crystalline symmetry is examined with use of two independent approaches. In both approaches the radiation stress is found to depend directly on an acoustic nonlinearity parameter which characterizes a spatially extended acoustic-radiation-induced static strain, a stress-generalized nonlinearity parameter which characterizes the stress nonlinearity in the solid, and the energy density of the propagating wave. Application of the Boltzmann-Ehrenfest principle of adiabatic invariance to a self-constrained system represented by the nonlinear equations of motion reveals that the radiation-induced static strain results from the time-averaged product of the internal energy density and the displacement gradient. The time-averaged product is scaled by the acoustic nonlinearity parameter and represents the first-order nonlinearity in the macroscopic virial theorem for solids. The term representing the stress nonlinearity is shown to result from the adiabatic variation in the natural velocity and produces a contribution to the total acoustic-radiation stress which is opposite in sign to that of the radiation-induced static strain. Finally, the relationship between the Boussinesq radiation stress and the Cauchy radiation stress is obtained in an exact three-dimensional form.

I. INTRODUCTION

Several recent investigations of nonlinear acoustic wave propagation in solids have dealt with the existence of a spatially extended static strain or "dc shift" associated with the propagating wave. Thurston and Shapiro[1] were the first to obtain a solution to the nonlinear acoustic wave equation that predicted such a phenomenon. Thompson and Tiersten[2] in a different approach to the problem made a similar prediction; they suggested that the static strain is related to the acoustic-radiation stress but did not elaborate. In a further study Perrin[3] pointed out an analogy to optical rectification and again emphasized the relationship to the radiation stress. Indirect experimental evidence for the existence of the static strain was reported by Carr and Slobodnik[4] for piezoelectric quartz and zinc oxide, and by Cantrell and Winfree[5] for single-crystal germanium. Yost and Cantrell[6] in the following paper report the first direct experimental confirmation of the static strain. The work is based on the theory to be presented here.

Although Brillouin[7,8] in the 1920's and 30's published an extensive theoretical study of acoustic-radiation stress in isotropic solids, his results do not predict the static strain. As we shall see, the failure of Brillouin's theory results from his neglect of the nonlinearity of the elastic wave equation itself while considering only the nonlinearity in the stress-diplacement gradient relationship. In the present treatment we consider the general case of acoustic-radiation stress associated with quasicompressional and quasishear waves propagating in infinite and semi-infinite lossless solids of arbitrary crystalline symmetry. We first obtain the nonlinear equations of motion in anisotropic solids in a decoupled form having a parametrized natural velocity. A periodic particle-velocity solution to the nonlinear wave equation in which sinusoidal boundary conditions are employed leads to a reexamination of the relationship between the particle velocity and the displacement gradient which is used in determining the acoustic-radiation stress. The Boussinesq radiation stress is then defined and found to depend directly on an acoustic nonlinearity parameter which characterizes the radiation-induced static strain, a stress-generalized nonlinearity parameter which characterizes the stress nonlinearity, and the energy density of the propagating wave. Further aspects of the acoustic-radiation stress are revealed by applying the Boltzmann-Ehrenfest principle of adiabatic invariance to a self-constrained system described by the nonlinear equations of motion. The more general and powerful adiabatic principle, which is independent of solutions to the wave equation, not only confirms our previously derived results but allows us to identify the acoustic-radiation-induced static strain with a self-constrained variation in the time-averaged product of the internal energy density and displacement gradient. The time-averaged product is scaled by the acoustic nonlinearity parameter and represents the first-order nonlinearity in the virial theorem. Finally, the relationship between the Boussinesq and the Cauchy radiation stress is obtained in a closed three-dimensional form.

II. NONLINEAR EQUATIONS OF MOTION IN ANISOTROPIC SOLIDS

We consider the propagation of an elastic wave in a lossless semi-infinite solid of arbitrary crystalline symmetry. The equations of motion in Lagrangian coordinates are[9] (we assume Einstein summation convention)

$$\rho_0 \frac{\partial^2 \bar{u}_i}{\partial t^2} = \frac{\partial \bar{\sigma}_{ij}}{\partial \bar{a}_j} , \tag{1}$$

where $\bar{a}_j$ are the Lagrangian (material) coordinates, $\bar{u}_i$ are the components of the wave displacement vector, and $\bar{\sigma}_{ij}$ are components of the stress tensor. If the stress tensor is defined in terms of Lagrangian strain derivatives of the internal energy per unit mass $U(a_k, n_{kl}, S)$, the stress tensor is commonly called the first Piola-Kirchhoff tensor. Formally, the first Piola-Kirchhoff tensor is defined by[10]

$$\bar{\sigma}_{ij} = \rho_0 \bar{\alpha}_{ik} \frac{\partial U}{\partial \bar{\eta}_{jk}} , \tag{2}$$

where S is the entropy, ρ_0 is the unperturbed mass density of the solid, $\bar{\alpha}_{ik}$ are transformation coefficients defined by

$$\bar{\alpha}_{ik} = \delta_{ik} + \frac{\partial \bar{u}_i}{\partial \bar{a}_k} , \tag{3}$$

δ_{ik} are Kronecker deltas, and $\bar{\eta}_{kl}$ are the Lagrangian strains defined by

$$\bar{\eta}_{jk} = \tfrac{1}{2}(\bar{\alpha}_{lj}\bar{\alpha}_{lk} - \delta_{jk}) . \tag{4}$$

