Shiba-Rusinov theory of magnetic impurities in anisotropic superconductors: Eliashberg formalism

S. Yoksan* and A. D. S. Nagi
Guelph-Waterloo Program for Graduate Work in Physics, Department of Physics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

(Received 30 January 1984)

The effect of paramagnetic impurities in anisotropic superconductors is investigated with use of the Shiba-Rusinov theory for the impurities and the Eliashberg formalism for the host superconductor. The analytical expressions for the transition temperature T_c and the specific-heat jump ΔC are derived with the use of the square-well model for the electron-phonon interaction. Considered as a function of the spin-flip scattering rate α, the quantities T_c/T_{c0} and $\Delta C/\Delta C_0$ depend on the microscopic parameters λ, ω_D, and μ^* of the host material. The dependence on the material parameters becomes insignificant if the above properties are plotted versus α/α_{cr} or if $\Delta C/\Delta C_0$ versus T_c/T_{c0} is studied. (T_{c0} and ΔC_0 are values of T_c and Δ_C, respectively, in the absence of impurities, α_{cr} is the value of α for which T_c becomes zero, λ is electron-phonon—interaction parameter, ω_D is the Debye frequency, and μ^* is the Coulomb pseudopotential.)

I. INTRODUCTION

Recently, Okabe and Nagi[1] have studied the effect of paramagnetic impurities on the anisotropic superconductors by treating the impurities within the Shiba[2]-Rusinov[3] (SR) model. In the SR theory the electron-impurity scattering is calculated exactly assuming a classical spin and the well-known Abrikosov-Gor'kov[4] (AG) theory is a limiting case of this model. In Refs. 1–4 the host metal is described by using the Bardeen-Cooper-Schrieffer[5] (BCS) formalism of the theory of superconductivity and as such the properties do not depend on the microscopic parameters λ, ω_D, and μ^* of the host (λ is the electron-phonon—interaction parameter, ω_D is the Debye cutoff frequency, and μ^* is the Coulomb pseudopotential).

It is important to consider the problem of paramagnetic impurities in superconductors by treating the host metal using the Eliashberg[6] formalism (EF) of the theory of superconductivity.[7] The transition temperature T_c for the case of AG impurities in isotropic superconductors using EF and the square-well model of the electron-phonon interaction (or the $\lambda^{\theta\theta}$ model) was investigated by Allen.[8] Detailed numerical study using $\alpha^2F(\omega)$ of lead in the AG case was done by Schachinger, Daams, and Carbotte[9] [$\alpha^2F(\omega)$ is the electron-phonon spectral density]. The T_c and some tunneling properties for the case of SR impurities were considered by Schachinger.[10] The specific-heat jump of anisotropic superconductors with AG impurities using EF and the $\lambda^{\theta\theta}$ model has been given by Zarate and Carbotte.[11].

The purpose of the present paper is to generalize the results of Ref. 1 by using the Eliashberg formalism. We are interested in the analytical expressions for the transition temperature and the specific-heat jump and as such we use the $\lambda^{\theta\theta}$ model. In appropriate limits the results of Refs. 1 and 11 are retrieved from our calculations.

The plan of the paper is as follows. In Sec. II we outline our general formalism. The transition temperature T_c and the specific-heat jump ΔC are discussed in Secs. III and IV, respectively. In Sec. V we give conclusions.

II. FORMALISM

Assuming a low concentration of magnetic impurities in a strong-coupling superconductor, the single-particle Green's function for the conduction electrons, averaged over the positions and the spin directions of the impurities is given by

$$G_{\vec{k}}(n)=[i\tilde{\omega}_{\vec{k}}(n)\rho_3-\epsilon_{\vec{k}}-\tilde{\Delta}_{\vec{k}}(n)\rho_2\sigma_2]^{-1}\,, \tag{2.1}$$

where

$$\tilde{\omega}_{\vec{k}}(n)=\omega_n+\pi T\left\langle \sum_m \lambda_{\vec{k}\vec{k}'}(n-m)\frac{U_{\vec{k}'}(m)}{[U^2_{\vec{k}'}(m)+1]^{1/2}}\right\rangle' +\Gamma_1\left\langle\frac{U_{\vec{k}'}(n)[U^2_{\vec{k}'}(n)+1]^{1/2}}{U^2_{\vec{k}'}(n)+\epsilon_0^2}\right\rangle'\,, \tag{2.2}$$

$$\tilde{\Delta}_{\vec{k}}(n)=\pi T\left\langle \sum_m [\lambda_{\vec{k}\vec{k}'}(n-m)-\mu^*]\frac{1}{[U^2_{\vec{k}'}(m)+1]^{1/2}}\right\rangle' +\Gamma_2\left\langle\frac{[U^2_{\vec{k}'}(n)+1]^{1/2}}{U^2_{\vec{k}'}(n)+\epsilon_0^2}\right\rangle'\,. \tag{2.3}$$

