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Random-field Ising model on a Bethe lattice

R. Bruinsma
IBM Thomas J. Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598
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The ground state of the random-field Ising ferromagnet on a Bethe lattice is found. With decreasing random-field strength an infinite series of spin-flip transitions precedes the onset of ferromagnetism. The $T=0$ critical behavior is not mean-field-like. At finite temperatures a series of Griffiths singularities precedes the phase transition. As $T\to 0$, the Griffiths singularities terminate at the spin-flip transitions. The $T\neq 0$ critical behavior is argued to be mean-field-like.

I. INTRODUCTION

Disorder in magnetic systems is caused both by thermal fluctuations and quenched impurities. Impurities can give rise to randomness in the exchange interaction as well as to a random magnetic field. If the randomness in the exchange interaction is not too large it only reduces the transition temperature for the onset of long-range magnetic order.[1] A random magnetic field though has a more pronounced effect. The field couples directly to the order parameter and it can completely destroy long-range order.[2] Among theorists, this is believed to happen for the two-dimensional ($d=2$) random-field Ising model (RFIM), but not for the three-dimensional case.[3] The RFIM is realized in experiment[4] by applying a uniform field to a randomly diluted antiferromagnet. Among experimentalists there is disagreement on the interpretation of the data in $d=3$. Long-time relaxation and hysteresis, as observed[5] in $Mg_xFe_{1-x}Cl_2$, may be important. One of the ways the hysteresis could come about is by having a set of degenerate ground states separated by finite energy barriers. The zero-temperature ($T=0$) entropy S_0 would be finite in that case.

The simplest mean-field theory of the RFIM assumes that the thermal expectation $\langle S(\vec{R})\rangle$ of a spin at site $\vec{R}$ is independent of $\vec{R}$. It predicts a phase transition from a paramagnet (PM) into a ferromagnet (FM) for a sufficiently small random field strength h. For random fields with a discrete distribution ($\pm h$) the transition is first order for low temperatures and second order for higher temperatures with a tricritical point in between.[6] The $T=0$ entropy S_0 and susceptibility χ_0 are zero. The RFIM with infinite-range interaction[7] has the same properties.

However, the $d=1$ RFIM can be solved at $T=0$,[8] and partially for $T\neq 0$.[9] It *does* have a finite S_0. As a function of h there is a series of spin-flip transitions at $T=0$, although, of course, no PM-to-FM transition occurs. Such transitions can be explained using a cluster expansion[10] and are observed in experiment.[11] Another question about mean-field theory is that it predicts that the free energy is analytic in T and H outside the phase-transition region. Griffiths[12] has shown that a diluted Ising ferromagnet has essential singularities in the free energy in a range of temperatures above the critical temperature. It has been argued[13] that such singularities also occur in the RFIM.

The main problem with mean-field theory is not that it ignores *thermal* fluctuations, but that it does not allow $\langle S(\vec{R})\rangle$ to vary with position. In other words, $\langle S(\vec{R})\rangle$ should be treated as a random variable with its own distribution. A measure of the importance of these *quenched* fluctuations is the width of the distribution: $Q\equiv[\langle S(\vec{R})\rangle^2]_{qu}$, the Edwards-Anderson order parameter.[14] Note that Q is finite both in the PM and the FM phase. $[\;]_{qu}$ in the definition of Q denotes a quenched average.

The quenched fluctuations have been considered using domain-wall arguments[2,3] and field-theoretic methods[15] based on a Ginzburg-Landau analog of the RFIM. Both methods were used, with varying results, to find the lower critical dimension of the RFIM, but they are less suitable to study low-temperature properties such as the $T=0$ entropy and susceptibility (however, see J. Cardy[15]). Recently, Parisi[13] argued that Griffiths singularities lead to difficulties at low temperatures in the present field-theoretic results.

In this paper we will discuss the RFIM on a Bethe lattice (Fig. 1) with a discrete random-field distribution ($\pm h$). We will not use domain-wall arguments or field-theoretic methods, but instead we will use a generalized version of the solution method[9,16] of the $d=1$ RFIM. The solution method only applies to lattices without "circuits," which is the main reason for using the Bethe lattice. The method will allow us to find the ground state and some of the finite-temperature properties which can be used as a test case for more approximate methods such as domain-wall arguments or cluster calculations. As we shall see, both make valid predictions for the Bethe-lattice RFIM.

The *pure* Ising model ($h=0$) on a Bethe lattice undergoes a PM-to-FM transition at $T=T_c^0$ with mean-field critical behavior[17] (for interior spins) so that, as in the case of previous calculations,[15] the *thermal* fluctuations do not play a role. The *quenched* fluctuations, however, do contribute: For $h\neq 0$, we find a self-consistent integral equation for the distribution of $\langle S(\vec{R})\rangle$. Solving this equation at $T=0$ allows us to discuss the ground-state properties and the results are quite different from mean-

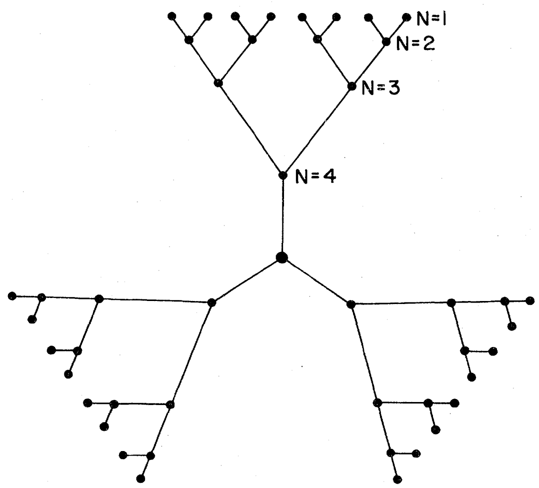


FIG. 1. Bethe lattice with connectivity $K=2$. The site marked $N=4$ is a fourth-generation site. Sites closer to the surface and connected to $N=4$, such as $N=1$, 2, and 3, together form a fourth-generation *branch*. There are three fourth-generation branches connected to the central site.

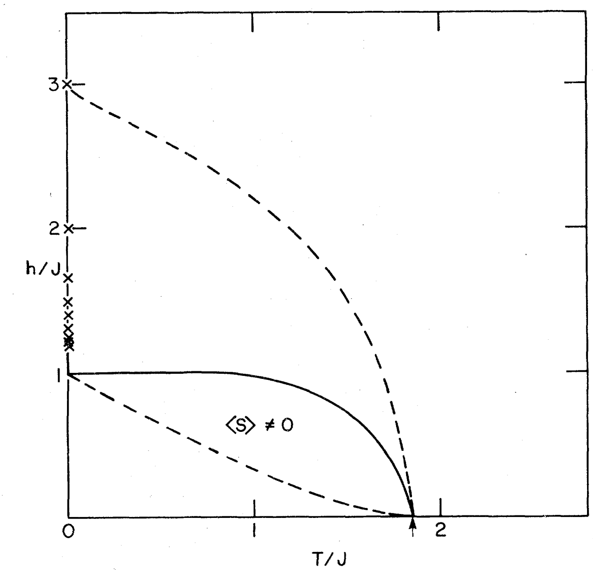


FIG. 2. Phase diagram of the RFIM on a Bethe lattice. The solid curve is a lower bound for the onset of ferromagnetism. Dashed lines are Griffiths singularities terminating at the $T=0$ spin-flip transitions $h/J=1+2/M$, indicated by crosses. h is the random-field strength, J is the exchange constant, and T is the temperature.