Brillouin[7] preferred to use the Boussinesq stress tensor $\mathscr{B}_{ij}$ which is defined in terms of derivatives of the internal energy per unit mass with respect to the displacement gradient $\partial \bar{u}_i / \partial \bar{a}_j = \bar{u}_{ij}$. It is straightforward to show the equivalence of the two stress tensors by writing

$$\mathscr{B}_{ij} = \rho_0 \frac{\partial U}{\partial \bar{u}_{ij}} = \rho_0 \frac{\partial \bar{\eta}_{kl}}{\partial \bar{u}_{ij}} \frac{\partial U}{\partial \bar{\eta}_{kl}} = \rho_0 \bar{\alpha}_{ik} \frac{\partial U}{\partial \bar{\eta}_{kj}} = \bar{\sigma}_{ij} , \tag{5}$$

where we have used Eqs. (2)–(4). In the spirit of Brillouin's pioneering work, we shall refer to stresses in the Lagrangian coordinate frame as Boussinesq stresses, but shall retain the symbol $\bar{\sigma}_{ij}$ to emphasize equivalence to the first Piola-Kirchhoff tensor.

Huang[11] expanded $\bar{\sigma}_{ij}$ in terms of the displacement gradients as (assuming no initial stress)

$$\bar{\sigma}_{ij} = \bar{A}_{ijkl}\bar{u}_{kl} + \tfrac{1}{2}\bar{A}_{ijklmn}\bar{u}_{kl}\bar{u}_{mn} + \cdots , \tag{6}$$

where $\bar{A}_{ijkl}$ and $\bar{A}_{ijklmn}$ are the Huang (or propagation) coefficients. The relationship between the Huang coefficients and the elastic coefficients referred to the Lagrangian strain measure was found by Wallace[9] to be (no initial stress)

$$\begin{aligned} \bar{A}_{ijkl} &= \bar{C}_{ijkl} , \\ \bar{A}_{ijklmn} &= \bar{C}_{jlmn}\delta_{ik} + \bar{C}_{ijnl}\delta_{km} + \bar{C}_{jknl}\delta_{im} + \bar{C}_{ijklmn} . \end{aligned} \tag{7}$$

The $\bar{C}_{ijkl}$ and $\bar{C}_{ijklmn}$ are the second- and third-order elastic coefficients of Brugger[12] defined for adiabatic conditions by appropriate Lagrangian strain derivatives of the internal energy per unit mass

$$\begin{aligned} \bar{C}_{ijkl} &= \rho_0 \left[\frac{\partial^2 U}{\partial \bar{\eta}_{ij} \partial \bar{\eta}_{kl}} \right]_{S,\,\eta=0} , \\ \bar{C}_{ijklmn} &= \rho_0 \left[\frac{\partial^3 U}{\partial \bar{\eta}_{ij} \partial \bar{\eta}_{kl} \partial \bar{\eta}_{mn}} \right]_{S,\,\eta=0} . \end{aligned} \tag{8}$$

Substituting Eq. (6) into Eq. (1) and keeping terms to third order in derivatives of $\bar{u}_i$, we obtain the nonlinear wave equation

$$\rho_0 \frac{\partial^2 \bar{u}_i}{\partial t^2} = \left[\bar{A}_{ijkl} + \bar{A}_{ijklmn} \frac{\partial \bar{u}_m}{\partial \bar{a}_n} \right] \frac{\partial^2 \bar{u}_k}{\partial \bar{a}_j \partial \bar{a}_l} . \tag{9}$$

We may simplify Eq. (9) by introducing an orthogonal transformation R defined by

$$a_i = R_{ij}\bar{a}_j \tag{10}$$

which rotates the $\bar{a}_1$ axis into a_1 such that a_1 is parallel to the direction of wave propagation. Applying this transformation to Eq. (9) we obtain

$$\rho_0 \frac{\partial^2 u_p}{\partial t^2} = B_{pr} \frac{\partial^2 u_r}{\partial a_1^2} + B_{prt} \left[\frac{\partial u_t}{\partial a_1} \right] \frac{\partial^2 u_r}{\partial a_1^2} , \tag{11}$$

where B_{pr} and B_{prt} are known linear combinations of the appropriate Huang coefficients.

Equation (11) may be simplified further by introducing an orthogonal transformation S defined by

$$u_i = S_{ik} P_k \tag{12}$$

such that the symmetric matrix of coefficients B_{pr} is diagonalized. The transformation S reduces Eq. (11) to the form (no sum on j)

$$\rho_0 \frac{\partial^2 P_j}{\partial t^2} = \mu_j \frac{\partial^2 P_j}{\partial a_1^2} + \nu_{jkl} \frac{\partial P_l}{\partial a_1} \frac{\partial^2 P_k}{\partial a_1^2} . \tag{13}$$