In above equations $\epsilon_{\vec{k}}$ is the single-particle energy, σ_i and ρ_i ($i=1,2,3$), respectively, are Pauli matrices operating on the ordinary spin states and the electron-hole spin states, ω_n is Matsubara frequency [$\omega_n=\pi T(2n+1)$, where T is temperature and n is an integer], $U_{\vec{k}}(n)=\tilde{\omega}_{\vec{k}}(n)/\tilde{\Delta}_{\vec{k}}(n)$, ϵ_0 is the normalized position of a bound state within the BCS gap, and

$$2\Gamma_1=1/\tau_1+1/\tau_2\,, \tag{2.4a}$$

$$2\Gamma_2=1/\tau_1-1/\tau_2\,, \tag{2.4b}$$

where $1/\tau_2$ ($1/\tau_1$) is the spin-flip (non-spin-flip) scattering rate from the magnetic impurities. Further $\langle\ \rangle'$ indicates the Fermi-surface averaging over the electron states $\vec{k}'$, the parameter μ^* is the Coulomb pseudopotential, and $\lambda_{\vec{k}\vec{k}'}$ is the electron-phonon—interaction parameter.

We use the square-well model (or the $\lambda^{\theta\theta}$ model) of the electron-phonon interaction and take

$$\lambda_{\vec{k}\vec{k}'}(n-m)=\lambda_{\vec{k}\vec{k}'}\theta(\omega_D-|\omega_n|)\theta(\omega_D-|\omega_m|)\ , \quad (2.5)$$

where ω_D is the Debye cutoff frequency. The quantities Γ_1, Γ_2, and μ^* are taken to be isotropic. For anisotropy, we use the separable model of Ref. 12 and write

$$\lambda_{\vec{k}\vec{k}'}=\lambda(1+a_{\vec{k}})(1+a_{\vec{k}'})\ , \quad (2.6)$$

$$\Delta_{\vec{k}}=\phi_0+a_{\vec{k}}\phi_1\ , \quad (2.7)$$

where $a_{\vec{k}}$ is the anisotropy parameter.

Using the above models, Eqs. (2.2) and (2.3) give

$$\frac{\omega_n}{U_{\vec{k}}(n)}=\frac{\phi_{0\lambda}+a_{\vec{k}}\phi_{1\lambda}+\Gamma_{2\lambda}\phi_n}{1+\Gamma_{1\lambda}C_n+a_{\vec{k}}\lambda/(1+\lambda)}\ , \quad (2.8)$$

where

$$\phi_{0\lambda}=\frac{\phi_0}{1+\lambda}=\frac{2\pi T}{1+\lambda}\sum_{m\geq 0}\left[\lambda\left\langle\frac{(1+a_{\vec{k}'})}{[U^2_{\vec{k}'}(m)+1]^{1/2}}\right\rangle'-\mu^*\left\langle\frac{1}{[U^2_{\vec{k}'}(m)+1]^{1/2}}\right\rangle'\right]\ , \quad (2.9)$$

$$\phi_{1\lambda}=\frac{\phi_1}{1+\lambda}=\frac{2\lambda\pi T}{1+\lambda}\sum_{m\geq 0}\left\langle\frac{(1+a_{\vec{k}'})}{[U^2_{\vec{k}'}(m)+1]^{1/2}}\right\rangle'\ , \quad (2.10)$$

$$\phi_n=\left\langle\frac{[U^2_{\vec{k}'}(n)+1]^{1/2}}{U^2_{\vec{k}'}(n)+\epsilon_0^2}\right\rangle'\ , \quad (2.11)$$

$$C_n\omega_n=\left\langle\frac{U_{\vec{k}'}(n)}{[U^2_{\vec{k}'}(n)+1]^{1/2}}\right\rangle'\ , \quad (2.12)$$

$$\Gamma_{1\lambda}=\frac{\Gamma_1}{1+\lambda}\ , \quad (2.13)$$

$$\Gamma_{2\lambda}=\frac{\Gamma_2}{1+\lambda}\ . \quad (2.14)$$

From now on various sums go up to $N=(\omega_D/2\pi T)-\frac{1}{2}$.