field theory: The $T=0$ entropy S_0 is finite for $1 \le h/J \le 3$ (J is the exchange constant). An infinite series of spin-flip transitions occurs at $h/J=1+2/M$ ($M=1,2,\ldots$). At these values of h, S_0 is exceptionally large. The ground state is ferromagnetic for $h/J<1$, in agreement with a domain-wall argument. The critical behavior at $h/J=1$ is not mean-field-like: The susceptibility diverges as $1/(h/J-1)^2$. At finite temperatures, a series of Griffiths singularities precedes the PM-to-FM transition. At $T=0$, the singular lines terminate at $h/J=1+2/M$ (see Fig. 2). There is no tricritical point. The $T \neq 0$ critical behavior is mean-field-like for small H and $T \cong T_c^0$. A $T=0$ cluster-expansion calculation[10] shows also an infinite series of spin-flip transitions. For the Bethe lattice a cluster calculation finds critical values $h/J=1+2/M$, in agreement with our results. This success of the cluster expansion is interesting. For the $d=2$ RFIM, the critical values are[10] $h/J=2+2/M$, which would imply a phase transition at $h/J=2$, although domain-wall calculations[2,3] argue against long-range magnetic order for $d=2$.

A disadvantage of the Bethe lattice is that a majority of spins reside at or near the surface. Only the *interior* spins can have a finite magnetization. The bulk magnetization is always zero.[18] In the following we will only discuss the properties of the interior spins; the surface spins are fixed.

An important problem is the appropriate choice of the random-field distribution. Within mean-field theory, the tricritical point is absent for a Gaussian distribution. The solution method discussed in this paper unfortunately does not yield results for a Gaussian distribution. The distribution appropriate for experiment is not known.

The paper is organized as follows. In Sec. II we will review the recursion-relation method. In Sec. III this method will be applied to find the ground state, and in Sec. IV it will be applied to find some of the finite-temperature properties, including the Griffiths singularities. Section IV also contains a short conclusion.

II. RECURSION-RELATION METHOD

We start with the Hamiltonian $\mathscr{H}$ of the RFIM,

$$\mathscr{H}=\frac{1}{2}\sum_{\{\vec{R}\},\{\vec{\delta}\}} S(\vec{R})S(\vec{R}+\vec{\delta})+\sum_{\{\vec{R}\}} h(\vec{R})S(\vec{R}) , \quad (1)$$

where $\{\vec{R}\}$ runs over a Bethe lattice with connectivity $K=2$ (Fig. 1) and $\vec{\delta}$ runs over the three nearest neighbors of a given site. The random field $h(\vec{R})$ takes on the values $\pm h$ with equal probability and the exchange constant is set to 1.

To find the free energy, we develop, in this section, the recursion-relation formalism[16] used previously[9] to solve the $d=1$ RFIM (a Bethe lattice with $K=1$) and the pure Ising model on a Bethe lattice.[18] A related method was used in numerical studies[19] of similar problems.

The principal idea is to find the dependence of the partition function on the system size N and to derive a recursion relation with respect to N. First, we give a number of definitions concerning the Bethe lattice: An nth-generation site is a site which is n layers away from the surface. It is directly connected to two $(n-1)$th-generation sites. All sites which are less than n layers from the surface and are connected to one nth-generation site constitute an nth-generation *branch*. The central site of the lattice (Fig. 1) is connected to three Nth-generation branches. N will be called the system size. Now, if we fix the spin at a nth-generation site $\vec{R}$ to be either up or down ($\pm$), then we can express the partition function $Z_n^{\pm}(\vec{R})$ of the associated nth-generation branch in terms of the partition function of two $(n-1)$th-generation branches,

$$Z_n^{\pm}(\vec{R})=e^{\pm\beta h(\vec{R})}\prod_{k=1,2}[Z_{n-1}^{+}(\vec{R}+\vec{\delta}_k)e^{\pm\beta}+Z_{n-1}^{-}(\vec{R}+\vec{\delta}_k)e^{\mp\beta}] , \quad (2)$$

where $\beta=1/T$ and $\vec{R}+\vec{\delta}_1$ and $\vec{R}+\vec{\delta}_2$ are the two $(n-1)$th-generation sites connected to $\vec{R}$. In the limit of large n, $Z_n^{\pm}$ diverge exponentially but the ratio Z_n^{+}/Z_n^{-} remains finite. From Eq. (2) we can find a recursion relation for the free energy $F_n(\vec{R})$ of an nth-generation branch terminating at $\vec{R}$,

$$F_n(\vec{R})=-T\ln[Z_n^{+}(\vec{R})+Z_n^{-}(\vec{R})] \quad (3a)$$

$$\cong-\tfrac{1}{2}T\ln[Z_n^{+}(\vec{R})Z_n^{-}(\vec{R})] \quad (3b)$$

$$=\sum_{k=1,2}(F_{n-1}(\vec{R}+\vec{\delta}_k)-\tfrac{1}{2}T\ln\{2\cosh(2\beta)+2\cosh[2\beta x_{n-1}(\vec{R}+\vec{\delta}_k)]\}) , \quad (3c)$$

where

$$x_n(\vec{R})=\tfrac{1}{2}T\ln[Z_n^{+}(\vec{R})/Z_n^{-}(\vec{R})] .$$

Equation (3c) was derived for the pure Ising model in Ref. 20. In going from (3a) to (3b) we used the fact that Z_n^{+}/Z_n^{-} is finite although Z_n^{+} diverges. From Eq. 3(c) one can identify a free energy per spin $f(\vec{R})$,

$$f(\vec{R})=-\frac{T}{2}\ln\{2\cosh(2\beta)+2\cosh[2\beta x_n(\vec{R})]\} , \quad (4)$$

in the sense that the total free energy of the branch is

$$F_n(\vec{R})=\sum_{\{\vec{R}'\}}f(\vec{R}') , \quad (5)$$

where $\{\vec{R}'\}$ runs over the nth-generation branch. The random variable $x_n(\vec{R})$, which determines $f(\vec{R})$, obeys a *stochastic* recursion relation,

$$x_n(\vec{R})=h_n(\vec{R})+\sum_{k=1,2}g(x_{n-1}(\vec{R}+\vec{\delta}_k)) , \quad (6)$$

where

$$g(x)=\frac{T}{2}\ln[(e^{2\beta(1+x)}+1)/(e^{2\beta x}+e^{2\beta})] . \quad (7)$$

Since $|g(x)|\le 1$, we see that $|x_n(\vec{R})|\le h+2$, so that Z_n^{+}/Z_n^{-} indeed remains finite. For $h=0$, Eq. (6) has a fixed point x_∞,

$$x_\infty=2g(x_\infty) . \quad (8)$$

If the slope $2g'(0)<1$, then $x_\infty=0$. For $2g'(0)>1$ there are two stable solutions $\pm x_\infty$ associated with ferromagnetic order. The resulting free energy is identical to that of the Bethe-Peierls approximation.[17,18] The critical value $2g'(0)=1$ corresponds to a PM-FM transition temperature $T_c^0=2/\ln(3)$.

