Monte Carlo renormalization-group study of the rectangular Ising ferromagnet: Universality and a fixed line

D. P. Landau
Department of Physics and Astronomy, The University of Georgia, Athens, Georgia 30602

R. H. Swendsen*
IBM Zurich Research Laboratory, CH-8803 Ruschlikon, Switzerland
(Received 13 December 1983)

The renormalization-group description of the anisotropic two-dimensional Ising model has been investigated with the use of Monte Carlo renormalization-group methods. We have been able to demonstrate the existence of a fixed line with universal critical exponents, in contrast to the nonuniversality of the Baxter fixed line. The possibility of a model with any given anisotropy being described by any point on the fixed line is demonstrated by explicit calculations.

I. INTRODUCTION

The Ising square lattice has been the object of intense scrutiny for over half a century, and this activity has included both efforts to provide analytic solutions as well as to test various approximation methods. Exact solutions have been obtained for the thermal properties and order parameter for the general model[1,2]

$$\mathcal{H}=J_x \sum_{(ij)} \sigma_i\sigma_j + J_y \sum_{(ik)} \sigma_i\sigma_k \ , \tag{1}$$

where the first sum is over nearest-neighbor (NN) pairs in the x direction and the second sum is over NN pairs in the y direction. These solutions[1,2] show that the critical exponents are identical for all finite, nonzero values of $R=J_x/J_y$. Real-space renormalization-group methods have also been used to examine the fixed-point structure of the isotropic ($R=1$) square lattice.[3] It has recently been conjectured, however, that spatial anisotropy is a marginal operator and that the critical behavior for general R should be described by a line of fixed points.[4] In this paper we shall describe the results of a Monte Carlo renormalization-group (MCRG) study of the anisotropic ($R\neq 1$) Ising square lattice.

We are able to present the first direct demonstration of the existence of a fixed line describing the *universal* critical behavior of rectangular Ising models. The evidence is of two types. First, from MCRG analyses of both the isotropic and anisotropic models, we can explicitly exhibit the marginal operator arising from odd combinations of exchange interactions in the x and y directions. Secondly, the renormalized correlation functions from an anisotropic simulation ($R=0.3$) show that the fixed point itself is anisotropic and is therefore located at a different point on the fixed line.

The situation must be clearly distinguished from that of the Baxter model, in which the fixed line describes nonuniversal behavior. In the Baxter case, models with different coupling constants renormalize to different points on the fixed line, corresponding to different critical exponents. In the anisotropic Ising case, transformations can be constructed to allow models with any given anisotropy to converge to (and be described by) *any* point along the fixed line, resulting in the same critical exponents for any anisotropy. This can be accomplished by preliminary anisotropic rescaling transformations that change the ratio of the correlation lengths. The MCRG method enables us to demonstrate this effect explicitly and we present an analysis of an anisotropic model with $R=3.0$, which converges to the isotropic fixed point after a 1×2 rescaling transformation.

II. METHOD

We have used the Monte Carlo renormalization-group method proposed by Swendsen.[5] A standard Monte Carlo method was used to simulate Ising models [see Eq. (1)] on $L\times M$ lattices with periodic boundary conditions. We then used the same 2×2 block-spin transformation applied to the generated configurations with ties decided by a random number instead of a tie breaker.[6] In addition, "block shifting" was used[6] to improve the accuracy of the results. Using correlation functions determined from both simulated and transformed lattices, we estimated the elements of the linearized transformation matrix[3] $T^{*}_{\alpha\beta}$ and the eigenvalues of this matrix were subsequently determined. [The Hamiltonian is rewritten as a sum of terms $K_\alpha S_\alpha$ where the S_α are sums of products of spin operators and the K_α are the associated couplings (see Fig. 1)]. The analysis allowed for the generation of a number of couplings (see Fig. 1) as a result of the RG transformation; the matrix was block diagonalized by examining the symmetry characteristics of these couplings, and even and odd symmetry eigenvalues, y^e and y^0, respectively, were determined separately. It is important to remember that couplings in the x and y directions are no longer required to be equal by symmetry and must therefore be treated separately. Data were obtained for $R=0.3$ and 0.1 with $\frac{1}{2}\leq L/M\leq 4$. Data were taken only at the critical temperatures as obtained from Ref. 1. Usually at least 10^4 Monte Carlo steps/spin (MCS/spin) were retained for computing averages, and correlation functions were deter-