In Eq. (13) and all following equations there is no sum over repeated j indices. The $j = 1,2,3$ are subscripts indicating polarization direction, μ_j are the eigenvalues of the B_{pr} matrix satisfying the similitude equation

$$\mu_j = (S^{-1})_{jp} B_{pr} S_{rj} , \tag{14}$$

and ν_{jkl} are linear combinations of third-order Huang coefficients satisfying the equation

$$\nu_{jkl} = (S^{-1})_{jp} B_{prt} S_{rk} S_{tl} . \tag{15}$$

The resonant terms of Eq. (13) corresponding to the condition $j = k = l$ are several orders of magnitude larger than the nonresonant terms.[13] If we retain only the resonant terms in Eq. (13), the nonlinear differential equation describing acoustic wave propagation of polarization j in the direction $\vec{\mathrm{N}}$ is written in the decoupled form

$$\rho_0 \frac{\partial^2 P_j}{\partial t^2} = \mu_j \frac{\partial^2 P_j}{\partial a_1^2} + \nu_{jjj} \frac{\partial P_j}{\partial a_1} \frac{\partial^2 P_j}{\partial a_1^2} . \tag{16}$$

III. PARTICLE-VELOCITY SOLUTION TO WAVE EQUATION AND RELATION TO DISPLACEMENT GRADIENT

It is convenient to write Eq. (16) in the form

$$\frac{\partial^2 P_j}{\partial t^2} = c_0^2 \left[1 - \beta_j \frac{\partial P_j}{\partial a_1} \right] \frac{\partial^2 P_j}{\partial a_1^2} , \tag{17}$$

where we define the acoustic nonlinearity parameter for solids β_j by

$$\beta_j = -\nu_{jjj} / \mu_j \tag{18}$$

and the small amplitude wave velocity c_0 by

$$c_0 = \left[\frac{\mu_j}{\rho_0} \right]^{1/2} . \tag{19}$$

The natural velocity W_j of the nonlinear elastic or finite amplitude wave is defined by

$$W_j = c_0 \left[1 - \beta_j \frac{\partial P_j}{\partial a_1} \right]^{1/2} . \tag{20}$$

Fubini[14] solved an equation similar to Eq. (17) by assuming an initially sinusoidal disturbance

$$\frac{\partial P_j}{\partial t} = \left[\frac{\partial P_j}{\partial t} \right]_0 \sin(\omega t) \tag{21}$$

at $a_1 = 0$ and obtained the particle-velocity $\partial P_j / \partial t$ solution

$$\frac{\partial P_j}{\partial t} = \sum_{n=1}^{\infty} \frac{2J_n(n\Omega)}{n\Omega} \left[\frac{\partial P_j}{\partial t} \right]_0 \sin n(\omega t - \kappa a_1) , \tag{22}$$

where ω is the angular frequency $\kappa = \omega / c_0$, J_n are the Bessel functions of the first kind, and

$$\Omega = \frac{1}{2} \left[\frac{\partial P_j}{\partial t} \right]_0 \frac{\omega}{c_0^2} \beta_j a_1 . \tag{23}$$

Equation (22) is valid for wave propagation distances less than the discontinuity distance L_j where

$$L_j = c_0^2 \left[\beta_j \omega \left[\frac{\partial P_j}{\partial t} \right]_0 \right]^{-1} . \tag{24}$$

For common solids the discontinuity distance for a 30-MHz sound wave having a typical particle velocity of 2 $\text{cm}\,\text{s}^{-1}$ is of the order 1 m. Since the dimensions of laboratory samples are usually a few centimeters or less along the wave propagation direction, Ω is small and we may approximate Eq. (22) as

$$\frac{\partial P_j}{\partial t} = \left[\frac{\partial P_j}{\partial t} \right]_0 \sin(\omega t - \kappa a_1) + \frac{1}{4} \left[\frac{\partial P_j}{\partial t} \right]_0^2 \frac{\kappa}{c_0} \beta_j a_1 \sin 2(\omega t - \kappa a_1) + \cdots . \tag{25}$$

We note from Eq. (6) that the Boussinesq stress field depends on the elastic wave motion through the displacement gradients. Our solution to the wave equation is in terms of the particle velocity. Anticipating that Eq. (6) enters prominently in the defining equation for the acoustic-radiation stress, we shall need then the relationship between the displacement gradient and the particle velocity.

For $\beta_j = 0$ the nonlinear wave equation (17) reduces to the linear wave equation

$$\frac{\partial^2 P_j}{\partial t^2} - c_0^2 \frac{\partial^2 P_j}{\partial a_1^2} = \left[\frac{\partial}{\partial t} - c_0 \frac{\partial}{\partial a_1} \right] \left[\frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial a_1} \right] P_j = 0 \tag{26}$$

and Eqs. (22) and (25) reduce to the linear wave solution. For such waves the relationship between the particle velocity and the displacement gradient for propagation in the positive coordinate direction is obtained from Eq. (26) to be

$$\frac{\partial P_j}{\partial t} = -c_0 \frac{\partial P_j}{\partial a_1} . \tag{27}$$

For nonzero β_j, however, one cannot generally assume that Eq. (27) holds. Following the approach of Earnshaw[15] in his studies of wave distortion, the most expedient relationship one may assume between the particle velocity and the displacement gradient is of the form

$$\frac{\partial P_j}{\partial t} = f \left[\frac{\partial P_j}{\partial a_1} \right] , \tag{28}$$

where f is some continuous differentiable function of $\partial P_j / \partial a_1$. Differentiating Eq. (28) with respect to time we obtain

$$\frac{\partial^2 P_j}{\partial t^2} = f' \frac{\partial^2 P_j}{\partial t \, \partial a_1} , \tag{29}$$

where the prime denotes differentiation with respect to $\partial P_j / \partial a_1$. But differentiating Eq. (28) with respect to the Lagrangian coordinate we obtain

$$\frac{\partial^2 P_j}{\partial t \, \partial a_1} = f' \frac{\partial^2 P_j}{\partial a_1^2} . \tag{30}$$