So far, we have only performed a thermal average. To do the quenched average over all possible configurations of $h(\vec{R})$, we associate a distribution $W_n(x)$ with the random variable x_n,

$$W_n(x)\Delta=\mathrm{Prob}(x<x_n<x+\Delta) . \quad (9)$$

Now, $x_n(\vec{R})$ depends on $x_{n-1}(\vec{R}+\vec{\delta}_k)$ through the stochastic recursion relation (6). Because there are no cross links between the two branches with $k=1$ and 2, the random variables $x_{n-1}(\vec{R}+\vec{\delta}_k)$ are independent so that x_n is the sum of three independent random variables,

$$W_{n+1}(x)=\int_{-\infty}^{+\infty}dh\,P(h)\int_{-\infty}^{+\infty}dx_1W_n(x_1)\int_{-\infty}^{+\infty}dx_2W_n(x_2)\delta(x-h-g(x_1)-g(x_2)) , \quad (10)$$

where $P(h)$ is the random-field distribution.

Equation (10) is the central result of this section. This *functional* recursion relation drives us to a fixed point $W_\infty(x)$, obeying a self-consistent integral equation,

$$W_\infty(x)=\tfrac{1}{2}\int_{-\infty}^{+\infty}dx_1\int_{-\infty}^{+\infty}dx_2W_\infty(x_1)W_\infty(x_2)[\delta(x-h-H-g(x_1)-g(x_2))+\delta(x+h-H-g(x_1)-g(x_2))] , \quad (11)$$

for a discrete distribution $h(\vec{R})=\pm h$ and where we added a small uniform field H. This integral equation is the random-field analog of the usual mean-field self-consistency equation [such as Eq. (8)] for nonrandom problems. It must be solved under the constraints $\int_{-\infty}^{+\infty}W_\infty(x)dx=1$ and $W_\infty(x)\ge 0$ for all x. For the $d=1$ RFIM one finds a similar recursion relation,[16,9] only linear. For a Bethe lattice of connectivity K, there will be a convolution of K distributions $W_\infty(x)$ on the right-hand side of Eq. (11).

Once we know $W_\infty(x)$, we can find the mean energy per spin $\bar{f}$, for interior spins,

$$\bar{f}=-\frac{T}{2}\int_{-\infty}^{+\infty}W_\infty(x)\ln[2\cosh(2\beta)+2\cosh(2\beta x)]dx , \quad (12)$$

while the magnetization and susceptibility follow from the field dependence of $\bar{f}$. To find the Edwards-Anderson order parameter Q we compute the magnetization $\langle S_0 \rangle$ of the spin at the central site. $\langle S_0 \rangle$ depends on the random field h_0 at the origin and on the partition function of the three Nth-generation branches connected to the origin,

$$\langle S_0 \rangle = \tanh \left[\beta \left[h_0 + \sum_{k=1}^{3} g(x_N(\vec{R}_k)) \right] \right] , \tag{13}$$

where $\vec{R}_k$ refers to the three Nth-generation sites connected to the origin. Notice that $[\langle S_0 \rangle]_{\text{qu}}$ is zero if the distribution $W_N(x)$ is even in x since $g(x)$ is odd in x. From Eq. (13) we can compute $Q = [\langle S_0 \rangle^2]_{\text{qu}}$ once $W_N(x)$ is known. Equation (13) gives physical meaning to the random variable x_N: $g(x_N(\vec{R}_k))$ is the effective field at the origin induced by site $\vec{R}_k$.

III. GROUND-STATE PROPERTIES

The ground state of the RFIM is believed[2,3] to be ferromagnetic, for weak random-field strength, if the dimension is larger than two. Even at $T=0$ though, there are still finite-sized minority regions with spins pointing in a direction opposite to the magnetization. The interfacial energy of a finite domain of size N in an infinite Bethe lattice is of order 2^N, while the gain in energy due to the random field is of order $h\,2^{N/2}$ since there are 2^N spins in the domain [Fig. 3(c)]. Minimizing the total energy with respect to N gives $N \sim \ln(h)$. This would imply that the ground state is ferromagnetic and there should be *no* domains of minority spins for $h \lesssim 1$. On the other hand, for $h > 3$, every spin must follow its local random field so there is no magnetic order at all. At $h = 3$, spins at sites surrounded by three neighboring sites, with random fields opposite to their own [Figs. 3(a) and 3(b)], are frustrated in the sense that $\langle S \rangle = 0$. For $h < 3$ there must be some short-range magnetic order. In general, spin-flip transitions of this kind are expected when[10]

$$Nh = p_N, \quad N = 1,2,3,\ldots \tag{14}$$

where p_N is the bond perimeter of a fully ramified cluster containing N sites. A fully ramified cluster is a cluster of N spins with the longest possible perimeter. For a Bethe lattice, $p_N = N + 2$ since a fully ramified cluster is just a straight line of neighboring sites [Fig. 3(b)]. The critical field strength for spin-flip transitions is thus

$$h = 1 + 2/N . \tag{15}$$

In this section we will use the recursion-relation method of Sec. II to verify these intuitive results and discuss the phase transition expected at $h = 1$. First we compute $W_\infty(x)$ as a function of h. The stochastic recursion relation

$$x_n(\vec{R}) = h_n(\vec{R}) + H + \sum_{k=1,2} g(x_{n-1}(\vec{R} + \vec{\delta}_k))$$

simplifies in the $T \to 0$ limit since

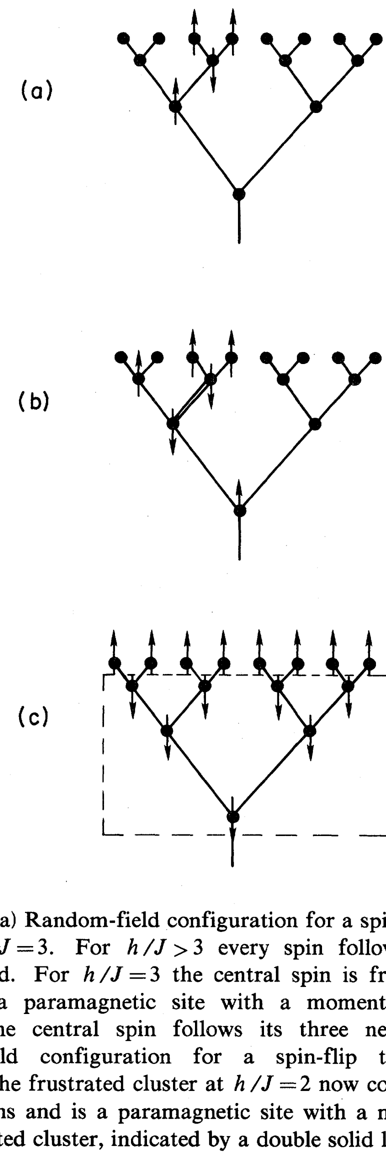


FIG. 3. (a) Random-field configuration for a spin-flip transition at $h/J = 3$. For $h/J > 3$ every spin follows the local random-field. For $h/J = 3$ the central spin is frustrated and represents a paramagnetic site with a moment of 1. For $h/J < 3$, the central spin follows its three neighbors. (b) Random-field configuration for a spin-flip transition at $h/J = 2$. The frustrated cluster at $h/J = 2$ now consists of two parallel spins and is a paramagnetic site with a moment of 2. The frustrated cluster, indicated by a double solid line, is a fully ramified cluster of two sites. (c) Domain-wall argument for the RFIM on a Bethe lattice. A domain of size $N = 3$, indicated by a dashed line, has its spins reversed. The interfacial energy is of order $J\,2^3$, while the volume energy is of order $h\,(2^3)^{1/2}$.

$$g(x) = \begin{cases} 1, & x > 1 \\ x, & -1 \le x \le 1 \\ -1, & x < -1 \end{cases} \tag{16}$$

for $T \to 0$.