For temperature near the superconducting transition temperature T_c, the superconducting order parameter becomes very small and $U_{\vec{k}}(n)\gg 1$. Furthermore, the anisotropy parameter is assumed to be small. Thus for T near T_c, Eq. (2.8) can be rewritten as

$$\omega_n/U_{\vec{k}}(n)=A_{0n}+a_{\vec{k}}A_{1n}+a^2_{\vec{k}}A_{2n}\ , \quad (2.15)$$

where after lengthy algebra, we find

$$A_{0n}=A_{0n}|_{T=T_c}+\frac{2\epsilon_0^2-1}{2}\frac{\phi^3_{0\lambda}\alpha_\lambda\omega_n}{(\omega_n+\alpha_\lambda)^4}+\frac{2\epsilon_0^2-1}{2}\phi^3_{0\lambda}a^2H_{2n}\ , \quad (2.16)$$

$$A_{0n}|_{T=T_c}=\frac{\phi_{0\lambda}\omega_n}{\omega_n+\alpha_\lambda}-\frac{\lambda}{1+\lambda}\frac{a^2\phi_{0\lambda}\Gamma_{2\lambda}\omega_n^2\tilde{h}(\omega_n,\alpha_\lambda)}{(\omega_n+\alpha_\lambda)^2(\omega_n+\beta_\lambda)^2}\ , \quad (2.17)$$

$$A_{1n}=\frac{\phi_{0\lambda}\omega_n\tilde{h}(\omega_n,\alpha_\lambda)}{(\omega_n+\alpha_\lambda)(\omega_n+\beta_\lambda)}+\frac{\phi^3_{0\lambda}(2\epsilon_0^2-1)}{2(\omega_n+\alpha_\lambda)^2(\omega_n+\beta_\lambda)}\left[\frac{\omega_n\beta_\lambda\tilde{h}(\omega_n,\alpha_\lambda)}{(\omega_n+\beta_\lambda)(\omega_n+\alpha_\lambda)}-\frac{\lambda}{1+\lambda}\frac{\alpha_\lambda\omega_n^2}{(\omega_n+\alpha_\lambda)^2}\right]\ , \quad (2.18)$$

$$A_{2n}=-\phi_{0\lambda}\frac{\lambda}{1+\lambda}\frac{1}{(\omega_n+\alpha_\lambda)(\omega_n+\beta_\lambda)^2}\left[\omega_n^2\tilde{h}(\omega_n,\alpha_\lambda)-\frac{2\epsilon_0^2-1}{2}\frac{\lambda}{1+\lambda}\phi^2_{0\lambda}\frac{\alpha_\lambda\omega_n^3}{(\omega_n+\alpha_\lambda)^3}+\phi^2_{0\lambda}(2\epsilon_0^2-1)\frac{\omega_n^2\tilde{h}(\omega_n,\alpha_\lambda)\beta_\lambda}{(\omega_n+\alpha_\lambda)^2(\omega_n+\beta_\lambda)}\right]\ , \quad (2.19)$$

$$H_{2n}=\frac{1}{(\omega_n+\beta_\lambda)(\omega_n+\alpha_\lambda)^4}\left[(3\alpha_\lambda-2\beta_\lambda)\frac{\omega_n\tilde{h}^2(\omega_n,\alpha_\lambda)}{(\omega_n+\beta_\lambda)}+\frac{\lambda}{1+\lambda}\left[\beta_\lambda-3\alpha_\lambda-\frac{\beta_\lambda\Gamma_{2\lambda}}{\omega_n+\beta_\lambda}\right]\frac{\omega_n^2\tilde{h}(\omega_n,\alpha_\lambda)}{(\omega_n+\alpha_\lambda)}-2\frac{\lambda}{1+\lambda}\frac{\omega_n^2\Gamma_{2\lambda}\beta_\lambda\tilde{h}(\omega_n,\alpha_\lambda)}{(\omega_n+\beta_\lambda)^2}+\left[\frac{\lambda}{1+\lambda}\right]^2\frac{\omega_n^3\Gamma_{2\lambda}\alpha_\lambda}{(\omega_n+\alpha_\lambda)(\omega_n+\beta_\lambda)}\right]\ , \quad (2.20)$$

where we have introduced

$$a^2=\langle a^2_{\vec{k}}\rangle\ , \quad (2.21)$$

$$\alpha_\lambda = \Gamma_{1\lambda} - \Gamma_{2\lambda} , \tag{2.22}$$

$$\beta_\lambda = \Gamma_{1\lambda} = \tfrac{1}{2}\alpha_\lambda(1+\delta), \quad \delta = \tau_2/\tau_1 , \tag{2.23}$$

$$\tilde{h}(\omega_n,\alpha_\lambda) = \left[h\omega_n + \frac{\lambda}{\lambda-\mu^*}\alpha_\lambda \right] , \tag{2.24}$$

$$h = \frac{\lambda(1+\mu^*)}{(1+\lambda)(\lambda-\mu^*)} . \tag{2.25}$$

Now we proceed to Eq. (2.9) for $\phi_{0\lambda}$. Expanding the square root in the denominator for small $1/U_{\vec{k}'}(m)$, then substituting Eqs. (2.15)–(2.20) and evaluating the Fermi-surface averages, we obtain

