Percolation in dimensions $d \geq 4$

Joan Adler
Solid State Institute and Physics Department, Technion, Israel Institute of Technology, Haifa 32000, Israel

Amnon Aharony
Department of Physics and Astronomy, University of Tel-Aviv, Ramat Aviv, Tel-Aviv 69978, Israel

A. Brooks Harris
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
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In this paper we present extended mean cluster size series for the bond percolation problem for hypercubic lattices in d dimensions and analyze them for $4 \leq d \leq 9$. Values of the critical exponent γ and the leading correction exponent Δ_1 are deduced, and the Δ_1 values for $d > 6$ are compared with new analytic results.

There has recently been a great deal of interest in various aspects of the critical behavior of the percolation problem. In $d=2$ dimensions, exact and conjectured exact results[1] have recently been obtained. For both $d=2$ and $d=3$ dimensions, extensive Monte Carlo,[2] field-theoretic,[3,4] and series-expansion[5,6] calculations have recently been undertaken. Particular emphasis, in the $d=3$ dimensions studies, has been placed on the evaluation of p_c and of the correction to scaling exponent Δ_1. With the exception of the vicinity of the upper critical dimension ($=6$, for percolation) relatively little attention has been paid to percolation in $d \geq 4$ in the last five years. In this paper we remedy this defect by presenting and analyzing new mean size series for hypercubic lattices in general dimensions.

We assume critical behavior of the form

$$S(p) \sim (p-p_c)^{-\gamma}[1+a_1(p-p_c)^{\Delta_1}+b(p-p_c) + a_2(p-p_c)^{\Delta_2}+\cdots], \quad (1)$$

where a_1 and a_2 are the amplitudes of the nonanalytic corrections and b is the amplitude of the analytic correction, for $d \neq 6$ and

$$S(p) \sim (p-p_c)^{-\gamma}[\ln(p-p_c)]^{\theta} \quad (2)$$

for $d=6$ and obtain estimates of p_c, γ, and Δ_1. New analytic results for Δ_1 in $d > 6$ are also deduced and a comparison with existing estimates is made.

The new series have been obtained for both the mean-square number of sites in a bond cluster (SB) and the mean-square number of bonds in a bond cluster (BB). Both series are to order p^{11}, the former being a completely new series and the latter extending the series of Gaunt and Ruskin[7] (GR) and Fisch and Harris[8] (FH) by one term. The series can be built from the coefficients $A(m,1)$ that are presented in Table I via the formula

$$\chi(p) = 1 + \sum_{m,l} A(m,L) d^l p^m . \quad (3)$$

We note that these series are generated via lattice animal

TABLE I. Coefficients $A(m,1)$ in the mean cluster size expansions. The numbers enclosed in parentheses represent powers of 10 by which the numbers preceding should be multiplied.

(a) No. of bonds in a bond cluster (BB)	(b) No. of sites in a bond cluster (SB)
$A(1,1)=0.100\,000\,000\,000(+1)$	$A(1,1)=0.200\,000\,000\,000(+1)$
$A(2,2)=0.400\,000\,000\,000(+1)$	$A(2,2)=0.400\,000\,000\,000(+1)$
$A(2,1)=-0.200\,000\,000\,000(+1)$	$A(2,1)=-0.200\,000\,000\,000(+1)$
$A(3,3)=0.800\,000\,000\,000(+1)$	$A(3,3)=0.800\,000\,000\,000(+1)$
$A(3,2)=-0.800\,000\,000\,000(+1)$	$A(3,2)=-0.800\,000\,000\,000(+1)$
$A(3,1)=0.200\,000\,000\,000(+1)$	$A(3,1)=0.200\,000\,000\,000(+1)$
$A(4,4)=0.160\,000\,000\,000(+2)$	$A(4,4)=0.160\,000\,000\,000(+2)$
$A(4,3)=-0.240\,000\,000\,000(+2)$	$A(4,3)=-0.240\,000\,000\,000(+2)$
$A(4,2)=0.600\,000\,000\,000(+1)$	$A(4,2)=0.200\,000\,000\,000(+1)$
$A(4,1)=0.400\,000\,000\,000(+1)$	$A(4,1)=0.800\,000\,000\,000(+1)$
$A(5,5)=0.320\,000\,000\,000(+2)$	$A(5,5)=0.320\,000\,000\,000(+2)$
$A(5,4)=-0.640\,000\,000\,000(+2)$	$A(5,4)=-0.640\,000\,000\,000(+2)$
$A(5,3)=0.160\,000\,000\,000(+2)$	$A(5,3)=0.800\,000\,000\,000(+1)$
$A(5,2)=0.440\,000\,000\,000(+2)$	$A(5,2)=0.600\,000\,000\,000(+2)$
$A(5,1)=-0.260\,000\,000\,000(+2)$	$A(5,1)=-0.340\,000\,000\,000(+2)$

TABLE I. (*Continued*).

