RAPID COMMUNICATIONS

PHYSICAL REVIEW B VOLUME 30, NUMBER 5 1 SEPTEMBER 1984

Dynamic scaling for dilute magnets near percolation threshold

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(Received 31 May 1984)

A convenient formulation of the principle of dynamic scaling for multicritical points is presented and applied to the dynamics of dilute magnets at the percolation threshold. The dynamic exponents observed along the temperature axis near the multicritical point are related to percolationlike exponents and a crossover exponent. Scaling also enables us to see the role of fractal geometry of the percolation cluster on spin dynamics in a transparent way.

The principle of dynamic scaling has been used with notable success in a number of systems undergoing second-order phase transitions.[1,2] Recently it has also been used for spin-wave excitations in dilute magnets near their percolation threshold p_c.[3-6] Harris and Stinchcombe[6] have verified the validity of the principle for ferromagnetic spin waves at zero temperature by explicit calculations on fractal and dilute systems. The purpose of this Rapid Communication is to extend the application of this principle to the multicritical phenomena at the percolation threshold of dilute magnets and relate the dynamic exponent along the temperature axis with the percolation-type exponents and the percolation-thermal crossover index. Another advantage of this approach is that by relating the behavior between small $\vec{k}$ regime, and the large $\vec{k}$ regime, it becomes possible to incorporate the geometrical effects due to fractal nature of the infinite cluster on the spin dynamics,[7] in a transparent way. Thus we find that we can extend and throw more light on the recent stimulating arguments of Aeppli, Guggenheim, and Uemura[7] concerning the role of fractal geometry on the dynamic exponent of an Ising antiferromagnet near its p_c.

Recently Harris[8] has also presented a generalization of the dynamic scaling principle to multicritical behavior. Here we present a somewhat different generalization, which seems to us more convenient and more in keeping with the basic idea of having a single correlation length and a single relaxation time to which all other lengths and times scale. We write for the Fourier transform of the order-parameter correlation function $C(\vec{k},\omega)$ as

$$C(\vec{k},\omega)=\frac{2\pi}{\omega(\vec{k},\xi)}C(\vec{k})F(\vec{k}\xi,\omega/\omega(\vec{k},\xi)) \ , \tag{1}$$

where ξ is the correlation length, and $\omega(\vec{k},\xi)$ is the typical relaxation (or spin wave) frequency associated with the order parameter. $\omega(\vec{k},\xi)$ has the following scaling form:[1]

$$\omega(\vec{k},\xi)=k^z f(k\xi) \ , \tag{2}$$

where z is the dynamic exponent. Near the multicritical point $p=p_c$ and $T=0$, the correlation length ξ can be written as[9]

$$\xi=|\Delta p|^{-\nu_p}X\left(\frac{\epsilon}{\Delta p^{\phi}}\right) \ , \tag{3}$$

where $\Delta p=p-p_c$, and ϵ is the appropriate temperature variable $\epsilon=\exp(-2J/k_BT)$ for an Ising system and $\epsilon=k_BT/J$ for a Heisenberg-type system. ν_p is the index characterizing the divergence of the percolation-correlation length, and ϕ denotes the percolation-thermal crossover index.

The asymptotic forms for the function $X(y)$ are

$$\begin{aligned} X(y)&=X_0(1+a_1y+\cdots), \quad y\to 0 \\ &=X_\infty y^{-\nu_p/\phi}, \quad y\to\infty \ . \end{aligned} \tag{4}$$

As a result

$$\begin{aligned} \xi&=X_0|\Delta p|^{-\nu_p}\left(1+a_1\frac{\epsilon}{(\Delta p)^{\phi}}+\cdots\right), \quad \epsilon\to 0, \ \Delta p\neq 0 \\ &=X_\infty\epsilon^{-\nu_p/\phi}=X_\infty\epsilon^{-\nu_T}, \quad \Delta p=0, \ \epsilon\to 0 \ . \end{aligned} \tag{5}$$