Substituting Eq. (30) into Eq. (29), we may write

$$\frac{\partial^2 P_j}{\partial t^2} = (f')^2 \frac{\partial^2 P_j}{\partial a_1^2} . \tag{31}$$

From Eqs. (17) and (31) we identify $(f')^2$ as

$$(f')^2 = c_0^2 \left[1 - \beta_j \frac{\partial P_j}{\partial a_1} \right] . \tag{32}$$

Taking the negative square root of Eq. (32) for waves propagating in the positive coordinate direction and then integrating with respect to $\partial P_j / \partial a_1$, we find

$$f = \frac{\partial P_j}{\partial t} = \frac{2c_0}{3\beta_j} \left[1 - \beta_j \frac{\partial P_j}{\partial a_1} \right]^{3/2} - \frac{2c_0}{3\beta_j} , \tag{33}$$

where the last term in Eq. (33) is the constant of integration evaluated for the condition that when no wave is present both the particle velocity and the displacement gradient are zero. Solving Eq. (33) for $\partial P_j / \partial a_1$ in terms of $\partial P_j / \partial t$ by means of a power-series expansion, we obtain

$$\frac{\partial P_j}{\partial a_1} = -\frac{1}{c_0} \frac{\partial P_j}{\partial t} + \frac{\beta_j}{4c_0^2} \left[\frac{\partial P_j}{\partial t} \right]^2 \tag{34}$$

to second order. For $\beta_j = 0$ Eq. (34) reduces to (27). Equation (34) will be used directly in the determination of the acoustic-radiation stress.

IV. BOUSSINESQ RADIATION STRESS

The Boussinesq radiation stress at a given fixed point a_1 along the wave propagation direction is the time-averaged Boussinesq stress $\langle \bar{\sigma}_{ij} \rangle$ defined[7] by

$$\langle \bar{\sigma}_{ij} \rangle = \lim_{t\to\infty} \frac{1}{t} \int_0^t \bar{\sigma}_{ij}(a_1,t')dt' \ . \tag{35}$$

In order to use the solutions to the decoupled wave equations we must transform the stress-displacement-gradient expression [Eq. (6)] into a form compatible with the decoupled equations. Performing the transformation R on Eq. (6), we obtain

$$\sigma_{pq} = D_{pqr} \frac{\partial u_r}{\partial a_1} + \tfrac{1}{2} D_{pqrt} \frac{\partial u_r}{\partial a_1} \frac{\partial u_t}{\partial a_1} \ , \tag{36}$$

where D_{pqr} and D_{pqrt} are known linear combinations of appropriate Huang coefficients. Performing the transformation S on Eq. (36), we obtain

$$\tau_{vq} = \mu_j^{vq} \frac{\partial P_j}{\partial a_1} + \tfrac{1}{2} \nu_{jj}^{vq} \left[\frac{\partial P_j}{\partial a_1} \right]^2 , \tag{37}$$

where

$$\mu_j^{vq} = (S^{-1})_{vp} D_{pqr} S_{rj} \ ,$$
$$\nu_{jj}^{vq} = (S^{-1})_{vp} D_{pqrt} S_{rj} S_{tj} \ . \tag{38}$$

By using Eq. (37) in the defining Eq. (35), the Boussinesq radiation stress may be written as

$$\langle \tau_{vq} \rangle = \mu_j^{vq} \left[\left\langle \frac{\partial P_j}{\partial a_1} \right\rangle - \tfrac{1}{2} \beta_j^{vq} \left\langle \left[\frac{\partial P_j}{\partial a_1} \right]^2 \right\rangle \right] , \tag{39}$$

where β_j^{vq} are the stress-generalized nonlinearity parameters of the solid defined by

$$\beta_j^{vq} = -\frac{\nu_{jj}^{vq}}{\mu_j^{vq}} \ . \tag{40}$$

The time-averaged values of the displacement gradient and the square of the displacement gradient to first order in the nonlinearity are easily obtained from Eqs. (22) or (25) and (34) as

$$\left\langle \frac{\partial P_j}{\partial a_1} \right\rangle = \frac{1}{8} \frac{\beta_j}{c_0^2} \left[\frac{\partial P_j}{\partial t} \right]_0^2 , \tag{41}$$

$$\left\langle \left[\frac{\partial P_j}{\partial a_1} \right]^2 \right\rangle = \frac{1}{2c_0^2} \left[\frac{\partial P_j}{\partial t} \right]_0^2 . \tag{42}$$

It is important to point out that in his pioneering studies of acoustic-radiation stress, Brillouin[7,8] assumed that $\langle \partial P_j / \partial a_1 \rangle$ is zero. Such an assumption is tantamount to setting the acoustic nonlinearity parameter β_j equal to zero in Eq. (41). The acoustic nonlinearity parameters have been measured directly from ultrasonic harmonic-generation experiments for some solids;[16] for others, they have been calculated[17] from published measurements of the second- and third-order Brugger elastic coefficients. Most solids of cubic symmetry have acoustic nonlinearity parameters ranging in value from approximately 2 to 20; hence, the value of $\langle \partial P_j / \partial a_1 \rangle$ [Eq. (41)] is comparable to that of $\langle (\partial P_j / \partial a_1)^2 \rangle$ [Eq. (42)] and cannot be neglected. The term $\langle \partial P_j / \partial a_1 \rangle$ is the acoustic-radiation-induced strain of the solid and can be identified with the term obtained by Thurston and Shapiro[1] in their displacement amplitude solution to the nonlinear wave equation. The term $\langle (\partial P_j / \partial a_1)^2 \rangle$ arises from the nonlinearity in the stress—displacement-gradient relationship of Eq. (37) and is the only term considered by Brillouin in his studies.