(a) No. of bonds in a bond cluster (BB)	(b) No. of sites in a bond cluster (SB)
$A(6,6)=0.640\,000\,000\,000(+2)$	$A(6,6)=0.640\,000\,000\,000(+2)$
$A(6,5)=-0.160\,000\,000\,000(+3)$	$A(6,5)=-0.160\,000\,000\,000(+3)$
$A(6,4)=0.560\,000\,000\,000(+2)$	$A(6,4)=0.400\,000\,000\,000(+2)$
$A(6,3)=0.104\,000\,000\,000(+3)$	$A(6,3)=0.112\,000\,000\,000(+3)$
$A(6,2)=0.100\,000\,000\,000(+2)$	$A(6,2)=0.620\,000\,000\,000(+2)$
$A(6,1)=-0.720\,000\,000\,000(+2)$	$A(6,1)=-0.116\,000\,000\,000(+3)$
$A(7,7)=0.128\,000\,000\,000(+3)$	$A(7,7)=0.128\,000\,000\,000(+3)$
$A(7,6)=-0.384\,000\,000\,000(+3)$	$A(7,6)=-0.384\,000\,000\,000(+3)$
$A(7,5)=0.192\,000\,000\,000(+3)$	$A(7,5)=0.160\,000\,000\,000(+3)$
$A(7,4)=0.192\,000\,000\,000(+3)$	$A(7,4)=0.224\,000\,000\,000(+3)$
$A(7,3)=0.600\,000\,000\,000(+3)$	$A(7,3)=0.788\,000\,000\,000(+3)$
$A(7,2)=-0.142\,400\,000\,000(+4)$	$A(7,2)=-0.179\,800\,000\,000(+4)$
$A(7,1)=0.698\,000\,000\,000(+3)$	$A(7,1)=0.884\,000\,000\,000(+3)$
$A(8,8)=0.256\,000\,000\,000(+3)$	$A(8,8)=0.256\,000\,000\,000(+3)$
$A(8,7)=-0.896\,000\,000\,000(+3)$	$A(8,7)=-0.896\,000\,000\,000(+3)$
$A(8,6)=0.608\,000\,000\,000(+3)$	$A(8,6)=0.544\,000\,000\,000(+3)$
$A(8,5)=0.352\,000\,000\,000(+3)$	$A(8,5)=0.448\,000\,000\,000(+3)$
$A(8,4)=0.106\,400\,000\,000(+4)$	$A(8,4)=0.102\,000\,000\,000(+4)$
$A(8,3)=-0.840\,000\,000\,000(+2)$	$A(8,3)=0.996\,000\,000\,000(+3)$
$A(8,2)=-0.572\,200\,000\,000(+4)$	$A(8,2)=-0.836\,400\,000\,000(+4)$
$A(8,1)=0.442\,400\,000\,000(+4)$	$A(8,1)=0.599\,800\,000\,000(+4)$
$A(9,9)=0.512\,000\,000\,000(+3)$	$A(9,9)=0.512\,000\,000\,000(+3)$
$A(9,8)=-0.204\,800\,000\,000(+4)$	$A(9,8)=-0.204\,800\,000\,000(+4)$
$A(9,7)=0.179\,200\,000\,000(+4)$	$A(9,7)=0.166\,400\,000\,000(+4)$
$A(9,6)=0.576\,000\,000\,000(+3)$	$A(9,6)=0.832\,000\,000\,000(+3)$