$$\frac{1}{2\pi T}\left[\frac{1+\lambda}{\lambda-\mu^*}\right] = \sum_{n\geq 0}\left[\frac{1}{\omega_n+\alpha_\lambda} + a^2\frac{\tilde{h}^2(\omega_n,\alpha_\lambda)}{(\omega_n+\alpha_\lambda)^2(\omega_n+\beta_\lambda)}\right]$$
$$-\tfrac{1}{2}\phi_{0\lambda}^2 \sum_{n\geq 0}\frac{1}{(\omega_n+\alpha_\lambda)^4}\left[\omega_n + 2(1-\epsilon_0^2)\alpha_\lambda + 2a^2\left[[3\omega_n + 3\alpha_\lambda(1-\epsilon_0^2)+\beta_\lambda(1+\epsilon_0^2)]\frac{\tilde{h}^2(\omega_n,\alpha_\lambda)}{(\omega_n+\beta_\lambda)^2}\right.\right.$$
$$\left.\left. + \frac{2\lambda}{1+\lambda}(2\epsilon_0^2-1)\alpha_\lambda\frac{\omega_n\tilde{h}(\omega_n,\alpha_\lambda)}{(\omega_n+\alpha_\lambda)(\omega_n+\beta_\lambda)}\right]\right] . \tag{2.26}$$

III. TRANSITION TEMPERATURE

At the transition temperature, the order parameter becomes zero. Setting $\phi_{0\lambda}=0$ in Eq. (2.26), and following a standard procedure, we obtain

$$(1+h^2a^2)\ln\left[\frac{T_c}{T_{c0}}\right] = [1+a^2h_1(2h-h_1)]\left[\psi\left[\frac{1}{2}\right] - \psi\left[\frac{1}{2}+\frac{\alpha_\lambda}{2\pi T_c}\right]\right]$$
$$+a^2(h-h_1)^2\left[\psi\left[\frac{1}{2}\right] - \psi\left[\frac{1}{2}+\frac{\beta_\lambda}{2\pi T_c}\right]\right] + a^2h_1^2\left[\frac{\beta_\lambda}{2\pi T_c} - \frac{\alpha_\lambda}{2\pi T_c}\right]\psi^{(1)}\left[\frac{1}{2}+\frac{\alpha_\lambda}{2\pi T_c}\right] , \tag{3.1}$$

where

$$h_1 = \frac{\lambda}{1+\lambda}\frac{\alpha_\lambda}{\beta_\lambda-\alpha_\lambda} , \tag{3.2}$$

$\psi^{(n)}(z)$ are polygamma functions,[13] and T_{c0} is the transition temperature of a pure anisotropic superconductor given by

$$\frac{1+\lambda}{\lambda-\mu^*} = (1+a^2h^2)\ln(2e^C\omega_D/\pi T_{c0}) , \tag{3.3}$$

with C as the Euler's constant. Equation (3.1) is our general T_c equation and agrees with Eq. (49) of Ref. 11. Note that the difference between the AG- and SR-approximation results is only through the definitions of Γ_1 and Γ_2. Using the values of these quantities given in Table I of Ref. 1, we obtain T_c in the AG and SR models. Note also that Eq. (3.1) of Ref. 1 is retrieved from our equations by taking $h_1 \to 0$, $h \to 1$, and $\alpha_\lambda \to \alpha$.

In the limit of low impurity concentration $n_i \to 0$, Eq. (3.1) gives

$$\frac{T_c}{T_{c0}} = 1 - \frac{\pi P}{4T_{c0}(1+a^2h^2)} , \tag{3.4}$$

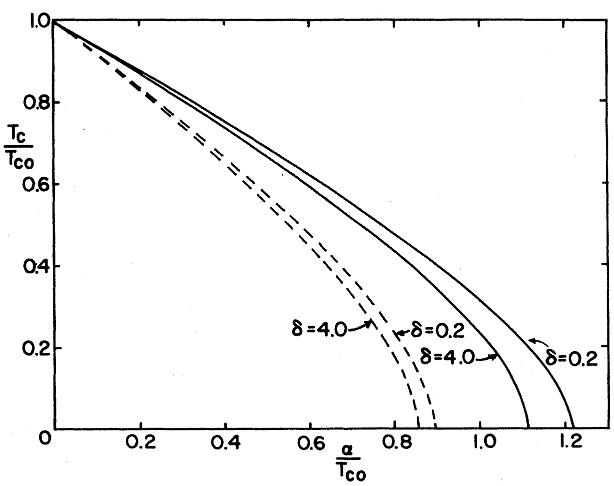

FIG. 1. Normalized transition temperature T_c/T_{c0} vs α/T_{c0} for $a^2=0.03$ and $\delta=0.2$ and 4.0. The quantity α is proportional to the impurity concentration, a^2 is the anisotropy parameter, and $\delta=\tau_2/\tau_1$ with $1/\tau_2$ $(1/\tau_1)$ as the spin-flip (non-spin-flip) scattering rate. The microscopic parameters used for solid curves are $\lambda=0.313$ [appropriate for Zn (Ref. 14)] and $\mu^*=0.1$. The dashed curves denote the BCS results.