As long as $h > 3$, it follows from Eq. (16) that $|x_n| > 1$, so that x_n can only assume the values $\pm h \pm 1 \pm 1$ (for $H = 0$). This shows that $W_\infty(x)$ is a sum of δ functions. To find $W_\infty(x)$ we start with $W_1(x) = \delta(x - x_0)$, with x_0 arbitrary, and use W_1 as a starting distribution for the recursion relation (10). For $h > 3$, we need only two iterates to find W_∞ with result

$$W_\infty(x)=\frac{1}{8}\sum_{\substack{\epsilon_1=\pm1,\\ \epsilon_2=\pm1,\\ \epsilon_3=\pm1}}\delta(x+\epsilon_1 h+\epsilon_2+\epsilon_3),\quad h\geq 3\ . \tag{17}$$

From Eq. (16), it is clear that this cannot be anymore a solution if a δ function occurs in the interval $[-1,1]$. This happens when $h-1=2$. For $2<h<3$ we see, from Eq. (6), that $W_\infty(x)$ is series of δ functions now located at

$$\pm h,\pm h\pm 2,\pm h\pm 2(h-2),\pm h\pm(h-2)\pm 1\ .$$

Inserting this into the fixed-point equation for $W_\infty(x)$, Eq. (11), gives the weights of the δ functions,

$$W_\infty(x)=\frac{1}{2}\sum_{\substack{\epsilon_1=\pm1,\\ \epsilon_2=\pm1,\\ \epsilon_3=\pm1}}[p^2\delta(x+\epsilon_1 h+\epsilon_2+\epsilon_3)+2pq\,\delta(x+\epsilon_1 h+\epsilon_2+\epsilon_3(2-h))+q^2\delta(x+\epsilon_1 h+\epsilon_2(2-h)+\epsilon_3(2-h))],\quad 2<h<3 \tag{18}$$

where $p=-1+\sqrt{2}$ and $q=\frac{1}{2}-p$. The second δ function occurs in the $[-1,1]$ interval when $2(h-1)<2$. For $h<2$, W_∞ is the sum of an even longer series of δ functions, and a δ function of this series acquires an argument in the $[-1,1]$ interval when $3(h-1)<2$. For smaller values of h, the calculation becomes laborious, but, in general, when

$$N(h-1)=2,\quad N=1,2,3,\ldots \tag{19}$$

we add another series of δ functions to $W_\infty(x)$. Equation (19) agrees with the cluster-calculation result. The necessary number of iterations to find $W_\infty(x)$ diverges when h approaches 1. Right at $h=1$, the solution of Eq. (11) follows by inspection [see Eq. (6)],

$$W_\infty(x)=\sum_{\epsilon=\pm1}[\tfrac{3}{8}\delta(x-\epsilon)+\tfrac{1}{8}\delta(x-3\epsilon)],\quad h=1 \tag{20}$$

and for $h<1$, there are two obvious solutions,

$$W_\infty^{\pm}(x)=\frac{1}{2}\sum_{\epsilon=\pm1}\delta(x\mp 2-\epsilon h),\quad h<1\ . \tag{21}$$

The limit of $W_\infty(x)$ for $h\to 1^+$ is found in Appendix A,

$$W^\infty(x)\cong A\sum_{n=1}^{1/\epsilon}\frac{1}{n^\alpha}\delta(x-1+n\epsilon),\quad 0<x<1 \tag{22a}$$

$$W^\infty(x)\cong\tfrac{1}{2}W(2-x),\quad 1<x<2 \tag{22b}$$

$$W^\infty(x)\cong\tfrac{1}{2}W(x-2),\quad 2<x<3 \tag{22c}$$

where $\epsilon=h-1$, $A\simeq 0.14$ and $\alpha\simeq 1.7$, while $W^\infty(x)=W^\infty(-x)$. For $h<1$, the solution (21) is not unique. There is an infinite series of solutions, all with zero magnetization and with a higher free energy. They are unstable under application of a uniform field and will be ignored in the following.

Having found $W_\infty(x)$, we can compute $\bar{f}$. We first note that

$$S_0=\lim_{T\to 0}-\frac{\partial\bar{f}}{\partial T}$$
$$=\lim_{\epsilon\to 0}\ln 2\int_{1-\epsilon}^{1+\epsilon}dx\,W_\infty(x)+\int_0^1 dx\frac{\partial}{\partial T}W_\infty(x)$$
$$+\int_1^\infty dx\frac{\partial}{\partial T}W_\infty(x)x\ . \tag{23}$$

For the case of $d=1$, it can be shown[8,20] that keeping the first term only is a good approximation. The uniform susceptibility χ is computed in Appendix B. The result is

$$\bar{f}=-h,\quad \chi_0=0,\quad S_0=0,\quad M=0,\quad Q=1\ , \tag{24a}$$

for $h>3$;

$$\bar{f}=-3,\quad \chi_0=(1/T)\tfrac{29}{128},\quad S_0\cong\tfrac{1}{8}\ln 2,\quad M=0,\quad Q=\tfrac{7}{8}\ , \tag{24b}$$

for $h=3$; and

$$\bar{f}=-3p^2-(1-p^2)h,\quad \chi_0=\frac{1}{2T}(4p^3+\tfrac{15}{4}p^5+\cdots)\ ,$$
$$S_0\cong pq\ln 2,\quad M=0,\quad Q=1-\tfrac{3}{8}p^2(1-p^2)^2\ , \tag{24c}$$

for $2<h<3$; where $p=-1+\sqrt{2}$ and $q=\frac{1}{2}-p$. Note that the entropy S_0 at $h=3$ is much larger than for $2<h<3$. S_0 is shown as a function of h in Fig. 4. We found that a cluster-calculation predicts the positions of the spin-flip transitions correctly. However, it also claims that S_0 is zero *between* the critical values, which we find not to be the case. The Curie-like temperature dependence of the susceptibility indicates that the low-lying excitations of the system are due to small clusters of aligned

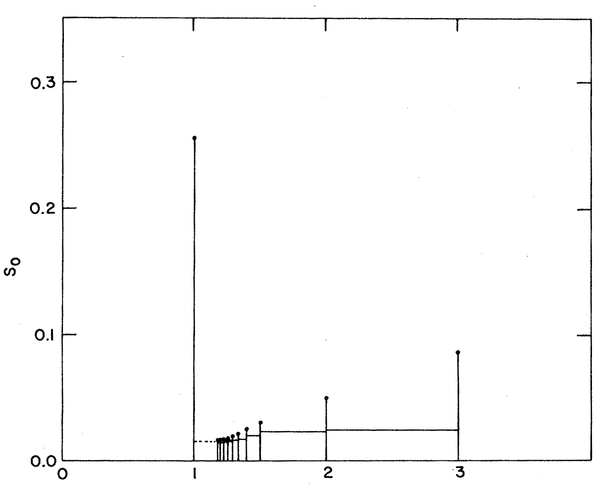


FIG. 4. Zero-temperature entropy per spin S_0 as a function of the random-field strength h ($J=1$). For $h>3$ and $h<1$, $S_0=0$. For $h<1$, the ground state is fully aligned. In computing S_0, only the first term of Eq. (23) was included.

spins having two equivalent orientations ("superparamagnetism").