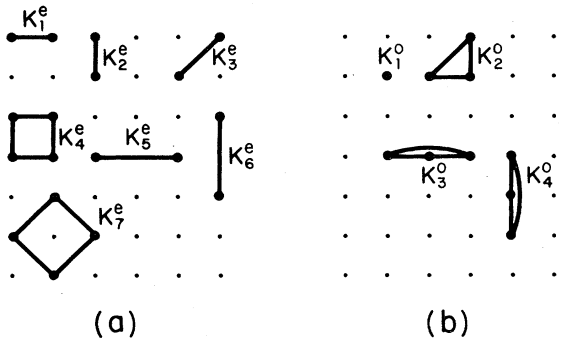

FIG. 1. Couplings used in the renormalization-group analysis: (a) even-even symmetry couplings; (b) odd-odd symmetry couplings (see Refs. 3 and 5).

mined every 10 MCS/spin.

Errors were determined by reanalyzing subsets of the data and calculating the spread in the estimates for correlation functions and eigenvalues. The error estimates are shown in the tables in terms of a one-standard-deviation uncertainty in the last digit.

In the interpretation of the data presented in the following sections, it is important to note that $T_{\alpha\beta}$ is, in principle, an infinite matrix. Since we can only do numerical calculations on finite parts of this matrix, corresponding to a finite number of interactions in the eigenvectors, it is important to report the trend as the number of interactions is increased. The entries in each column for a given level of renormalization do not represent independent estimates of the eigenvalue exponents, but instead present a sequence, which should converge to the true value at that level of renormalization. The successive iterations of the renormalization transformation should then give the critical exponents.

III. THE EXISTENCE OF A UNIVERSAL FIXED LINE

A. Isotropic model: $R=1.0$

The isotropic, two-dimensional Ising model has previously been studied in detail using MCRG with *isotropic* interactions for the analysis.[5,7] This work had shown that deviations from linearity in the region of the fixed point were already negligibly small at the second iteration of the RG transformation. Furthermore, the *effects* of the finite lattice were also shown to be negligible if the renormalized lattice is at least 8×8, and the effect of reducing the size of the normalized lattice to 4×4 only produced a 2% bias on the leading thermal eigenvalue exponent.

This led us to begin the study by simulating isotropic Ising models on 16×16 and 32×32 lattices and analyzing the data using the anisotropic operators illustrated in Fig. 1. Data on the MC simulations are given in Table I. A number of consistency checks were made based on data available from previous work. For example, the correlation functions in the x and y directions were monitored to make sure that they retained the symmetry of the model

TABLE I. MC simulation data for MCRG calculations for the $d=2$ Ising model with $R=1.0$. Data for correlation functions taken every 10 MCS/site.

Lattice size (L)	32	16
10^3 MCS/site used	360	23
10^3 MCS/site discarded	30	6
Fraction of successful MCS	0.192	0.185
Approximation correlation time in units of MCS/site	53	33

and Table II shows the accuracy to which this was achieved. The usual magnetic eigenvalues were also calculated, with results that were identical to those obtained previously within the accuracy of the calculation.

The results of the new MCRG analysis differed qualitatively from previous work in that a new eigenvalue exponent appeared. It is located between the two previously obtained leading eigenvalues of $+1.0$ and -1.0 and corresponds to eigenvectors in which couplings in the x and y directions have opposite sign. This is the only new eigenvalue to appear; the fourth-largest eigenvalue is located, as for the isotropic analysis, close to the expected value of -2 (with large error bars).