Let us consider now the case of the kinetic Ising model. The relaxation function $f(x)$ in Eq. (2) has the form

$$\begin{aligned} f(x)&=f_\infty, \quad x\gg 1 \\ &=f_0x^{-z}, \quad x\ll 1 \ , \end{aligned} \tag{6}$$

which yields the relaxation behavior

$$\omega=f_0\xi^{-z}, \quad \text{for } k\ll\xi^{-1} \tag{7a}$$

$$=f_\infty k^z, \quad \text{for } k\gg\xi^{-1} \ . \tag{7b}$$

We can consider the long wavelength relaxation behavior in two limits (7a) $\Delta p=0$ and $\epsilon\ll 1$, and (7b) $\Delta p\neq 0$ $\epsilon\simeq 0$. For case (7a) we have

$$\omega=f_0X_\infty^{-z}(\epsilon)^{z\nu_p/\phi_I} \ , \tag{8}$$

while for case (7b)

$$\omega=f_0X_0\Delta p^{z\nu_p} \ , \tag{9}$$

where ϕ_I is the percolation-thermal crossover index for the Ising model, and its value is exactly known to be unity.[10] In the spirit of the argument due to Aeppli *et al.*[7] we try to estimate z in terms of its value, z^*, for the pure system and the fractal effects. For this purpose, we make use of the large wave-vector form of ω, i.e., Eq. (7b). For length scales smaller than the correlation length ξ, the wave vector, K, on the fractal lattice is related to the Euclidean wave vector, k, as[5,11]

$$K=k^{d/\bar{d}} \ , \tag{10}$$

where $\bar{d}$ is the fractal dimension of the infinite cluster. $\bar{d}=d-\beta_p/\nu_p$, where β_p is the index associated with the

fraction of sites in the infinite cluster. The use of Eq. (10) has led to a rather nice prediction for the index of spin-wave stiffness.[11] Now we write Eq. (7b) as

$$\omega = f_\infty K^{z^*} = f_\infty k^{z^* d/\bar{d}} \tag{11}$$

and identify $z = z^* d/\bar{d}$. For the pure kinetic Ising model in two dimensions, $z^* = 2.32$ (Ref. 12) and $d = 1.896$, which yields $z = 2.45$. This agrees well with the measured value of 2.42 for $Rb_2(CoMg)F_4$, but this agreement is somewhat fortuitous as the exponent z^* for pure Rb_2CoF_4 is 1.67, and it is not clear what kinetics is applicable to this system. Aeppli *et al.*[7] have suggested the relation $z = z^* + \theta = 2 - \eta + \theta$ for this system, where θ is the index of anomalous diffusion on fractal lattices.[13] Though this expression is in excellent agreement with the measured value, it is difficult to understand at first sight how diffusion enters Ising kinetics without conserved order parameter. Below, we see how θ enters in the kinetics of models with conserved order parameter. The above arguments, of course, assume that the kinetics is affected only by geometrical effects, which may be a good approximation at very low temperatures in the multicritical region, but can only be justified by a microscopic calculation.

Let us now consider one component system with a conserved order parameter. For such systems, there is a regime of diffusive relaxation above the transition temperature. For crystalline systems, we have the usual $\omega(k) = \Gamma k^2$, but near the percolation threshold, the basic geometry on which the diffusion is occurring is fractal, and we expect for certain range of k

$$\omega(k) = \Gamma k^{2+\theta}, \quad \xi_p^{-1} < k < \xi^{-1} , \tag{12}$$

where $\xi_p = X_0 |\Delta p|^{-\nu_p}$. For percolation clusters, $\theta = (t - \beta_p)/\nu_p$, where t is the index associated with the conductivity. Now if one assumes that, just as for the pure systems, the conventional Van Hove theory is right, $\Gamma \propto \chi^{-1} \propto \xi^{\eta-2}$, and

$$\omega(k) \propto \xi^{-(4+\theta-\eta)} (k\xi)^2, \quad \xi_p^{-1} < k < \xi^{-1} . \tag{13}$$

This yields $z = 4 + \theta - \eta$ and a critical mode

$$\omega(k) \propto k^{(4+\theta-\eta)}, \quad k >> \xi^{-1} . \tag{14}$$

We can now use Eq. (5) to write relaxation rate for $\Delta p \simeq 0$ and $\epsilon << 1$. In the anomalous diffusion regime we have

$$\omega(k) = \Gamma_0 (\epsilon)^{\nu_p (2-\eta)/\phi_I} k^{2+\theta}, \quad \xi_p^{-1} < k < \xi , \tag{15a}$$

and in the normal diffusive regime

$$\omega(k) = \Gamma_0 \epsilon^{\nu_p (2+\theta-\eta)/\phi_i} k^2, \quad k < \xi_p^{-1} . \tag{15b}$$