Next we turn to the critical behavior, for $h \to 1^+$,

$$\bar{f} \cong A_1(h-1) - \tfrac{3}{2} , \tag{25a}$$

$$\chi_0 \cong A_2 \frac{1}{(h-1)^2} + \frac{A_3}{T} , \tag{25b}$$

$$S_0 \cong 0.47 \ln 2 , \tag{25c}$$

$$M = 0 , \tag{25d}$$

$$Q \cong A_5 + A_4(h-1)^{2\alpha-1} , \tag{25e}$$

where $A_1 - A_5$ are constants and $\alpha \simeq 1.7$. For the ordered phase ($h=1$),

$$\bar{f} = -2, \ \chi_0 = 0, \ S_0 = 0, \ M = \pm 1, \ Q = 1 . \tag{26}$$

The result that $M = \pm 1$ for $h < 1$ agrees with the domain-wall argument. Mean-field theory, for Gaussian randomness, predicts $\chi \sim 1/(h-1)$, and thus the critical behavior is not mean-field-like. Furthermore, the susceptibility is not Curie-like, and thus the "superparamagnet" description fails near $h=1$. Equations (25a) and (25b) must, however, be considered with caution: $\bar{f}$ is *discontinuous* at $h=1$. The bulk free energy, computed numerically from Eq. (3), *is* continuous. This problem is due to the diverging correlation length. For $h \le 1$, we cannot consider the interior spins separately from the surface, and our definition of $\bar{f}$ becomes questionable. Summarizing, we have found that both the domain-wall argument and the cluster calculation are valid for the RFIM. The critical behavior near $h=1$ is not mean-field-like, and since the $T=0$ susceptibility diverges at $h=1$, we can rule out a tricritical point.

IV. FINITE-TEMPERATURE PROPERTIES

For $T \neq 0$, $W_\infty(x)$ is not anymore a finite series of δ functions and it is not possible to give explicit expressions for the fixed-point distribution. In this section we first derive some general properties of $W_\infty(x)$ and then apply the results to compute the free energy $\bar{f}$.

We begin in the same way as in Sec. III: The initial condition is $W_1(x) = \delta(x - x_0)$, with x_0 arbitrary, and then use W_1 as a starting distribution for the recursion relation (10),

$$\begin{aligned}
W_1(x) &= \delta(x - x_0) , \\
W_2(x) &= \tfrac{1}{2}\delta(x - h - 2g(x_0)) + \tfrac{1}{2}\delta(x + h - 2g(x_0)) , \\
W_3(x) &= \tfrac{1}{8}\delta(x - h - 2g(-h + 2g(x_0))) + \tfrac{1}{8}\delta(x - h - 2g(h + 2g(x_0))) + \tfrac{1}{4}\delta(x - h - g(h + 2g(x_0)) - g(-h + 2g(x_0))) \\
&\quad + \tfrac{1}{4}\delta(x + h - g(h + 2g(x_0)) - g(-h + 2g(x_0))) + \tfrac{1}{8}\delta(x + h - 2g(-h + 2g(x_0))) + \tfrac{1}{8}\delta(x + h - 2g(h + 2g(x_0))) , \\
W_4(x) &= \tfrac{1}{128}\delta(x - h - 2g(-h + 2g(h + 2g(x_0)))) + \tfrac{1}{8}\delta(x - h + g(h + g(h + 2g(x_0)) + g(-h + 2g(x_0))) \\
&\qquad + g(-h + g(h + 2g(x_0)) + g(-h + 2g(x_0)))) \\
&\quad + \tfrac{1}{8}\delta(x + h + g(h + g(h + 2g((x_0)) + g(-h + 2g(x_0))) + g(-h + g(h + 2g(x_0)) + g(-h + 2g(x_0)))) \\
&\quad + \tfrac{1}{128}\delta(x + h - 2g(h + 2g(-h + 2g(x_0)))) .
\end{aligned} \tag{27}$$

Only the two largest and two of smallest terms of W_4 are given. In general, W_n is a sum of Z_n δ functions where

$$Z_{n+1} = Z_n^2 + Z_n , \tag{28}$$

and thus $Z_n \sim 2^{2^n}$ for $n \to \infty$. Since $|g(x)| < 1$, it follows that $W_n(x)$ has a bounded support so that the mean density of δ functions is of order 2^{2^n}. From Eq. (27) we see that the weights of the δ functions are maximal for the two δ functions with arguments $x - x_n^1$ and $x - x_n^2$ where

$$x_n^1 = h + g(x_{n-1}^1) + g(x_{n-1}^2) , \tag{29a}$$

$$x_n^2 = -h + g(x_{n-1}^1) + g(x_{n-1}^2) . \tag{29b}$$

This can be shown, using induction, by considering the two maxima at the nth iteration, both with weight A_n,

$$W_n(x) = A_n[\delta(x - x_n^1) + \delta(x - x_n^2)] + \cdots ,$$

where the ellipsis represents lower-weight δ functions, and using Eq. (10) to compute W_{n+1}. This also gives a recursion relation for the weights A_n,

$$A_{n+1} = A_n^2 ,$$

and so $A_n = 1/2^{2^n}$. In Appendix C we will show that $W_\infty(x)$ is a differentiable function, at least for small h. This implies that, for large n, the δ functions near x_n^1 and x_n^2 have weights comparable to A_n. Since the δ-function density is $\sim 2^{2^n}$ and $A_n = 1/2^{2^n}$, we conclude that $W_\infty(x)$ has two maxima at the fixed points of x_n^1 and x_n^2,

$$x_\infty^1 = h + g(x_\infty^1) + g(x_\infty^2) , \tag{30a}$$

$$x_\infty^2 = -h + g(x_\infty^2) + g(x_\infty^1) . \tag{30b}$$

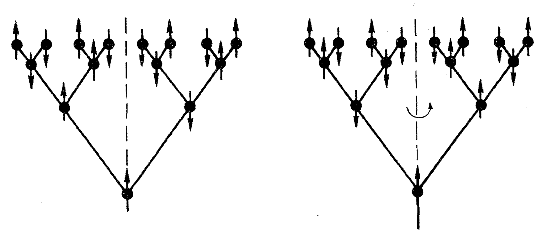
As long as $2g'(h)<1$, there is only one solution to Eqs. (30) ($x^1_\infty=-x^2_\infty=h$). However, if this condition is violated then there are two stable solutions and one unstable. For the stable solutions, $x^1_\infty \neq -x^2_\infty$ and so $W_\infty(x)$ is asymmetric in x. Now recall that a symmetric distribution would imply that $[\langle S_0 \rangle]_{qu}=0$. We conclude that we have found a lower bound for the onset of ferromagnetism: If

$$\cosh(2\beta)-2\exp(-2\beta)>\cosh(2\beta h) , \tag{30c}$$

then $[\langle S_0\rangle]_{qu}$ is finite. The line terminates at $h=1$ in the limit $T\to 0$, in agreement with the $T=0$ results. At $T=0$ and $h=1$, $x^1_\infty=1$ and $x^2_\infty=-1$, which is consistent with Eq. (20).