Quantitative results for the first three thermal eigenvalues are shown in Table III. The first and third eigenvalues are completely consistent with the first and second eigenvalues obtained from the isotropic analysis. The new eigenvalue is about 0.2 on the first iteration, but drops to 0.04(1) on the second iteration, which is known to be very close to the fixed point. The third RG transformation has a rather large statistical uncertainty, but again a very small value of the second eigenvalue exponent. All results, both qualitative and quantitative, are completely consistent with the existence of a marginal operator[8] for the anisotropy.

B. Anisotropic model: $R=0.3$

Again a 32×32 and a 16×16 lattice were simulated, using the same MC algorithm and analysis subroutines as for the isotropic case. Simulation data are given in Table IV and the leading thermal eigenvalues are given in Table V. Although the statistical uncertainty is somewhat larger due to the long relaxation times in the anisotropic system, the eigenvalues are completely consistent with those obtained for the isotropic model.

However, the correlation functions reveal a different condition. Table VI shows the effect of renormalization on the nearest-neighbor correlations in the x and y directions. The anisotropy is clearly marked, indicating that the fixed-point Hamiltonian is also anisotropic. This is also clear evidence for a fixed line, since models with differing anisotropy are shown to converge to different fixed points.

IV. FINITE-SIZE AND SHAPE EFFECTS

Different size lattices with different shapes were studied for $R=0.1$. The results of an analysis of data ob-

TABLE II. Correlations (see Fig. 1) as a function of iteration for $R=1$ and $L=32$. Error estimates in the last digits are given in parentheses.

Iter.	$\langle S_1 \rangle$	$\langle S_2 \rangle$	$\langle S_5 \rangle$	$\langle S_6 \rangle$
1	0.7176(8)	0.7178(8)	0.6138(13)	0.6141(15)
2	0.7167(15)	0.7169(16)	0.624(3)	0.625(3)
3	0.7362(4)	0.737(4)	0.661(5)	0.663(5)
4	0.7802(6)	0.782(6)	0.742(7)	0.746(9)

TABLE III. Estimates for the three largest thermal eigenvalues for $R=1$. Errors in the last digits are given in parentheses.

		$L=16$			$L=32$		
Iter.	No. of interact.	y_1^e	y_2^e	y_3^e	y_1^e	y_2^3	y_3^e
1	1	0.846(7)			0.859(5)		
	2	0.893(7)	−0.34(3)		0.899(4)	−0.291(3)	
	3	0.952(9)	−0.34(3)	−1.95(5)	0.958(4)	−0.291(4)	−2.13(6)
	4	0.953(9)	−0.34(3)	−1.86(5)	0.960(5)	−0.291(4)	−2.09(6)
	5	0.951(9)	−0.4(2)	−1.50(9)	0.962(4)	−0.056(15)	−1.56(7)
	6	0.951(9)	0.11(2)	−1.00(3)	0.961(4)	0.143(24)	−1.06(7)
	7	0.946(10)	0.13(2)	−0.84(5)	0.960(6)	0.154(24)	−0.92(6)
	8	0.946(10)	0.19(4)	−0.72(8)	0.959(6)	0.192(14)	−0.86(4)
	9	0.946(10)	0.17(4)	−0.78(8)	0.960(6)	0.192(14)	−0.99(7)
	10	0.939(10)	0.17(4)	−0.66(13)	0.961(6)	0.192(14)	−1.13(16)
2	1	0.874(12)			0.900(8)		
	2	0.942(10)	−0.54(4)		0.952(8)	−0.548(27)	
	3	1.003(11)	−0.54(4)	−2.44(27)	1.002(5)	−0.548(27)	−2.32(5)
	4	1.008(11)	−0.54(4)	−2.58(20)	1.005(5)	−0.547(26)	−2.65(9)
	5	1.003(15)	−0.19(4)	−1.70(18)	1.002(3)	−0.223(10)	−1.59(9)
	6	0.991(14)	−0.07(2)	−1.04(4)	1.006(5)	−0.036(13)	−1.09(5)
	7	0.995(13)	−0.00(4)	−0.84(15)	1.003(6)	0.000(11)	−0.95(4)
	8	0.994(12)	0.06(5)	−0.65(4)	1.003(6)	0.036(2)	−0.87(5)
	9	0.993(13)	0.06(5)	−0.68(4)	1.002(6)	0.036(2)	−0.94(9)
	10	0.986(13)	0.06(4)	−0.63(17)	1.005(7)	0.037(6)	−1.26(12)
3	1				0.872(6)		
	2				0.935(6)	−0.53(3)	
	3				0.990(7)	−0.53(3)	−2.37(11)
	4				0.991(9)	−0.53(2)	−2.53(15)
	5				0.986(9)	−0.31(3)	−1.71(17)
	6				0.977(6)	−0.18(5)	−1.19(19)
	7				0.972(6)	−0.10(3)	−0.93(8)
	8				0.970(7)	−0.01(7)	−0.74(8)
	9				0.968(9)	−0.01(7)	−0.74(8)
	10				0.968(8)	−0.01(7)	−0.73(10)