$A(9,5)=0.107\,200\,000\,000(+4)$	$A(9,5)=0.968\,000\,000\,000(+3)$
$A(9,4)=0.162\,400\,000\,000(+5)$	$A(9,4)=0.198\,106\,666\,667(+5)$
$A(9,3)=-0.726\,720\,000\,000(+5)$	$A(9,3)=-0.864\,893\,333\,333(+5)$
$A(9,2)=0.912\,540\,000\,000(+5)$	$A(9,2)=0.108\,987\,333\,333(+6)$
$A(9,1)=-0.367\,240\,000\,000(+5)$	$A(9,1)=-0.442\,346\,666\,667(+5)$
$A(10,10)=0.102\,400\,000\,000(+4)$	$A(10,10)=0.102\,400\,000\,000(+4)$
$A(10,9)=-0.460\,800\,000\,000(+4)$	$A(10,9)=-0.460\,800\,000\,000(+4)$
$A(10,8)=0.499\,200\,000\,000(+4)$	$A(10,8)=0.473\,600\,000\,000(+4)$
$A(10,7)=0.640\,000\,000\,000(+3)$	$A(10,7)=0.128\,000\,000\,000(+4)$
$A(10,6)=0.800\,000\,000\,000(+2)$	$A(10,6)=-0.224\,000\,000\,000(+3)$
$A(10,5)=0.341\,280\,000\,000(+5)$	$A(10,5)=0.335\,066\,666\,667(+5)$
$A(10,4)=-0.470\,880\,000\,000(+5)$	$A(10,4)=-0.186\,213\,333\,333(+5)$
$A(10,3)=-0.288\,498\,000\,000(+6)$	$A(10,3)=-0.417\,486\,666\,667(+6)$
$A(10,2)=0.666\,848\,000\,000(+6)$	$A(10,2)=0.863\,971\,333\,333(+6)$
$A(10,1)=-0.367\,516\,000\,000(+6)$	$A(10,1)=-0.463\,576\,000\,000(+6)$
$A(11,11)=0.204\,800\,000\,000(+4)$	$A(11,11)=0.204\,800\,000\,000(+4)$
$A(11,10)=-0.102\,400\,000\,000(+5)$	$A(11,10)=-0.102\,400\,000\,000(+5)$
$A(11,9)=0.133\,120\,000\,000(+5)$	$A(11,9)=0.128\,000\,000\,000(+5)$
$A(11,8)=-0.512\,000\,000\,000(+3)$	$A(11,8)=0.102\,400\,000\,000(+4)$
$A(11,7)=-0.393\,600\,000\,000(+4)$	$A(11,7)=-0.486\,400\,000\,000(+4)$
$A(11,6)=0.593\,706\,666\,667(+5)$	$A(11,6)=0.582\,720\,000\,000(+5)$
$A(11,5)=0.384\,704\,000\,000(+6)$	$A(11,5)=0.470\,893\,333\,333(+6)$
$A(11,4)=-0.336\,477\,600\,000(+7)$	$A(11,4)=-0.388\,022\,933\,333(+7)$
$A(11,3)=0.784\,808\,400\,000(+7)$	$A(11,3)=0.901\,120\,666\,667(+7)$
$A(11,2)=-0.752\,391\,666\,667(+7)$	$A(11,2)=-0.867\,225\,466\,667(+7)$
$A(11,1)=0.259\,586\,400\,000(+7)$	$A(11,1)=0.301\,134\,600\,000(+7)$