Substituting Eqs. (41) and (42) into Eq. (39), we find that the components of the Boussinesq radiation stress tensor for acoustic waves of polarization j propagating in any direction $\vec{N}$ of a solid of arbitrary crystalline symmetry may be written in the transformed frame as

$$\langle \tau_{vq} \rangle = \frac{\mu_j^{vq}}{\mu_j} (\tfrac{1}{4}\beta_j - \tfrac{1}{2}\beta_j^{vq}) \langle E_0 \rangle \ , \tag{43}$$

where $\langle E_0 \rangle$ is the time-averaged total-energy density of the initial sinusoidal waveform [see Eq. (21)] given by

$$\langle E_0 \rangle = \frac{1}{2} \frac{\mu_j}{c_0^2} \left[\frac{\partial P_j}{\partial t} \right]_0^2 . \tag{44}$$

For a wave of polarization j the $\langle \tau_{j1} \rangle$ component of the radiation stress may be written as

$$\langle \tau_{j1} \rangle = (\tfrac{1}{4}\beta_j - \tfrac{1}{2}\beta_j) \langle E_0 \rangle = -\tfrac{1}{4}\beta_j \langle E_0 \rangle \ , \tag{45}$$

where we have used the easily verified relations

$$\mu_j^{j1} = \mu_j, \quad \nu_{jj}^{j1} = \nu_{jjj}, \quad \beta_j^{j1} = \beta_j \tag{46}$$

in Eq. (43).

V. BOUSSINESQ RADIATION STRESS AND THE PRINCIPLE OF ADIABATIC INVARIANCE

Equations (43) and (45) which express the Boussinesq radiation stress in terms of the nonlinearity parameters of the solid and the energy density of the wave were obtained by using the Fubini particle-velocity solution of the nonlinear acoustic wave equation (17) in the nonlinear particle-velocity—displacement-gradient equation (34). We show here that Eq. (45) can be obtained directly from the Boltzmann-Enrenfest principle of adiabatic invariance without having to solve the nonlinear wave equation at all.

The Boltzmann-Ehrenfest adiabatic principle states[8,18,19] that if the constraints of a periodic system are allowed to vary sufficiently slowly, then the product of the period T and the mean (time-averaged) kinetic-energy density $\langle K \rangle$ of the system is a constant or adiabatic invariant of the motion,

$$T\langle K \rangle = \text{const} \ . \tag{47}$$

Taking the logarithmic variational derivative of Eq. (47), we obtain

$$\delta\langle K \rangle = -\langle K \rangle \frac{\delta T}{T} \ . \tag{48}$$

The internal energy density Φ of the system may be expressed as a power series in the displacement gradient as[11] (assuming no initial stress)

$$\Phi=\rho_0 U=\rho_0 U_0+\tfrac{1}{2}\bar{A}_{ijkl}\bar{u}_{ij}\bar{u}_{kl} + \frac{1}{3!}\bar{A}_{ijklmn}\bar{u}_{ij}\bar{u}_{kl}\bar{u}_{mn}+\cdots , \tag{49}$$

where U_0 is the initial internal energy per unit mass, ρ_0 is the initial mass density, and the coefficients of the displacement gradients are the Huang coefficients. In the transformed coordinate frame defined by the transformations R [Eq. (10)] and S [Eq. (12)] the internal energy density may be written to third order as

$$\begin{aligned}\Phi&=\tfrac{1}{2}\mu_j\left[\frac{\partial P_j}{\partial a_1}\right]^2+\frac{1}{3!}\nu_{jjj}\left[\frac{\partial P_j}{\partial a_1}\right]^3\\ &=\tfrac{1}{2}\mu_j\left[\frac{\partial P_j}{\partial a_1}\right]^2-\frac{1}{3!}\beta_j\mu_j\left[\frac{\partial P_j}{\partial a_1}\right]^3 ,\end{aligned} \tag{50}$$

where we have used Eq. (18). The constant initial internal energy density and the nonresonant terms have been dropped in the transformation since they do not affect the final results to the approximation assumed.