TABLE IV. MC simulation data for MCRG calculations for the $d=2$ Ising model with $R=0.3$ Data for correlation functions taken every 20 MCS/site.

Lattice size (L)	32	16
10^3 MCS/site used	786	420
10^3 MCS/site discarded	14	10
Fraction of successful MCS	0.156	0.152
Approximate correlation time in units of MCS/site	120	64

TABLE V. Estimates for the two largest thermal eigenvalues for $R=0.3$. Errors in the last digits are given in parentheses.

Iter.	No. of interact.	$L=16$ y	$L=16$ y_2^e	$L=32$ y_1^e	$L=32$ y_2^e
1	1	1.035(4)		1.060(4)	
	2	0.899(4)	−0.60(5)	0.918(6)	−0.67(2)
	3	0.951(4)	−0.58(4)	0.964(4)	−0.65(2)
	4	0.954(5)	−0.58(4)	0.966(5)	−0.66(2)
	5	0.897(5)	−0.15(5)	0.937(7)	−0.62(7)
	6	0.911(4)	0.10(6)	0.925(7)	−0.00(6)
	7	0.905(5)	0.11(5)	0.919(8)	−0.08(6)
	8	0.912(5)	0.17(4)	0.923(8)	−0.11(5)
	9	0.920(5)	0.14(3)	0.935(7)	−0.10(5)
	10	0.922(4)	0.15(2)	0.932(9)	−0.10(4)
2	1	0.927(8)		0.974(6)	
	2	0.926(4)	−0.74(4)	0.957(4)	−0.79(5)
	3	0.961(5)	−0.71(5)	0.989(4)	−0.77(4)
	4	0.962(5)	−0.74(6)	0.991(4)	−0.75(4)
	5	0.909(19)	−0.27(12)	0.961(5)	−0.32(5)
	6	0.937(16)	−0.15(11)	0.977(5)	−0.11(2)
	7	0.911(35)	−0.06(12)	0.976(5)	−0.11(2)
	8	0.919(36)	−0.01(11)	0.983(5)	−0.02(2)
	9	0.902(37)	−0.04(11)	0.981(5)	−0.03(3)
	10	0.899(51)	−0.04(12)	0.980(5)	−0.03(3)
3	1			0.911(7)	
	2			0.924(5)	−0.68(3)
	3			0.956(6)	−0.67(3)
	4			0.957(6)	−0.68(3)
	5			0.876(10)	−0.09(9)
	6			0.916(13)	−0.02(7)
	7			0.885(16)	0.11(6)
	8			0.892(19)	0.15(6)
	9			0.862(19)	0.15(6)
	10			0.863(18)	0.16(6)

tained on $L\times L$ lattices are shown in Table VII. (Improvement of these data would require very long runs. For $R=0.1$ ferromagnetic chains tend to form in the strong-coupling direction and the subsequent slow relaxation gives poorer statistics.) These results show that the eigenvalue estimates after one iteration are not only strongly dependent upon the number of couplings included in the analysis but also show very pronounced finite-size effects. Even with a 32×32 lattice the results are not good; after three iterations y_1^e is still about 10% too low and finite-size effects begin to appear. y_2^e is also not well behaved. y_1^0 is much better behaved and quickly approaches the exact value for $L=32$, although here also finite-size effects become noticeable quite quickly. Standard Monte Carlo calculations have shown that finite-size rounding in anisotropic simple-cubic models[9] can be reduced by changing sample shape, so we also simulated $L\times M$ lattices. The results shown in Table VIII do