data (unlike GR), and both series should have the same p_c and γ values.

The series have been analyzed both by the usual D log Padé technique and by the methods of Adler *et al.*[5] that are specifically designed to estimate the correction to scaling exponents Δ_1 (Ref. 5) and θ (Ref. 9). These methods have recently been reviewed.[5] Their basis is the construction of functions that give γ as a function of Δ or θ for a given choice of p_c.

Behavior such as that of Eq. (2) can be analyzed by the

method of Adler and Privman.[9] This method was developed to prove the absence of logarithmic corrections in $d=2$ dimensions percolation, but is equally suitable for demonstrating their presence. We write $\theta=z\gamma$ and derive the series for

$$g(p)=\frac{1}{-\gamma}(p-p_c)\ln(p_c-p)\left[\frac{S'(p)}{S(p)}-\frac{\gamma}{p_c-p}\right]. \quad (4)$$

We can show that

$$\lim_{p\to p_c} g(p)=z \quad (5)$$

and form Padé approximants to $g(p)$ in order to evaluate θ as a function of γ, graph γ as a function of θ for different p_c values, and note that the θ value is extremely sensitive to p_c. Thus for $d=6$ dimensions percolation,

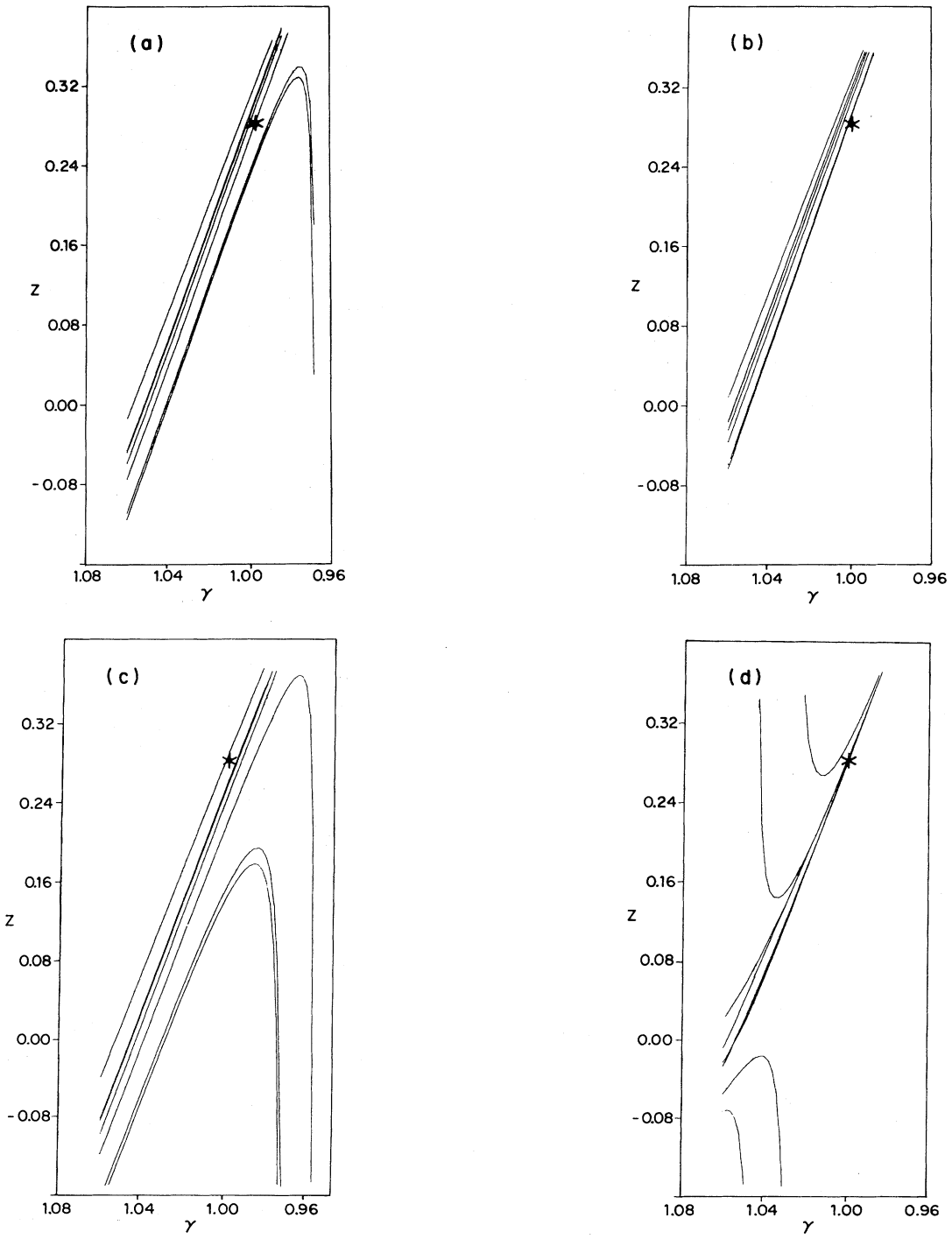

FIG. 1. Graphs of γ versus z for (a) BB percolation, $p_c=0.094\,075$, (b) BB percolation, $p_c=0.094\,110$, (c) BB percolation, $p_c=0.094\,040$, (d) SB percolation, $p_c=0.094\,100$ in six dimensions. These figures illustrate the variation in convergence for different p_c values. The RG (Ref. 10) exponents are illustrated by an asterisk.

where we know both $\theta(=\frac{2}{7})$ and $\gamma(=1)$ from the renormalization-group[10] (RG) results, we can evaluate p_c by scanning over different p_c values and observing which give the best agreement with the RG values. We find for BB percolation [Fig. 1(a)] that $p_c=0.094\,075$ is the best value and illustrate the weaker agreement found for $p_c=0.094\,11$ [Fig. 1(b)] and $p_c=0.094\,04$ [Fig. 1(c)]. In all graphs the RG values are indicated by an asterisk. For SB percolation all the approximants pass close to the point $(\frac{2}{7},1)$ for $p_c=0.094\,075$ but closer convergence can be observed for other values. We choose $p_c=0.094\,075\pm0.0001$ within the error bounds of GR and just within those of FH. This can be compared (see Table II) with the series results of Ref. 10, where they imposed $p_c=0.0941$ from the Padé result of Ref. 7, to find $\theta=0.28\pm0.07$.