The fixed-point equations for x^1_∞ and x^2_∞ resemble that of the pure Ising model [Eq. (8)], and we could ask whether there is some nonrandom system, with a given configuration $h(\vec{R})$, obeying the recursion relations (29). This is the case, and a typical $h(\vec{R})$ is shown in Fig. 5(a). For an Nth-generation branch there are $2^{2^{N-1}}$ possible configurations of $h(\vec{R})$ obeying Eqs. (29), out of a total of 2^{2^N}. This explains the maxima of W_∞ at x^1_∞ and x^2_∞. However, as we saw in Sec. III, contributions to W_∞ away from the maxima are important in understanding the $T=0$ spin-flip transitions. For this reason we now consider some of the *least* likely contributions to W_∞: the first

(a)



(b)

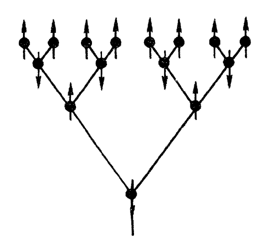


FIG. 5. (a) Two random-field configurations where every Nth-generation site is connected to two $(N-1)$th-generation sites with opposite random fields. The two configurations shown are related by mirror reflection around the center line and have the same free energy. There are $2^{2^{N-1}}$ equivalent states generated by mirror reflections around the 2^N sites of an Nth-generation branch. (b) A random-field configuration with only two equivalent realizations. The second realization has all random fields reversed.

and the last δ functions in the series for W_n [Eqs. (27)]. Their arguments, $x-x_n^+$ and $x-x_n^-$, obey the recursion relations

$$x_n^+=h+2g(x_{n-1}^-) , \tag{31a}$$

$$x_n^-=-h+2g(x_{n-1}^+) , \tag{31b}$$

with fixed points x^+_∞ and x^-_∞. The weights of the δ functions B_n are $1/(2^{2^N})^2$. Again for large h, $x^+_\infty=-x^-_\infty$ and $|2g'(x^+_\infty)|<1$. Below the line $|2g'(x^+_\infty)|=1$, however, $x^+_\infty\neq -x^-_\infty$. The line $|2g'(x^+_\infty)|=1$ terminates at $h=3$ for $T\to 0$ (see Fig. 2), and this would imply that $[\langle S_0\rangle]_{qu}$ is already finite for $h<3$. From Sec. III we know that is not the case. The reason is that, for large n, the contribution to $W_n(x)$ of δ functions near x_n^+ and x_n^- is proportional to the product of the δ-function density and the weights B_n which would vanish as $1/2^{2^n}$ for large n. The associated field configuration $h(\vec{R})$ is shown in Fig. 5. There are only two equivalent realizations.

This does not mean that the free energy $\bar{f}$ is analytic at $2g'(x^+_\infty)=1$. As $n\to\infty$, x_n^+ approaches x^+_∞ as

$$x_n^+\cong x^+_\infty+C\exp(-n/\xi) , \tag{32}$$

where C is a constant depending on the initial value x_0 and

$$\xi=1/|\ln[2g'(x^+_\infty)]| .$$

ξ diverges as $2g'(x^+_\infty)$ approaches 1. If the number of iterations n is larger than ξ, we may replace x_n^+ and x_n^- by x^+_∞ and x^-_∞ in Eqs. (27). However, even if only the last ξ iterations in the arguments of the δ functions in Eqs. (27) are the same as for x_n^+ and x_n^- [Eqs. (31)], then we may still replace the arguments of the δ functions by x^+_∞ and x^-_∞. The δ functions with the largest possible weights for which this remains valid are

$$W_n(x)\sim\frac{1}{2^{2^n}}\frac{1}{2^{2^\xi}}[\delta(x-x^+_\infty)+\delta(x-x^-_\infty)]+\cdots . \tag{33}$$

In the limit $n\to\infty$, one must multiply W_n by the δ-function density 2^{2^n}. This shows that $W_\infty(x)$ is actually of order $1/2^{2^\xi}$ for x near x^+_∞ and x^-_∞. We can now discuss the analytical properties of $\bar{f}$. The maxima of $W_\infty(x)$ contribute an amount $\Delta\bar{f}_1$ to $\bar{f}$ which has mean-field critical behavior at $2g'(h)=1$,

$$\Delta\bar{f}_1\sim -T\{\ln[2\cosh(2\beta)+2\cosh(2\beta x^1_\infty)] + \ln[2\cosh(2\beta)+2\cosh(2\beta x^2_\infty)]\} , \tag{34}$$

while a typical contribution $\Delta\bar{f}_2$ of the "tail" of $W_\infty(x)$ is

$$\Delta\bar{f}_2\sim -\frac{T}{2^{2^\xi}}\{\ln[2\cosh(2\beta)+2\cosh(2\beta x^+_\infty)] + \ln[2\cosh(2\beta)+2\cosh(2\beta x^-_\infty)]\} , \tag{35}$$

and thus $\Delta\bar{f}_2$ has an *essential* singularity in h when ξ diverges. We have thus found the expected Griffiths singularities. For every configuration $h(\vec{R})$ that undergoes a phase transition at $h(T)$, $\bar{f}$ will be singular at $h(T)$. If there is only a finite number of equivalent configurations for $h(\vec{r})$, then it will be an essential singularity. All the lines of essential singularities that we considered terminated at $h=1+2/M$ in the $T\rightarrow 0$ limit (Fig. 2). The Griffiths singularities may be considered as the finite-temperature remnants of the $T=0$ spin-flip transitions.

Finally in the limits $h\rightarrow 0$ and $T\rightarrow T_c^0$, we can expand the recursion relation in powers of h and find a power series for $W_\infty(x)$ and similarly for $\bar{f}$,

$$\bar{f}(h)=\bar{f}(0)+C_1(T)h^2+C_2(T)h^4+\cdots , \tag{36}$$

where $C_1(T)$ and $C_2(T)$ are analytic in T near T_c^0. Equation (36) indicates that $\bar{f}(h)$ does not have a power-law singularity in h at T_c^0, and thus we expect mean-field critical behavior. In summary, we have found the ground state of the RFIM on a Bethe lattice and some of its finite-temperature properties. Both domain-wall arguments and cluster calculations were seen to be valid. Griffiths singularities are the finite-temperature analogs of the $T=0$ spin-flip transitions and there is no tricritical point.

The broken-symmetry phase is at $T=0$ fully aligned, either with all spins up or all spins down, and so at least for the special case of the Bethe lattice one does not need to consider different order parameters, such as Q, in addition to the magnetization. Nevertheless, Q shows nontrivial critical behavior at $T=0$.