TABLE VI. Correlations (see Fig. 1) as a function of iteration for $R=0.3$. Error estimates in the last digits are given in parentheses.

Iter.	$L=16$ $\langle S_1\rangle$	$L=16$ $\langle S_2\rangle$	$L=32$ $\langle S_1\rangle$	$L=32$ $\langle S_2\rangle$
0	0.676(3)	0.807(1)	0.678(1)	0.802(1)
1	0.675(4)	0.766(3)	0.662(2)	0.752(1)
2	0.691(8)	0.787(3)	0.675(4)	0.758(2)
			0.698(7)	0.788(4)

TABLE VII. Eigenvalue estimates for $L \times L$ lattices with $R = 0.1$. Data were obtained for 3×10^5 MCS after 10^4 MCS were discarded. Asterisks indicate complex estimates.

Iter.	No. of interact.	y_1^e 8	y_1^e 16	y_1^e 32	y_2^e 8	y_2^e 16	y_2^e 32	y_1^0 8	y_1^0 16	y_1^0 32
1	1	1.003	1.107	1.162				1.913	1.933	1.937
	2	0.740	0.827	0.876	−1.62	−1.10	−1.34	1.914	1.931	1.933
	3	0.769	0.856	0.905	−1.62	−1.09	−1.35	1.915	1.931	1.933
	4	0.776	0.865	0.909	−1.58	−1.14	−1.41	1.921	1.936	1.937
	5	*	*	0.707	*	*	0.18			
	6	*	*	0.648	*	*	0.284			
	7	*	*	0.629	*	*	0.29			
	8	*	*	0.612	*	*	0.42			
	9	*	0.772	0.832	*	0.07	0.12			
	10	*	0.769	0.829	*	0.06	0.12			
2	1		0.866	0.975					1.859	1.879
	2		0.866	0.943		−1.53	−1.17		1.859	1.879
	3		0.885	0.960		−1.54	−1.16		1.861	1.879
	4		0.884	0.962		−1.76	−1.10		1.860	1.879
	5		0.703	0.861		−0.22	−0.13			
	6		0.752	0.899		0.22	−0.13			
	7		0.721	0.892		−0.10	−0.10			
	8		0.719	0.899		−0.08	−0.04			
	9		0.683	0.897		−0.06	−0.09			
	10		0.677	0.895		−0.05	−0.08			
3	1			0.858						1.855
	2			0.870			−1.32			1.855
	3			0.895			−1.30			1.856
	4			0.895			−1.35			1.855
	5			0.697			−0.12			
	6			0.750			−0.14			
	7			0.723			−0.06			
	8			0.723			−0.05			
	9			0.655			0.05			
	10			0.653			0.05			

indeed show a substantial improvement in the convergence when rectangular lattices, with increased length in the strong-coupling direction, are used. In contrast, if the lattice is elongated in the weak-coupling direction the finite-size effects are enhanced. For the 16×64 lattice y_1^e and y_1^0 approach the exact result fairly smoothly. In addition, the results for y_2^e remain close to zero although noticeable statistical uncertainty remains. As shown in Table IX, however, the correlation functions remain anisotropic with increasing iteration.

V. ANISOTROPIC TRANSFORMATIONS AND THE FIXED LINE

Since the concept of a fixed line is normally associated with nonuniversal behavior of the sort seen in the Baxter model,[10] it is important to demonstrate the mechanism by which universality is maintained in the present case. The essential feature is the existence of rescaling transformations, which enable the critical behavior of any anisotropic Ising model to be described by any point along the fixed line.