The method developed to analyze behavior of the Eq. (1) type involves transforming the original series in p to one in

$$y=1-(1-p/p_c)^{\Delta}\,. \tag{6}$$

We then look at different Padé approximants to the function

$$G_{\Delta}(y)=\Delta(y-1)\frac{d}{dy}[\ln S(p)] =\gamma-x/(1+x)\,, \tag{7}$$

where $x=ap_c^{\Delta_1}\Delta_1(y-1)^{\Delta_1/\Delta}$. The correction term x becomes 0 when $p=p_c$ and $\Delta=\Delta_1$. Different Padé approximants to this function are graphed, giving lines of γ as a function of Δ. These should converge near the correct (Δ_1,γ) point for the correct p_c. For this model, where neither p_c nor Δ_1 are known exactly for $d\neq6$ dimensions, we search for the best convergence in the (p_c,γ,Δ_1) space[11] by considering different p_c planes. For each of these, the region where this convergence occurs is enclosed by a box in the figures; the box is subjective and gives error bounds on γ and Δ_1 for the particular p_c value chosen. The exponents given for $d=4$ and 5 dimensions in the following paragraph and in Table II are based on graphs for all p_c values within the p_c range; we present below some selected figures to illustrate specific points.

We first consider $d=4$ and 5 dimensions. Here we have no definite indications of exponent values beyond the knowledge that $\gamma>1$ and probably smaller than the currently accepted $3d$ value of $1.7\sim1.8$.[12] There are several predictions for γ, Δ_1, and p_c and these are listed in Table II, together with our results. Some selected graphs of our results for 4 and 5 dimensions, respectively, are presented in Figs. 2 and 3. In Fig. 2 we present graphs of γ versus Δ_1 for three different values of p_c for BB percolation and one value for SB. From these and other graphs we deduce $0.6<\Delta_1<1.0$, $1.39<\gamma<1.49$, and

TABLE II. Exponent and p_c values for $d\geq4$ dimensions.

d	γ	Δ_1	p_c
4	1.44[a]	0.88–1.03[b]	
	1.40±0.02[c]		0.1600±0.0002[c]
	1.48±0.08[d]		0.161 ±0.0015[d]
	1.44±0.05[e]	0.6 –1.0[e]	0.1603±0.0002[e]
	1.6 ±0.1[f]		
5	1.18[a]	0.42–0.45[b]	
	1.17±0.02[c]		0.1181±0.0002[c]
	1.18±0.07[d]		0.118 ±0.001[d]
	1.20±0.03[e]	0.45–0.9[e]	0.1182±0.0002[e]
	1.3 ±0.1[f]		
6	1.08[c]		0.094 3 ±0.0002[c]
	1.04±0.06[d]		0.094 1 ±0.0005[d]
	1.00±0.02[e]		0.094 075±0.0001[e]
	1.00±0.05[f]		
7	1.00±0.03[d]	$\frac{1}{2}$[g]	0.078 8 ±0.000 2[c]
			0.078 6 ±0.000 2[d]
			0.078 62±0.000 03[e]
8		1[g]	0.067 70±0.000 05[e]
9		$\frac{3}{2}$[g]	0.059 50[e]±0.000 05[e]

[a] ϵ expansion, Ref. 4.
[b] ϵ expansion, Ref. 13.
[c] Series, Ref. 8.
[d] Series, Ref. 7.
[e] This work series, see Ref. 10 for a discussion of error bounds on p_c.
[f] Monte Carlo simulation, Ref. 14.
[g] This work, analytic.