with

$$P=\alpha_\lambda+a^2h^2\beta_\lambda-2a^2h^2\alpha_\lambda\frac{\lambda-\mu^*}{1+\mu^*}\ . \tag{3.5}$$

The critical value of α which makes $T_c=0$ is obtained from Eq. (3.1),

$$\alpha_{\rm cr}=(1+\lambda)\frac{\pi T_{c0}}{2e^C}\left[\frac{2}{1+\delta}\right]^{a^2(h-h_1)^2/(1+a^2h^2)}$$

$$\times\exp\left[\frac{a^2h_1^2(\delta-1)}{2(1+a^2h^2)}\right]. \tag{3.6}$$

One should note the dependence of Eqs. (3.4) and (3.6) on the microscopic parameters by comparing these with Eqs. (3.2) and (3.3) of Ref. 1.

When $\alpha_\lambda=\beta_\lambda$ (i.e., $\delta=1.0$), Eqs. (3.1) and (3.6) become, respectively,

$$\ln\left[\frac{T_c}{T_{c0}}\right]=\left[\psi\left[\frac{1}{2}\right]-\psi\left[\frac{1}{2}+\frac{\alpha_\lambda}{2\pi T_c}\right]\right]$$

$$+2h\frac{\lambda}{1+\lambda}\frac{\alpha_\lambda}{2\pi T_c}\frac{a^2}{1+a^2h^2}\psi^{(1)}\left[\frac{1}{2}+\frac{\alpha_\lambda}{2\pi T_c}\right], \tag{3.7}$$

$$\alpha_{\rm cr}=(1+\lambda)\left[\frac{\pi T_{c0}}{2e^C}\right]\exp\left[\frac{2\lambda}{1+\lambda}\frac{a^2h}{1+a^2h^2}\right]. \tag{3.8}$$

The dependence of T_c/T_{c0} on the impurity concentration for the case of $\lambda=0.313$ [appropriate for Zn (Ref. 14)] $\mu^*=0.1$, and for the BCS case is shown in Fig. 1. We have taken $a^2=0.03$ and $\delta=0.2$ and 4.0. One notes a dependence on the microscopic parameters in the figure. However, this dependence becomes very small if we plot T_c/T_{c0} versus the normalized impurity concentration $\alpha/\alpha_{\rm cr}$.

IV. SPECIFIC-HEAT JUMP

In this section we discuss the specific-heat jump at T_c. Because the $\lambda^{\theta\theta}$ model is essentially a weak-coupling model, with the electron-phonon—interaction parameter regarded as a constant, we calculate[15] ΔC by following the same procedure as in Ref. 1. First we write Eq. (2.26) as

$$\ln\left[\frac{T_{c0}}{T}\right]=B_0(n_i,T)+\frac{1}{2}\left[\frac{\phi_0}{2\pi T}\right]^2\frac{B_1(n_i,T)}{(1+\lambda)^2}+\cdots\ , \tag{4.1}$$

where

$$B_0(n_i,T)=\frac{2\pi T}{1+a^2h^2}\sum_{n\geq 0}\left[\left[\frac{1}{\omega_n}-\frac{1}{\omega_n+\alpha_\lambda}\right]+a^2h^2\left[\frac{1}{\omega_n}-\frac{\{\omega_n+[(1+\lambda)/(1+\mu^*)]\alpha_\lambda\}^2}{(\omega_n+\alpha_\lambda)^2(\omega_n+\beta_\lambda)}\right]\right], \tag{4.2}$$

$$B_1(n_i,T)=\frac{1}{1+a^2h^2}[b_0(n_i,T)+a^2b_1(n_i,T)]\ , \tag{4.3}$$

$$b_0(n_i,T)=(2\pi T)^3\sum_{n\geq 0}\left[\frac{1}{(\omega_n+\alpha_\lambda)^3}-\frac{(2\epsilon_0^2-1)\alpha_\lambda}{(\omega_n+\alpha_\lambda)^4}\right], \tag{4.4}$$

$$b_1(n_i,T)=16\pi^3T^3\sum_{n\geq 0}\frac{1}{(\omega_n+\alpha_\lambda)^4}\left\{[3\omega_n+3\alpha_\lambda(1-\epsilon_0^2)+\beta_\lambda(1+\epsilon_0^2)]\frac{h^2\{\omega_n+[(1+\lambda)/(1+\mu^*)]\alpha_\lambda\}^2}{(\omega_n+\beta_\lambda)^2}\right.$$

$$\left.+\frac{2\lambda}{1+\lambda}(2\epsilon_0^2-1)\frac{\alpha_\lambda\omega_nh\{\omega_n+[(1+\lambda)/(1+\mu^*)]\alpha_\lambda\}}{(\omega_n+\alpha_\lambda)(\omega_n+\beta_\lambda)}\right]. \tag{4.5}$$

In BCS limit, Eqs. (4.2)—(4.5) reduce to Eqs. (2.16)—(2.19) of Ref. 1.