The relationship between the mean kinetic-energy density and the mean internal energy density can be obtained from the virial theorem[20]

$$\langle K\rangle=\frac{1}{2}\left\langle\frac{\partial\Phi}{\partial(\partial P_j/\partial a_1)}\frac{\partial P_j}{\partial a_1}\right\rangle . \tag{51}$$

From Eq. (50) we write

$$\begin{aligned}\frac{\partial\Phi}{\partial(\partial P_j/\partial a_1)}\frac{\partial P_j}{\partial a_1}&=\mu_j\left[\frac{\partial P_j}{\partial a_1}\right]^2-\tfrac{1}{2}\beta_j\mu_j\left[\frac{\partial P_j}{\partial a_1}\right]^3\\ &=2\Phi-\tfrac{1}{3}\beta_j\Phi\left[\frac{\partial P_j}{\partial a_1}\right]\\ &\quad+O(\beta_j^2(\partial P_j/\partial a_1)^4) ,\end{aligned} \tag{52}$$

where the last equality results from solving Eq. (50) for $\mu_j(\partial P_j/\partial a_1)^2$ and iteratively substituting into Eq. (52). Substituting Eq. (52) into Eq. (51), we find that to first order in the nonlinearity

$$\langle K\rangle=\langle\Phi\rangle-\tfrac{1}{6}\beta_j\left\langle\Phi\frac{\partial P_j}{\partial a_1}\right\rangle . \tag{53}$$

The work performed by a variation in the constraints of a system produces a change in the time-averaged total-energy density of the system. For a nonlinear acoustic wave propagating in a solid we consider that the constraint is self-imposed by the displacement gradient itself. Hence, the work is appropriately represented as the product of the time-averaged stress $\langle\tau_{j1}\rangle$ times the variation in the conjugate constraint parameter $(\partial P_j/\partial a_1)$

$$\delta\langle E\rangle=\langle\tau_{j1}\rangle\delta\left[\frac{\partial P_j}{\partial a_1}\right] . \tag{54}$$

The time-averaged total-energy density of the system can be obtained from Eq. (53) as

$$\langle E\rangle=\langle K\rangle+\langle\Phi\rangle=2\langle K\rangle+\tfrac{1}{6}\beta_j\left\langle\Phi\frac{\partial P_j}{\partial a_1}\right\rangle . \tag{55}$$

Substituting Eqs. (54) and (55) into Eq. (48), we obtain

$$\begin{aligned}\langle\tau_{j1}\rangle=&-\left[\langle E\rangle-\tfrac{1}{6}\beta_j\left\langle\Phi\frac{\partial P_j}{\partial a_1}\right\rangle\right]T^{-1}\frac{\delta T}{\delta(\partial P_j/\partial a_1)}\\ &+\tfrac{1}{6}\beta_j\frac{\delta\langle\Phi(\partial P_j/\partial a_1)\rangle}{\delta(\partial P_j/\partial a_1)} .\end{aligned} \tag{56}$$

For a wave of polarization j the fractional variation in the period $T^{-1}\delta T/\delta(\partial P_j/\partial a_1)$ may be obtained from the natural velocity W_j defined by Eq. (20). For a given length of solid l_0 the system period T is proportional to the length l_0 and we may write

$$T\propto l_0/W_j . \tag{57}$$

Thus, from Eqs. (20) and (57)

$$\begin{aligned}T^{-1}\frac{\delta T}{\delta(\partial P_j/\partial a_1)}&=-W_j^{-1}\frac{\delta W_j}{\delta(\partial P_j/\partial a_1)}\\ &=\tfrac{1}{2}\beta_j\left[1-\beta_j\frac{\partial P_j}{\partial a_1}\right]^{-1} .\end{aligned} \tag{58}$$

The last term in Eq. (56) may be evaluated to first order in the nonlinearity by writing

$$\begin{aligned}\frac{\delta\langle\Phi(\partial P_j/\partial a_1)\rangle}{\delta(\partial P_j/\partial a_1)}&=\left[\langle\Phi\rangle+\left\langle\frac{\partial\Phi}{\partial(\partial P_j/\partial a_1)}\frac{\partial P_j}{\partial a_1}\right\rangle\right]\\ &=\tfrac{3}{2}\langle E\rangle ,\end{aligned} \tag{59}$$

where the last equality follows from Eqs. (51) and (55). Substituting Eqs. (58) and (59) into Eq. (56), we find to first order in the nonlinearity

$$\langle\tau_{j1}\rangle=(\tfrac{1}{4}\beta_j-\tfrac{1}{2}\beta_j)\langle E\rangle=-\tfrac{1}{4}\beta_j\langle E\rangle . \tag{60}$$

Equation (60) is identical in form to Eq. (45). There is, however, a subtle difference. The time-averaged energy density $\langle E\rangle$ in Eq. (60) is the conserved time-averaged energy density of an isolated wave propagating through an infinite solid without regard to energy sources, waveform distortion, or resulting harmonic generation. The time-averaged energy density $\langle E_0\rangle$ in Eq. (45) is that obtained from a driving sinusoidal source at the origin of a semi-infinite solid. The driving term defines the initial shape of the propagating nonlinear wave. If we assume that the total-energy density possessed by the isolated wave is that obtained from a similar driving source in a given amount of time, then $\langle E\rangle=\langle E_0\rangle$ and Eqs. (45) and (60) become identical.

Finally, it should be mentioned that Brillouin[18] also applied the Boltzmann-Ehrenfest adiabatic principle to the radiation-stress problem. His approach, however, did not consider the nonlinearity in the time-averaged total-energy density of the propagating wave as obtained in Eq. (55). Consequently, the term corresponding to that of Eq. (59), which enters Eq. (60) as the component $\frac{1}{4}\beta_j\langle E\rangle$, did not appear in his results. From Eqs. (41) and (43) we see that the contribution of this term to the total Boussinesq radiation stress is that resulting from the time-averaged displacement gradient or the acoustic-radiation-induced static strain in the solid.