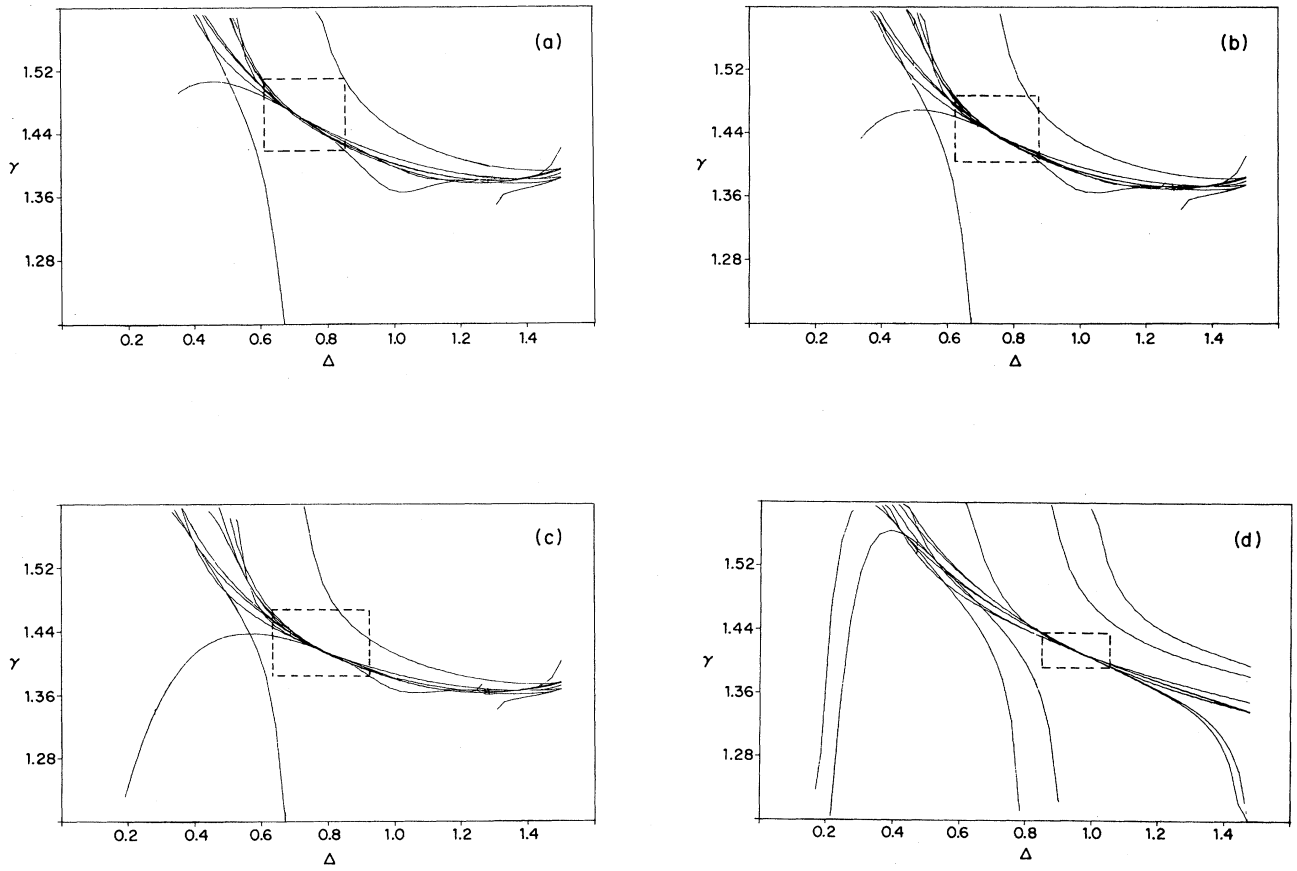

FIG. 2. Graphs of γ versus Δ_1 for (a) BB percolation, $p_c=0.1604$, (b) BB percolation, $p_c=0.1603$, (c) BB percolation, $p_c=0.1602$, (d) SB percolation, $p_c=0.1603$ in four dimensions. These figures illustrate the variation in intersection regions with p_c values within the error bounds and the consistency of BB and SB results.

$p_c=0.1603\pm0.0002$. These results are in excellent agreement with those given in Table II except for the p_c of FH. The five-dimensional results (Fig. 3) are less consistent with ϵ-expansion results. We present the graph for $p_c=0.1183$ [Fig. 3(a)] which is the value for which the clearest convergence is found for BB percolation; here $\gamma\sim1.20$ and $\Delta_1\sim0.5$. This particular γ value is not in good agreement with the ϵ-expansion result, $\gamma=1.18$, although the Δ_1 value is close to $\Delta_1=0.45$. For $p_c=0.182$ [Fig. 3(b)] we find that the approximants give consistency with the ϵ-expansion γ result but the intersection is much less clear. For SB percolation we have [Fig. 3(c)] consistency with the ϵ-expansion results for $p_c=0.118\,15$, but again the intersection is not strong. For $d=5$ dimensions we quote the overall estimates $\gamma=1.20\pm0.03$ and $0.5<\Delta_1<0.8$.