Now the specific-heat jump is given by

$$\Delta C=C_S-C_N=8\pi^2N(0)T_c(1+\lambda)\frac{1+a^2h^2}{B_1(n_i,T_c)}\left[1+T_c\frac{\partial}{\partial T}B_0(n_i,T)\bigg|_{T=T_c}\right]^2, \tag{4.6}$$

where C_S and C_N are the electronic specific heat in the superconducting and normal states, respectively. It is convenient to define

$$\frac{\Delta C}{\Delta C_0}=\frac{T_c}{T_{c0}}\frac{B_1(0,T_{c0})}{B_1(n_i,T_c)}\left[1+T_c\frac{\partial}{\partial T}B_0(n_i,T)\bigg|_{T=T_c}\right]^2, \tag{4.7}$$

where ΔC_0 is the value of ΔC for $n_i=0$ and is given by

$$\Delta C_0=\frac{8\pi^2N(0)T_{c0}(1+\lambda)(1+a^2h^2)^2}{8\lambda(3)(1+6a^2h^2)}\ , \tag{4.8}$$

with

$$\lambda(l)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^l}=\frac{1}{2^l(-1)^l(l-1)!}\psi^{(l-1)}(\tfrac{1}{2})\ . \tag{4.9}$$

First we calculate the initial depression of $\Delta C/\Delta C_0$ in the limit of $n_i\to 0$. Using Eqs. (4.7) and (4.2)—(4.5) we obtain

$$\frac{\Delta C}{\Delta C_0}=1-\frac{12\lambda(2)}{1+a^2h^2}\frac{P}{2\pi T_{c0}}+\frac{4\lambda(4)}{\lambda(3)(1+6a^2h^2)}\frac{Q}{2\pi T_{c0}}\ , \tag{4.10}$$

where

$$Q=\alpha_\lambda(1+\epsilon_0^2)+a^2h^2[\tfrac{9}{2}(\alpha_\lambda+\beta_\lambda)+\tfrac{1}{2}(2\epsilon_0^2-1)(3\alpha_\lambda-\beta_\lambda)]-4a^2h^2\alpha_\lambda(\epsilon_0^2+1)\frac{\lambda-\mu^*}{1+\mu^*}\ , \tag{4.11}$$

and P is defined in Eq. (3.5). One should note the dependence of Eq. (4.10) on the microscopic parameters.

It is also interesting to calculate the quantity

$$C^*=\left[\frac{\partial}{\partial n_i}\frac{\Delta C}{\Delta C_0}\Big/\frac{\partial}{\partial n_i}\frac{T_c}{T_{c0}}\right]\Bigg|_{n_i\to 0} \tag{4.12}$$

using Eqs. (3.4), (3.5), (4.10), and (4.11) in Eq. (4.12), thereby obtaining

$$C^*=3-\frac{1+a^2h^2}{1+6a^2h^2}\frac{\lambda(4)}{\lambda(2)\lambda(3)}\frac{\alpha(1+\epsilon_0^2)+a^2h^2[\tfrac{9}{2}(\alpha+\beta)+\tfrac{1}{2}(2\epsilon_0^2-1)(3\alpha-\beta)]-4a^2h^2\alpha(1+\epsilon_0^2)[(\lambda-\mu^*)/(1+\mu^*)]}{\alpha+a^2h^2\beta-2a^2h^2\alpha[(\lambda-\mu^*)/(1+\mu^*)]}\ . \tag{4.13}$$

Comparing Eq. (4.13) with Eq. (4.4) of Ref. 1, we note that the anisotropy parameter a^2 has been replaced by a^2h^2 and the terms proportional to $(\lambda-\mu^*)/(1+\mu^*)$ have been added in the numerator and the denominator. Taking $a^2=0.03$, $\epsilon_0^2=1.0$, and $0\le(\beta/\alpha)\le 5$, however, we find no significant difference between the BCS value of C^* and the value computed by taking $\lambda=0.313$ (appropriate for Zn) and $\mu^*=0.1$.