Let us consider an isotropic model at or near criticality, with equal correlation lengths in the x and y directions. Now we make a 1×2 block-spin transformation in which the two sites in each block are one lattice constant apart in the y direction. The correlation length of the transformed system will be unchanged in the x direction, but one-half as large in the y direction. If we now proceed with 2×2 (isotropic) block-spin transformations as described above, our original isotropic system will be described by an anisotropic fixed point.

Naturally, we can also take an anisotropic system and

TABLE VIII. Eigenvalue estimates for $L \times M$ lattices with $R = 0.1$. Data were obtained for 3×10^5 MCS after 10^4 MCS were discarded. Asterisks indicate complex estimates.

Iter.	No. of interact.	y_1^e 16×8	8×16	16×32	16×64	y_1^0 16×8	8×16	16×32	16×64
1	1	1.044	1.132	1.179	1.206	1.911	1.933	1.937	1.937
	2	0.804	0.822	0.873	0.920	1.912	1.930	1.935	1.933
	3	0.841	0.849	0.902	0.949	1.915	1.931	1.936	1.934
	4	0.847	0.850	0.904	0.950	1.921	1.934	1.939	1.937
	5	*	*	0.655	0.831				
	6	*	*	0.557	0.775				
	7	*	*	*	0.765				
	8	*	*	*	0.749				
	9	*	*	0.790	0.856				
	10	*	*	0.785	0.844				
2	1			1.017	1.118			1.879	1.882
	2			0.973	1.048			1.879	1.882
	3			0.990	1.063			1.879	1.882
	4			0.991	1.064			1.879	1.883
	5			0.819	0.998				
	6			0.861	0.987				
	7			0.817	0.978				
	8			0.787	0.963				
	9			0.834	0.977				
	10			0.819	0.970				

TABLE IX. Correlations as a function of iteration for $L \times L$ lattices with $R = 0.1$.

Iter.	$\langle S_1 \rangle$ $L=8$	16	32	$\langle S_2 \rangle$ $L=8$	16	32
0	0.5616	0.6237	0.6401	0.9040	0.8877	0.8834
1	0.4318	0.5389	0.5911	0.7551	0.7696	0.7720
2		0.4325	0.5494		0.7409	0.7541
3			0.4611			0.7534

TABLE X. MC simulation data for MCRG calculations for the $d = 2$ Ising model with $R = 3.0$. Data for correlation functions taken every 20 MCS/site.

Lattice size (L)	16×32
10^3 MCS/site used	450
10^3 MCS/site discarded	37
Fraction of successful MCS	0.158
Approximate correlation time in units of MCS/site	72

TABLE XI. Correlations (see Fig. 1) as a function of iteration for $R = 3.0$ and a 16×32 lattice. Note that the 0th iteration refers to the system resulting from a preliminary (1×2) rescaling transformation (which yields a 16×16 lattice).

Iter.	$\langle S_1 \rangle$	$\langle S_2 \rangle$
0	0.669(2)	0.705(2)
1	0.714(3)	0.719(3)
2	0.767(5)	0.767(5)

TABLE XII. Estimates for the three largest thermal eigenvalues for $R=3.0$ with a 16×32 lattice. Note that preliminary (1×2) rescaling transformation was performed prior to the RG analysis.