In both the $d=4$ and 5 figures there is some evidence of a second intersection region near $\Delta_2\sim1.3$ and $\Delta_2\sim1.2$, respectively. The existence of two correction terms in Eq. (1) is consistent with the results found from series analysis in three dimensions[15] and the results of the scaling-field method.[16] In the latter work two irrelevant operators are observed (see Fig. 5 of Ref. 16), one being larger than the other at $d=4$ and the opposite occurring at $d=5$ dimensions. The general trend of our values for Δ_1 and Δ_2 is in good agreement with those of Ref. 16; the range of uncertainty precluding more detailed comparison. We note that the scaling-field value of Δ_1 for $d=5$ dimensions is larger than the ϵ-expansion result as is our value.

We now consider $d>6$. Here, the renormalization-group analysis yields the general result[17]

$$S(p)^{-1}\propto(p-p_c)\tilde{W}^{-2/7} \tag{8}$$

with $\tilde{W}=A+(B/\epsilon)(p-p_c)^{-\epsilon/2}$ where $\epsilon=6-d$. At $\epsilon\to0$, Eq. (6) reduces to Eq. (2). For $\epsilon<0$, we may expand $\tilde{W}^{-2/7}$ and reproduce Eq. (1) with $\gamma=1$ and $\Delta_1=-\epsilon/2$. Thus, $\Delta_1=\frac{1}{2}$, 1, and $\frac{3}{2}$ for $d=7$, 8, and 9 dimensions, respectively.

There are, of course, also analytic corrections, and distinction between these and values of $\Delta_1=\frac{1}{2}$ or 1 can be very difficult. Analytic (and other) corrections can lead to "resonances"[18] which are intersections at values of Δ_1/k, where $k=2,3\ldots$ and the same γ as that of the first intersection region. Thus $\Delta_1=\frac{1}{2}$ could be a resonance from an analytic correction.