VI. RELATIONSHIP BETWEEN BOUSSINESQ AND CAUCHY RADIATION STRESSES

The derivations of the acoustic radiation to this point have been in the Lagrangian (material) frame of reference. Both the Boussinesq stress and Boussinesq radiation stress are referred to the Lagrangian frame. It is of interest to obtain the acoustic-radiation stress referred to the Eulerian (spatial) frame of reference and to find the relationship between the two formulations. The stress referred to the Eulerian frame is called the Cauchy stress.

The equations of motion in the Eulerian frame of reference are[9,10]

$$\rho \frac{dv_i}{dt} = \frac{\partial T_{ji}}{\partial x_j} , \tag{61}$$

where T_{ji} are the components of the symmetric Cauchy stress, x_j are the Eulerian coordinates, ρ is the mass density of the solid in the deformed state, and v_i are the components of the Eulerian particle velocity. The time derivative in Eq. (61) is the material derivative in the Eulerian frame and may be written in convective form as

$$\frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} . \tag{62}$$

With the use of Eq. (62), the conservation of mass density in the form

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_j)}{\partial x_j} \tag{63}$$

and the relation

$$\frac{\partial(\rho v_i)}{\partial t} = \rho \frac{\partial v_i}{\partial t} + v_i \frac{\partial \rho}{\partial t} , \tag{64}$$

we may write Eq. (61) in the form

$$\frac{\partial(\rho v_i)}{\partial t} = \frac{\partial(T_{ji} - \rho v_i v_j)}{\partial x_j} . \tag{65}$$

The term $\partial(\rho v_i)/\partial t$ in Eq. (65) is the local time variation in the momentum density ρv_i of the oscillating mass of the solid. Time averaging $\partial(\rho v_i)/\partial t$ according to Eq. (35) gives the value zero. This implies, as one would expect, that there is no net transport of mass across a fixed surface embedded in the solid after the steady state is established. T_{ji} are the elastic stresses in the Eulerian frame (Cauchy stress) and $\rho v_i v_j$ are components of the momentum flux across the fixed surface. The momentum flux represents the net transport of acoustic momentum and does not time average to zero.

Equation (65) is to be compared to the Lagrangian equations of motion [Eq. (1)] written in the form

$$\frac{\partial(\rho_0 \bar{v}_i)}{\partial t} = \frac{\partial \bar{\sigma}_{ij}}{\partial \bar{a}_j} . \tag{66}$$

The left-hand side of Eq. (66) represents the time variation in momentum density across a fixed surface in the solid referred to the Lagrangian frame. As with its Eulerian counterpart it time averages to zero by Eq. (35). Note, however, that a momentum flux term does not appear explicitly in Eq. (66) since the material derivative (time derivative following a particle) in the Lagrangian frame does not depend on spatial variations (convection) in the velocity field. One may still define a momentum flux in the Lagrangian frame but it does not enter the equations of motion in that frame.

We consider a fixed closed surface S bounding a volume V in the Eulerian frame and a fixed closed surface S^0 bounding a volume V^0 in the Lagrangian frame. The ith component of the time rate of change of momentum through S in the Eulerian frame is

$$\frac{\partial}{\partial t} \int_V (\rho v_i) dV = \int_V \frac{\partial(T_{ji} - \rho v_i v_j)}{\partial x_j} dV$$

$$= \oint_S (T_{ji} - \rho v_i v_j) dS_j , \tag{67}$$

where we have used Green's theorem in Eq. (67). Likewise, the ith component of the time rate of change of momentum through S^0 in the Lagrangian frame is

$$\frac{\partial}{\partial t} \int_{V^0} (\rho_0 \bar{v}_i) dV^0 = \int_{V^0} \frac{\partial \bar{\sigma}_{ik}}{\partial \bar{a}_k} dV^0 = \oint_{S^0} \bar{\sigma}_{ik} dS_k^0 . \tag{68}$$

Since Eqs. (67) and (68) both time average to zero, we may write

$$\oint_S \langle T_{ji} - \rho v_i v_j \rangle dS_j = \oint_{S^0} \langle \bar{\sigma}_{ik} \rangle dS_k^0 . \tag{69}$$

We consider the case in which the fixed surfaces in the two frames are coincident and deformationally equivalent (i.e., evaluated at the same coordinate values). The relationship between the Cauchy radiation stress $\langle T_{ji} \rangle$ and the Boussinesq radiation stress $\langle \bar{\sigma}_{ij} \rangle$ is then seen from Eq. (69) to be

$$\langle T_{ji} \rangle - \langle \rho v_i v_j \rangle = \langle \bar{\sigma}_{ij} \rangle . \tag{70}$$

The term $\langle \rho v_i v_j \rangle$ is the time-averaged acoustic momentum flux through the surface referred to the Eulerian frame. Equation (70) was originally obtained by Brillouin[7] by expanding the Eulerian quantities in a truncated power series of Lagrangian quantities and time averaging the results. Brillouin's derivation is thus an approximation, but Eq. (70) is not since it follows directly from the exact equation (69).