Next we consider the case of dilute Heisenberg magnets, which have propagating hydrodynamic modes below the transition temperature. Consider ferromagnets first. The low-temperature spin-wave mode has the frequency proportional to k^2, and can be generally written as[4,5]

$$\omega = k^z f^-(k\xi) \tag{16}$$

with the function $f^-(y)$ having the asymptotic forms

$$f^-(y) \to f_\infty^- , \quad \text{as } y \to \infty$$
$$\to f_0^- y^{2-z}, \quad \text{as } y \to 0 . \tag{17}$$

Noting that at $T = 0$, $\omega = D_{sw}(\Delta p) k^2$, with $D_{sw} \propto (\Delta p)^{t-\beta_p}$, one easily sees that[4-6,14]

$$z = 2 + (t - \beta_p)/\eta_p = 2 + \theta . \tag{18}$$

We can now relate these exponents to diffusive regime above the transition temperature. Writing $\omega(k) = D_T k^2$, one finds for $\Delta_p \simeq 0$ and $\epsilon << 1$

$$D_T = f_0 X_\infty^{2-z} \epsilon^{(z-2)\nu_p/\phi_H} \propto \epsilon^{\theta\nu_p/\phi_H} , \tag{19}$$

where ϕ_H is the percolation-thermal crossover index for the Heisenberg system. On the other hand, for $\Delta p \lesssim 0$ and $\epsilon \simeq 0$

$$D_T = f_0 x_0^{2-z} |\Delta p|^{(z-2)\nu_p} . \tag{20}$$

Equation (20) corresponds to diffusion in the large finite clusters.

Finally we discuss an isotropic dilute antiferromagnet. The hydrodynamic spin-wave mode governing transverse order-parameter correlations is given by[15]

$$\omega = cq = b(p - p_c)^{(t+\tau)/2}$$
$$\propto \xi^{-(1+(t+\tau)/2\nu_p)} (q\xi) , \tag{21}$$

where τ is an index governing the divergence of the transverse susceptibility $\chi_\perp$ of an antiferromagnet,[15] i.e., $\chi_\perp \propto (p - p_c)^{-\tau}$. For $d = 3$, the numerical estimate for τ is 0.5.

From (23), we identify $z = (1 + (t + \tau)/2\nu_p$ and write generally

$$\omega = k^z f^-(k\xi) . \tag{22}$$

For large k, we get a critical mode $\omega \propto k^z$. Considering the hydrodynamic regime above the transition point, we know that sublattice magnetization relaxes at a k-independent rate, which should depend on ξ as

$$\omega \propto \xi^{-z} . \tag{23}$$

Use of Eq. (5) now enables us to write

$$\omega \propto |\Delta p|^{\nu_p z}, \quad \epsilon \simeq 0, \ \Delta p < 0 \tag{24}$$

and

$$\omega \propto \epsilon^{\nu_p z/\phi_H}, \quad \Delta p = 0, \ \epsilon << 1 . \tag{25}$$

For convenience, we present the values of z for various models discussed here in Table I. Result of a similar analysis for a planar ferromagnet is also included.

In summary, we have presented a convenient formulation of the dynamic scaling principle at multicritical points and thereby related the dynamic exponents obtained when approaching the multicritical point along the temperature axis to the percolation-type exponents and other known critical exponents. The dynamic scaling principle also enables us to throw more light on the arguments of Aeppli *et al.* connecting fractal dimension and anomalous diffusion index to dynamic exponent. Our argument clearly shows that these relations should depend crucially on the basic dynamics of the system. We illustrate this for four cases.

TABLE I. Values of exponent z for the models discussed in the text.

Model / Dimension	Kinetic Ising	Kinetic Ising with conserved order parameter	Isotopic Ferromagnet	Isotopic Antiferromagnet	Planar Ferromagnet
	$z=z^*d/\bar{d}$	$z=4+\theta-\eta$	$z=2+(t-\beta)/\nu_p$	$z=1+(t+\tau)/2\nu_p$	$z=1+(t-\beta_p)/2\nu_p$
2 (Ref. 16)	2.24	4.47	2.72		
		$(\phi_I=1.0,\ \nu_p=4/3)$	$(\phi_H=1.5,\ \nu_p=4/3)$		
3	2.34	5.60	3.64	2.22	1.82
		$(\phi_I=1.0,\ \nu_p=0.86)$	$(\phi_H=1.2,\ \nu_p=0.86)$		

It is a pleasure to thank Professor D. L. Huber for discussions, suggestions, and hospitality at the University of Wisconsin—Madison. Early discussions with Professor R. B. Stinchcombe are also thankfully acknowledged. The research was supported by the National Science Foundation under Grant No. DMR-8203704.

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