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APPENDIX A: SOLVING THE $T=0$ FIXED-POINT EQUATION

In this appendix we will solve the $T=0$ fixed-point equation in the limit $h\rightarrow 1^+$,

$$W_\infty(x)=\tfrac{1}{2}\int_{-\infty}^{+\infty}dx_1 W_\infty(x_1)\int_{-\infty}^{+\infty}dx_2 W_\infty(x_2)[\delta(x-h-g(x_1)-g(x_2))+\delta(x+h-g(x_1)-g(x_2))] , \tag{A1}$$

with

$$g(x)=\begin{cases}1, & x>1\\ x, & -1\le x\le 1\\ -1, & x<-1 .\end{cases} \tag{A2}$$

Let $\epsilon=h-1$. For $\epsilon=0$, we know the solution,

$$W_\infty(x)=\sum_{r=\pm 1}[\tfrac{3}{8}\delta(x-r)+\tfrac{1}{8}\delta(x-3r)] . \tag{A3}$$

We expect, from Eq. (A3), that $W_\infty(x)=W_\infty(-x)$ for $\epsilon>0$, and that there are maxima at $x=\pm 1$ and $x=\pm 3$. We divide the interval $[0,(3+\epsilon)]$ into the regions I, II, and III, where region I is $[0,1]$, II is $[1,2]$, and III is $[2,(3+\epsilon)]$. Similarly, $W_\infty(x)$ equals $W_\infty^{\rm I}(x)$ in I, $W_\infty^{\rm II}(x)$ in II, and $W_\infty^{\rm III}(x)$ in III. The fixed-point equation (A1) breaks up into

$$W_\infty^{\rm I}(x)=\tfrac{1}{2}p^2\delta(x-1+\epsilon)+p[W_\infty^{\rm I}(x-\epsilon)+W_\infty^{\rm I}(x+\epsilon)]+\int_0^{x-\epsilon}dx_1 W_\infty^{\rm I}(x_1)W_\infty^{\rm I}(x-1-\epsilon-x_1)$$
$$+\int_{x+\epsilon}^{1}dx_1 W_\infty^{\rm I}(x_1)W_\infty^{\rm I}(x+1+\epsilon-x_1) , \tag{A4}$$

$$W_\infty^{\rm II}(x)=p^2\delta(x-1-\epsilon)+pW_\infty^{\rm I}(2+\epsilon-x)+\tfrac{1}{2}\int_{x-2-\epsilon}^{1}dx_1 W_\infty^{\rm I}(x_1)W_\infty^{\rm I}(x-1-\epsilon-x_1) , \tag{A5}$$

$$W_\infty^{\rm III}(x)=\tfrac{1}{2}p^2\delta(x-3-\epsilon)+pW_\infty^{\rm I}(x-2-\epsilon)+\tfrac{1}{2}\int_{x-2-\epsilon}^{1}dx_1 W_\infty^{\rm I}(x_1)W_\infty^{\rm I}(x-1-\epsilon-x_1) , \tag{A6}$$

where $p\equiv\int_1^\infty W_\infty(x)dx$. If, for a given p, we have solved Eq. (A4), then we can find $W^{\rm II}$ and $W^{\rm III}$. In addition, $W_\infty(x)$ is a normalized distribution,

$$\int_{-\infty}^{+\infty}W_\infty(x)dx=1 , \tag{A7}$$

and so

$$\int_0^1 W_\infty^{\rm I}(x)dx+\int_1^2 W_\infty^{\rm II}(x)dx+\int_2^{3+\epsilon}W_\infty^{\rm III}(x)dx=\tfrac{1}{2} . \tag{A8}$$

We now restrict ourselves to the special case $\epsilon=1/N$ with N being a large positive integer. The support of $W^\infty(x)$ is, in that case, the set of integer multiples of ϵ,

$$W_\infty^{\rm I}(x)=\sum_{n=1}^{N}W_n^{\rm I}\delta(x-1+\epsilon n)+W_0\delta(x-1) . \tag{A9}$$

Next, we make the ansatz

$$W_n^{\rm I}=A/n^\alpha , \tag{A10}$$

which was motivated by numerical results, and insert Eqs. (A10) and (A9) into Eq. (A4). The ansatz solves Eq. (A4) for $\epsilon\rightarrow 0$ if

$$p=\tfrac{1}{2}-\tfrac{1}{2}W_0 . \tag{A11}$$

From the definition of p and Eq. (A10),

$$p=\tfrac{1}{2}-A\sum_{n=1}^{N}\frac{1}{n^{\alpha}}$$

$$\cong\tfrac{1}{2}-A\zeta(\alpha)\ , \qquad \text{(A12)}$$

where $\zeta(\alpha)$ is the Riemann ζ function. We can now find $W^{\rm III}$ and $W^{\rm II}$, and use the normalization condition Eq. (A8), giving

$$W^{\rm III}(x)\cong\tfrac{1}{2}W^{\rm I}(x-2)\ , \qquad \text{(A13)}$$

$$W^{\rm II}(x)\cong\tfrac{1}{2}W^{\rm I}(2-x)\ , \qquad \text{(A14)}$$

$$2A\zeta(\alpha)+(1+p)W_0\cong 0.5\ , \qquad \text{(A15)}$$

and, finally, from the δ-function term in Eq. (A4),

$$A=\tfrac{1}{2}p^2\ . \qquad \text{(A16)}$$

From Eqs. (A11), (A12), and (A16) one finds

$$p=[-1+\sqrt{1+\zeta(\alpha)}]/\zeta(\alpha)\ , \qquad \text{(A17)}$$

and from Eqs. (A11), (A12), and (A15), one finds

$$p^2\zeta(\alpha)+[1+p]p^2\zeta(\alpha)\cong 0.5\ ,$$

with the solution $\alpha\simeq 1.7\ldots$, in reasonable agreement with numerical results. Collecting Eqs. (A10), (A13), (A14), and $\alpha\simeq 1.7$ produces Eq. (22) of Sec. III.

APPENDIX B: MAGNETIZATION AND SUSCEPTIBILITY

The magnetization M and susceptibility χ follow from the dependence of the free energy $\bar{f}$ on a uniform field H,

$$M=\frac{T}{2}\int_{-\infty}^{+\infty}dx\frac{\partial}{\partial H}W_{\infty}(x,H)\,|_{H=0}$$

$$\times\ln[2\cosh(2\beta)+2\cosh(2\beta x)]\ , \qquad \text{(B1)}$$

$$\chi=\frac{T}{2}\int_{-\infty}^{+\infty}dx\frac{\partial^2}{\partial H^2}W_{\infty}(x,H)\,|_{H=0}$$

$$\times\ln[2\cosh(2\beta)+2\cosh(2\beta x)]\ . \qquad \text{(B2)}$$

If $W_{\infty}(x,0)$ is a series of δ functions,

$$W_{\infty}(x,0)=\sum_{m}W_m\delta(x-x_m)\ , \qquad \text{(B3)}$$

then it follows from the recursion relation (10), with a field H added, that for $T\to 0$

$$W^{\infty}(x,H)=\sum_{m}\sum_{n=0}^{\infty}W_m(n)\delta(x-x_m-nH)\ . \qquad \text{(B4)}$$