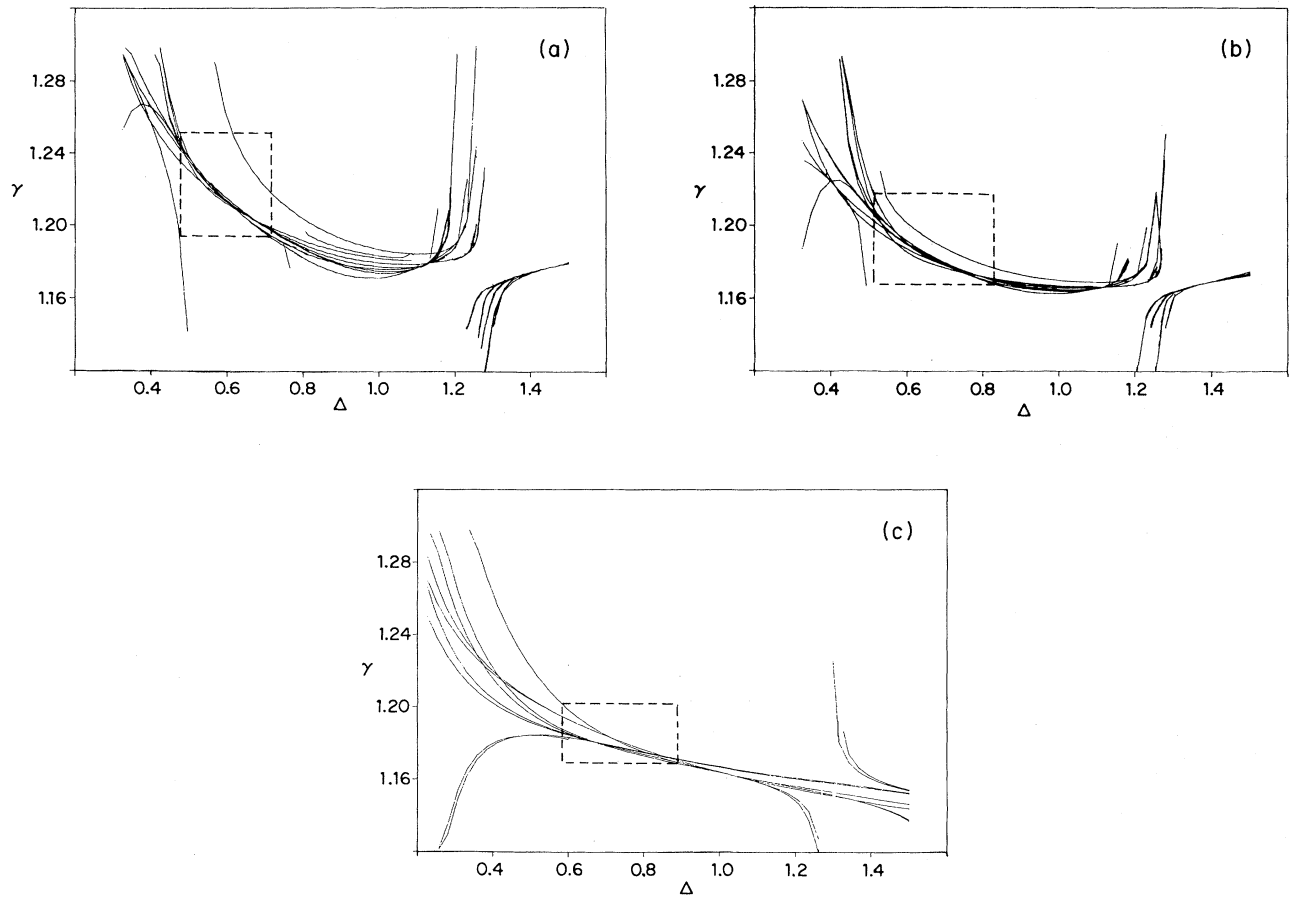

FIG. 3. Graphs of γ versus Δ_1 for (a) BB percolation, $p_c = 0.1183$, (b) BB percolation, $p_c = 0.1182$, (c) SB percolation, $p_c = 0.118\,15$ in five dimensions. These figures illustrate (a) the clearest intersection region and closest agreement with ϵ expansion, Δ_1 value, and (b) the best agreement with the ϵ expansion γ value.

We analyzed the series for $d = 7$, 8, and 9 dimensions, BB percolation. In each case we found an analytic ($\Delta_1 = 1$) intersection region and a possible intersection region near $\Delta_1 = \frac{1}{2}$; in both regions $\gamma \sim 1.0$ and we suspect that the $\Delta = \frac{1}{2}$ intersection region is a resonance.[10] For $d = 7$ and 8 dimensions these results are consistent with the above. For $d = 9$ dimensions (Fig. 4) no intersection at $\Delta_1 = 1.5$ is observed; however, a strong analytic term could be swamping this. We presume that this discrepancy is due to the length of the series, and quote p_c values in Table II. We note that unlike $d < 6$, where our p_c values are no more precise than previous estimates, for $d > 6$ we are able to determine p_c quite accurately by assuming $\gamma = 1$.

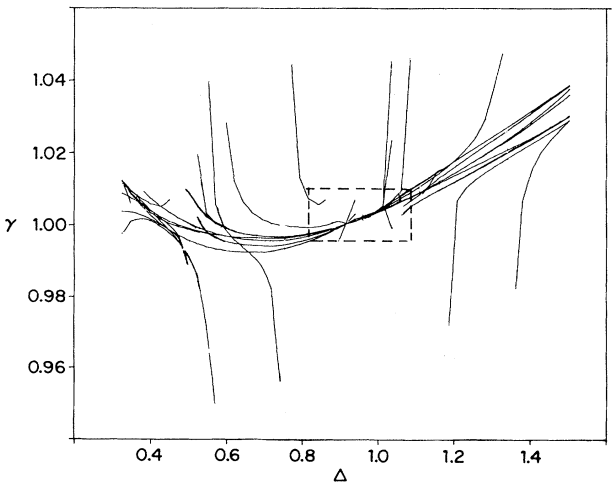

FIG. 4. Graph of γ versus Δ_1 for BB percolation in nine dimensions at $p_c = 0.059\,50$.

In conclusion, we have presented new mean cluster size series for percolation and analyzed them. For $d < 6$ our results are in basic agreement with ϵ-expansion results. Using RG values of γ and θ at $d = 6$ dimensions and the classical value of γ for $d > 6$, we were able to make very precise estimates of p_c in $d > 6$.

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