For the purpose of numerical evaluation of the specific-heat jump at different impurity concentrations, we use Eqs. (4.7) and (4.2)—(4.5) to write

$$\frac{\Delta C}{\Delta C_0}=8\lambda(3)\frac{(1+6a^2h^2)}{(1+a^2h^2)^2}\frac{T_c}{T_{c0}}\frac{[\underline{\bar{A}}(n_i,T_c)+a^2\underline{\bar{B}}(n_i,T_c)]^2}{[\underline{C}(n_i,T_c)+a^2\underline{D}(n_i,T_c)]}\ , \tag{4.14}$$

where

$$\underline{\bar{A}}(n_i,T_c)=1-\frac{\alpha_\lambda}{2\pi T_c}\psi^{(1)}\left[\frac{1}{2}+\frac{\alpha_\lambda}{2\pi T_c}\right]\ , \tag{4.15}$$

$$\begin{aligned}\underline{\bar{B}}(n_i,T_c)={}&h^2-(h-h_1)^2\frac{\beta_\lambda}{2\pi T_c}\psi^{(1)}\left[\frac{1}{2}+\frac{\beta_\lambda}{2\pi T_c}\right]-\left[h_1(2h-h_1)\frac{\alpha_\lambda}{2\pi T_c}+h_1^2\left[\frac{\alpha_\lambda}{2\pi T_c}-\frac{\beta_\lambda}{2\pi T_c}\right]\right]\psi^{(1)}\left[\frac{1}{2}+\frac{\alpha_\lambda}{2\pi T_c}\right]\\&+h_1^2\frac{\alpha_\lambda}{2\pi T_c}\left[\frac{\beta_\lambda}{2\pi T_c}-\frac{\alpha_\lambda}{2\pi T_c}\right]\psi^{(2)}\left[\frac{1}{2}+\frac{\alpha_\lambda}{2\pi T_c}\right]\ ,\end{aligned} \tag{4.16}$$

$$\underline{C}(n_i,T_c)=\sum_{n\ge 0}\frac{n+\tfrac{1}{2}+2(1-\epsilon_0^2)(\alpha_\lambda/2\pi T_c)}{[n+\tfrac{1}{2}+(\alpha_\lambda/2\pi T_c)]^4}\ , \tag{4.17}$$

$$\begin{aligned}\underline{D}(n_i,T_c)=2\sum_{n\ge 0}\frac{1}{\left[n+\frac{1}{2}+\frac{\alpha_\lambda}{2\pi T_c}\right]^4}&\left[\left[3(n+\tfrac{1}{2})+3\frac{\alpha_\lambda}{2\pi T_c}(1-\epsilon_0^2)+\frac{\beta_\lambda}{2\pi T_c}(1+\epsilon_0^2)\right]\frac{h^2\left[n+\frac{1}{2}+\frac{1+\lambda}{1+\mu^*}\frac{\alpha_\lambda}{2\pi T_c}\right]^2}{\left[n+\frac{1}{2}+\frac{\beta_\lambda}{2\pi T_c}\right]^2}\right.\\&\left.+\frac{2\lambda}{1+\lambda}(2\epsilon_0^2-1)\frac{\alpha_\lambda}{2\pi T_c}(n+\tfrac{1}{2})h\frac{\left[n+\frac{1}{2}+\frac{1+\lambda}{1+\mu^*}\frac{\alpha_\lambda}{2\pi T_c}\right]}{\left[n+\frac{1}{2}+\frac{\alpha_\lambda}{2\pi T_c}\right]\left[n+\frac{1}{2}+\frac{\beta_\lambda}{2\pi T_c}\right]}\right]\ .\end{aligned} \tag{4.18}$$

When $\alpha_\lambda=\beta_\lambda$ (i.e., $\delta=1.0$), Eqs. (4.15), (4.17), and (4.18) remain valid but Eq. (4.16) becomes

$$\underline{\bar{B}}(n_i,T_c)=h^2-\frac{\alpha_\lambda}{2\pi T_c}h\left[h-\frac{2\lambda}{1+\lambda}\right]\psi^{(1)}\left[\frac{1}{2}+\frac{\alpha_\lambda}{2\pi T_c}\right]-\frac{\lambda}{1+\lambda}\left[2h-\frac{\lambda}{1+\lambda}\right]\left[\frac{\alpha_\lambda}{2\pi T_c}\right]^2\psi^{(2)}\left[\frac{1}{2}+\frac{\alpha_\lambda}{2\pi T_c}\right]$$
$$-\frac{1}{2}\left[\frac{\lambda}{1+\lambda}\right]^2\left[\frac{\alpha_\lambda}{2\pi T_c}\right]^3\psi^{(3)}\left[\frac{1}{2}+\frac{\alpha_\lambda}{2\pi T_c}\right]. \qquad (4.19)$$

In the AG limit ($\epsilon_0=1.0$) the above equations agree with corresponding ones given in Ref. 11, and in the BCS limit the results of Ref. 1 are retrieved.