Iter.	No. of interact.	y_1^e	y_2^e	y_3^e
1	1	0.932(5)		
	2	0.895(2)	−0.48(4)	
	3	0.947(2)	−0.50(4)	−2.05(10)
	4	0.951(3)	−0.52(4)	−2.15(15)
	5	0.957(5)	−0.35(7)	−1.55(16)
	6	0.958(6)	−0.08(2)	−1.35(10)
	7	0.959(9)	−0.07(5)	−0.92(13)
	8	0.960(10)	0.00(3)	−0.91(8)
	9	0.961(10)	0.00(3)	−1.00(9)
	10	0.963(12)	0.01(2)	−1.01(10)
2	1	0.867(5)		
	2	0.924(4)	−0.49(4)	
	3	0.978(3)	−0.49(4)	−2.19(14)
	4	0.982(3)	−0.49(4)	−2.33(15)
	5	0.986(5)	−0.29(7)	−1.55(11)
	6	0.981(4)	−0.10(8)	−1.37(7)
	7	0.988(7)	−0.04(13)	−1.21(27)
	8	0.990(9)	0.08(13)	−1.11(19)
	9	0.986(9)	0.08(14)	−1.07(18)
	10	0.976(11)	0.08(10)	−1.02(13)

transform it into an isotropic one through a rescaling transformation. By carrying out such a calculation, the equality of the renormalized correlation functions in the x and y directions would directly demonstrate the correctness of this description.

The most convenient value of the anisotropy for the calculation outlined above is $R=3$, for which the ratio of the correlation functions[2,11] is known to be exactly 2. We have simulated such a model on a 16×32 lattice and give information on the simulations in Table X. A 1×2 block-spin transformation was carried out before proceeding with the RG analysis, to reduce the system size to 16×16. Although the short-range behavior was still anisotropic after the 1×2 transformation as shown by the data in Table XI, isotropy was rapidly restored by the RG transformation. By the second iteration, the correlations in the x and y directions are equal to well within the accuracy of the calculation. The thermal equivalences, shown in Table XII, also show excellent convergence to the correct values when the 1×2 rescaling transformation is carried out first.

VI. CONCLUSIONS

By straightforward MCRG analysis of isotropic and anisotropic Ising models, we have demonstrated the correctness of the conjecture that spatial anisotropy is a marginal operator and that the critical behavior of the general anisotropic model is governed by a line of fixed points. We have demonstrated that the universality of the Ising critical exponents is guaranteed by the possibility of using preliminary rescaling transformations to describe the critical behavior of any anisotropic model by any point on the universal fixed line.

ACKNOWLEDGMENTS

This research was supported in part by the National Science Foundation Grant No. DMR-79-26178. We wish to thank Professor R. K. P. Zia, Professor D. Stauffer, and Dr. H. J. Herrmann for helpful comments and discussion. One of us (R.H.S.) wishes to thank the Program in Simulational Physics at the University of Georgia for its hospitality during a portion of the time this work was carried out.

*Present address: Physics Department, Carnegie-Mellon University, Pittsburgh, PA 15213.

[1]L. Onsager, Phys. Rev. **65**, 117 (1944); C. N. Yang, *ibid.* **85**, 808 (1952).

[2]A complete review of analytic results and references may be found in B. M. McCoy and T. T. Wu, *The Two-Dimensional Ising Model* (Harvard, Cambridge, 1973).

[3]See Th. Niemeijer and J. M. J. van Leeuwen, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6, and references therein.

[4]A. D. Bruce, J. Phys. C **7**, 2089 (1974); A. Aharony and M. E. Fisher, Phys. Rev. Lett. **45**, 679 (1980).

[5]R. H. Swendsen, Phys. Rev. B **20**, 2080 (1979); J. Appl. Phys. **53**, 1920 (1982).

[6]C. Rebbi and R. H. Swendsen, Phys. Rev. B **21**, 4094 (1980).

[7]R. H. Swendsen, in *Real-Space Renormalization,* edited by Th. W. Burkhardt and J. M. J. van Leeuwen (Springer, Berlin, 1982), p. 57.

[8]When the exponent y_i of an operator θ_i^* vanishes the operator

is called marginal. See, for example, F. J. Wegner, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, London, 1976), Vol. 6.

[9]T. Graim and D. P. Landau, Phys. Rev. B **24**, 5156 (1981).

[10]R. H. Swendsen and S. Krinsky, Phys. Rev. Lett. **43**, 177 (1979).

[11]R. K. P. Zia and J. E. Avron, Phys. Rev. B **25**, 2042 (1982).