The (1,1) component of the acoustic-radiation stress is easily obtained from Eq. (70) as

$$\langle T_{11} \rangle - \langle E \rangle = \langle \bar{\sigma}_{11} \rangle , \tag{71}$$

where we write $\langle E \rangle = \langle \rho v_1^2 \rangle$. Equation (71) has been obtained by a number of authors[21,22] using a variation of Brillouin's procedure but recently it has been derived without approximation by Chu and Apfel[23] using an approach based on Leibnitz's rule.

VII. CONCLUSIONS

We have theoretically examined the acoustic-radiation stress in infinite and semi-infinite solids of arbitrary crystalline symmetry using two independent approaches and have obtained identical results. The first approach is based on a particle-velocity solution to the nonlinear acoustic wave equation which, after transforming to a displacement-gradient solution, is used directly in the nonlinear stress—displacement-gradient equation. We identify two contributions to the Boussinesq radiation stress. One component results from the nonlinearity in the stress—displacement-gradient equation and is the only term treated by Brillouin[7] in his pioneering studies of acoustic-radiation stress in solids. This component occurs even with a sinusoidally oscillating displacement gradient. The second component results from the nonlinearity in the wave equation itself and gives rise to an acoustic-radiation-induced static strain. It occurs even under the assumption of a sinusoidally oscillating particle velocity since, through the nonlinear wave equation, the displacement gradient does not time average to zero. The prediction of the experimentally verified[4–6] radiation-induced static strain is a significant triumph of the present theory. The magnitude of the predicted static strain is in agreement with the magnitude of the static displacement term found by Thurston and Shapiro[1] in their method-of-characteristics solution to the nonlinear wave equation.

The second approach is based on an application of the Boltzmann-Ehrenfest principle of adiabatic invariance to a self-constrained system represented by the nonlinear wave equation. We again obtain two components to the Boussinesq radiation stress. A comparison of the component terms in Eqs. (43) and (60) reveals that the radiation-induced static strain results from the variation in the time-averaged product of the internal energy density and the displacement gradient. The time-averaged product, which is scaled by the nonlinearity parameter β_j, represents the first-order nonlinearity in the kinetic-energy—potential-energy relationship [Eq. (53)] obtained directly from the macroscopic virial theorem for solids [Eq. (51)]. Indeed, if $\beta_j=0$ the mean kinetic-energy density would equal the mean potential- (internal) energy density as would be expected for a linear oscillating system. The term representing the stress nonlinearity in Eq. (60) results from the adiabatic variation in the natural velocity and the contribution of this term to the total radiation stress is opposite in sign to that giving rise to the radiation-induced static strain.

Finally, the relationship between the Boussinesq and Cauchy radiation stresses has been obtained in an exact closed three-dimensional form. The results are in agreement with the expression obtained by Brillouin, who used an approximation based on a power-series expansion of Eulerian quantities in terms of Lagrangian ones.

[1]R. N. Thurston and M. J. Shapiro, J. Acoust. Soc. Am. **41**, 1112 (1967).
[2]R. B. Thompson and H. F. Tiersten, J. Acoust. Soc. Am. **62**, 33 (1977).
[3]B. Perrin, J. Phys. (Paris) Colloq. **41**, C8-216 (1979).
[4]P. H. Carr and A. J. Slobodnik, Jr., J. Appl. Phys. **38**, 5153 (1967).
[5]J. H. Cantrell, Jr. and W. P. Winfree, Appl. Phys. Lett. **37**, 785 (1980).
[6]W. T. Yost and J. H. Cantrell, Jr., following paper [Phys. Rev. B **30**, 3221 (1984)].
[7]L. Brillouin, Ann. Phys. (Paris) **4**, 528 (1925); J. Phys. Radium **6**, 337 (1925); Rev. Acoust. **5**, 99 (1936).
[8]L. Brillouin, *Tensors in Mechanics and Elasticity* (Academic, New York, 1964) (English translation of *Les Tenseurs en Méchanique et Elasticité*).
[9]D. C. Wallace, Phys. Rev. **162**, 776 (1967).
[10]R. N. Thurston, in *Physical Acoustics,* edited by Warren P. Mason (Academic, New York, 1964), Vol. 1, pp. 2—109.
[11]K. Huang, Proc. R. Soc. London, Ser. A **203**, 178 (1950).
[12]K. Brugger, Phys. Rev. **133**, A1611 (1964).
[13]A. C. Holt and J. Ford, J. Appl. Phys. **38**, 42 (1967).
[14]E. Fubini Ghiron, Alta Freq. **4**, 530 (1935).
[15]S. Earnshaw, Philos. Trans. R. Soc. London **150**, 133 (1860).
[16]See references cited in W. T. Yost, J. H. Cantrell, Jr., and M. A. Breazeale, J. Appl. Phys. **52**, 126 (1981).
[17]J. H. Cantrell, Jr., J. Acoust. Soc. Am. Suppl. **70**, 589 (1981).
[18]L. Boltzmann, *Vorlesungen über die Prinzipe des Mechanik* (Barth, Leipzip, 1897).
[19]P. Ehrenfest, Philos. Mag. **33**, 500 (1917).
[20]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Cambridge, Massachusetts, 1950).
[21]F. E. Borgnis, Rev. Mod. Phys. **25**, 653 (1953).
[22]J. A. Rooney and W. L. Nyborg, Am. J. Phys. **40**, 1825 (1972).
[23]B.-T. Chu and R. E. Apfel, J. Acoust. Soc. Am. **72**, 1673 (1982).