Inserting Eq. (B4) into Eq. (B2) gives, in the low-temperature limit,

$$\chi_0=\frac{1}{2T}\left[\sum_{n=1}^{\infty}W_{m_0}(n)n^2+\sum_{n=1}^{\infty}W_{-m_0}(n)n^2\right]\ , \qquad \text{(B5)}$$

where $x_{m_0}=1$ and $x_{-m_0}=-1$. For $h\gg 1$, we use the recursion relation (10) with H finite and $W_{\infty}(x,0)$ as an initial condition. For $h>3$, one finds after only one iteration,

$$W_{\infty}(x,H)=\tfrac{1}{8}\sum_{\substack{\epsilon_1=\pm 1,\\ \epsilon_2=\pm 1,\\ \epsilon_3=\pm 1}}\delta(x+\epsilon_1 h+\epsilon_2+\epsilon_3+H)\ , \qquad \text{(B6)}$$

with H small. Since $W_{m_0}(n)=0$, we have $\chi=0$ for $h>3$. At $h=3$, we need two iterations,

$$\begin{aligned}W_{\infty}(x,H)=&\tfrac{1}{128}\delta(x+5-3H)+\tfrac{6}{128}\delta(x+5-2H)+\tfrac{9}{128}\delta(x+5-H)+\tfrac{8}{128}\delta(x+3-2H)+\tfrac{24}{128}\delta(x+3-H)\\&+\tfrac{16}{128}\delta(x+1-H)+\tfrac{9}{128}\delta(x-1-H)+\tfrac{6}{128}\delta(x-1-2H)+\tfrac{1}{128}\delta(x-1-3H)+\tfrac{24}{128}\delta(x-3-H)\\&+\tfrac{8}{128}\delta(x-3-2H)+\tfrac{16}{128}\delta(x-5-H)\ ,\end{aligned} \qquad \text{(B7)}$$

and so

$$\chi_0=(1/T)(\tfrac{29}{128}),\quad h=3\ . \qquad \text{(B8)}$$

For $2<h<3$, the recursion relation does not terminate (for $H\neq 0$), but corrections due to higher iterations become rapidly smaller. After three iterations the relevant δ functions (near ± 1) are

$$\begin{aligned}W_{\infty}(x,H)\cong&\tfrac{1}{2}p^3(1-q)^2\delta(x+1-2H)+p^3q(1-q)\delta(x+1-3H)\\&+\tfrac{1}{2}p^3q^2\delta(x+1-4H)+pq(1-q)\delta(x-1-2H)+pq^2\delta(x-1-3H)\ ,\end{aligned} \qquad \text{(B9)}$$

where $p=-1+\sqrt{2}$ and $q=\tfrac{1}{2}-p$ [see Eq. (18)]. From Eqs. (B9) and (B5) we find a power series of χ_0 in terms of p [Eq. 24(c)].

As we approach $h=1$, we again need many iterations, as for the $H=0$ case. We define

$$W^{\mathrm{I}}_{\infty}(x) \equiv \frac{\partial}{\partial H} W_{\infty}(x,H) \Big|_{H=0} , \tag{B10}$$

$$W^{\mathrm{II}}_{\infty}(x) \equiv \frac{\partial^2}{\partial H^2} W_{\infty}(x,H) \Big|_{H=0} . \tag{B11}$$

Differentiating the fixed-point equation [Eq. (11)] with respect to H gives a Fredholm equation of the second kind for $W^{\mathrm{I}}_{\infty}(x)$,

$$W^{\mathrm{I}}_{\infty}(x) = \frac{\partial}{\partial x} W_{\infty}(x) + \int dx_1 \int dx_2 W_{\infty}(x_1) W^{\mathrm{I}}_{\infty}(x_2)[\delta(x+h-g(x_1)-g(x_2)) + \delta(x-h-g(x_1)-g(x_2))] . \tag{B12}$$

The Neumann series[21] can be summed for $h \to 1^+$,

$$W^{\mathrm{I}}_{\infty}(x) \cong A^0 \frac{\partial}{\partial x} W_{\infty}(x) . \tag{B13}$$

with A^0 a constant. By twice differentiating the fixed-point equation with respect to H, we obtain in a similar way for $h \to 1^+$,

$$W^{\mathrm{II}}_{\infty}(x) \cong A^1 W_{\infty}(x) + A^2 \frac{\partial^2}{\partial x^2} W_{\infty}(x) , \tag{B14}$$

where A^1 and A^2 are constants. The use of the expression for $W_{\infty}(x)$ found in Appendix A in Eq. (B14) gives the susceptibility quoted in Eq. (25b).

APPENDIX C: DIFFERENTIABILITY OF $W_{\infty}(x)$

In discussing the finite-temperature properties we used the fact that $W_{\infty}(x)$ is a smooth function. From the $T=0$ solution in Sec. III, we know that this need not be the case. In this appendix we show that for small h and for T near T_c^0, $W_{\infty}(x)$ is indeed differentiable.

For $h=0$, the recursion relation for the pure system has a fixed point $x_{\infty}(T)$ which is zero for $T > T_c^0$. For h small but finite we expand $g(x)$ in powers of x. To lowest order, Eq. (6) becomes

$$x_{n+1}(\vec{R}) = h(\vec{R}) + g'(0)[x_n(\vec{R}+\vec{\delta}_1) + x_n(\vec{R}+\vec{\delta}_2)] , \tag{C1}$$

where $g'(0)$ is the temperature-dependent slope of $g(x)$ at $x=0$. We solve Eq. (C1) by iteration,

$$x_1 = \pm h , \tag{C2}$$

$$x_2 = \pm h + g'(0)[\pm h \pm h] , \tag{C3}$$

$$x_3 = \pm h + g'(0)[\pm h \pm h] + [g'(0)]^2[\pm h \pm h \pm h \pm h] , \tag{C4}$$

$$x_{\infty} = h \sum_{k=0}^{\infty} [g'(0)]^k \left[\sum_{l=1}^{2^k} \epsilon(k,l) \right] , \tag{C5}$$

where $\epsilon(k,l)$ are independent random variables assuming the values ± 1 with equal probability. Now, there is a theorem, due to Salem,[22] that states that if

$$x = \sum_{k=0}^{\infty} \epsilon(k) r_k , \tag{C6}$$

where $\epsilon(k)$ are again independent random variables with values ± 1, then the distribution of x, $W(x)$, is differentiable in x if

$$r_k < \sum_{\gamma=k+1}^{\infty} r_{\gamma} \tag{C7}$$

for all k. Conditions (C7) translate, for our case, into $g' > \frac{1}{4}$. The $h=0$ phase transition occurs when $g' = \frac{1}{2}$, and so in a finite-temperature range above T_c^0, $W_{\infty}(x)$ is indeed differentiable. A similar argument, but with $x_{\infty} \neq 0$, applies for $T < T_c^0$. For $g' < \frac{1}{4}$, $W_{\infty}(x)$ is not differentiable. A similar situation occurs for the $d=1$ RFIM and is discussed in more detail by Aeppli and Bruinsma (Ref. 9).

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