We show the dependence of $\Delta C/\Delta C_0$ versus α/T_{c0} for $\epsilon_0^2=0.0$ and 0.5, $a^2=0.03$, and $\delta=1.0$ in Fig. 2. The microscopic parameters used for the solid curves are $\lambda=0.313$, $\omega_D=26.3$ meV [appropriate for Zn (Ref. 14)], and $\mu^*=0.1$. The dashed curves denote the BCS results. Although one notes a dependence on the material parameters in the figure, this dependence becomes very small if we plot $\Delta C/\Delta C_0$ versus $\alpha/\alpha_{\mathrm{cr}}$. We have also found that the normalized plot of the specific-heat jump versus the transition temperature does not depend on the microscopic parameters significantly.

Now we give ΔC for the nonmagnetic impurities ($\alpha_\lambda=0$). In this limit, we obtain

$$\frac{\Delta C/T_c}{(\Delta C_0)_0/(T_{c0})_0}=\left[1+a^2h^2-a^2h^2\left[\frac{\beta_\lambda}{2\pi T_c}\right]\psi^{(1)}\left[\frac{1}{2}+\frac{\beta_\lambda}{2\pi T_c}\right]\right]^2$$
$$\times\left\{1-\frac{a^2h^2}{\frac{1}{4}\psi^{(2)}(\frac{1}{2})}\frac{2\pi T_c}{\beta_\lambda}\left[2\psi^{(1)}\left[\frac{1}{2}\right]-\psi^{(1)}\left[\frac{1}{2}+\frac{\beta_\lambda}{2\pi T_c}\right]\right.\right.$$
$$\left.\left.+\frac{2\pi T_c}{\beta_\lambda}\psi\left[\frac{1}{2}\right]-\frac{2\pi T_c}{\beta_\lambda}\psi\left[\frac{1}{2}+\frac{\beta_\lambda}{2\pi T_c}\right]\right]\right\}^{-1}, \qquad (4.20)$$

where $(\Delta C_0)_0$ and $(T_{c0})_0$ are the values of the corresponding quantities for $n_i=0$ and $a^2=0$. For $a^2\ll 1$, we have

$$\frac{\Delta C/T_c}{(\Delta C_0)_0/(T_{c0})_0}=1-2a^2h^2\left[\frac{\beta_\lambda}{2\pi T_c}\psi^{(1)}\left[\frac{1}{2}+\frac{\beta_\lambda}{2\pi T_c}\right]-1+[-\tfrac{1}{2}\psi^{(2)}(\tfrac{1}{2})]^{-1}\right.$$
$$\times\left\{\frac{2\pi T_c}{\beta_\lambda}\left[2\psi^{(1)}\left[\frac{1}{2}\right]-\psi^{(1)}\left[\frac{1}{2}+\frac{\beta_\lambda}{2\pi T_c}\right]\right]\right.$$
$$\left.\left.-\left[\frac{2\pi T_c}{\beta_\lambda}\right]^2\left[\psi\left[\frac{1}{2}+\frac{\beta_\lambda}{2\pi T_c}\right]-\psi\left[\frac{1}{2}\right]\right]\right\}\right] \qquad (4.21)$$

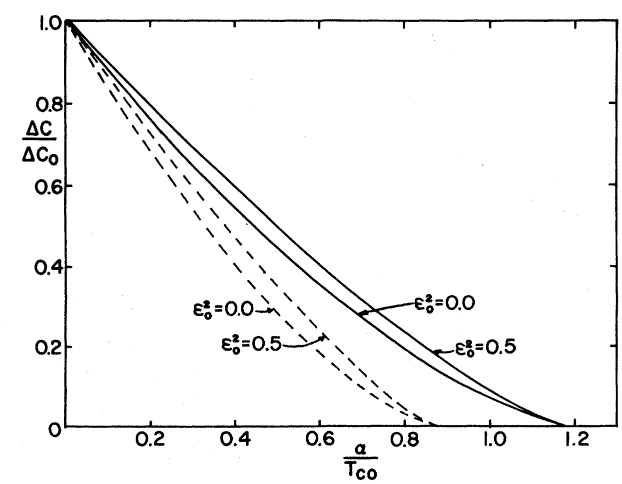

FIG. 2. Normalized specific-heat jump at T_c vs α/T_{c0} for $\epsilon_0^2=0.0$ and 0.5, $a^2=0.03$, and $\delta=1.0$. The microscopic parameters used for solid curves are $\lambda=0.313$, $\omega_D=26.3$ meV, and $\mu^*=0.1$. The dashed curves denote the BCS results.

Comparing Eq. (4.21) with Eq. (4.8) of Ref. 1 we note that the quantity a^2 has been replaced by a^2h^2 and β has been replaced by $\beta_\lambda=\beta/(1+\lambda)$. With a suitable scaling of a^2 and β, the dependence of $\Delta C/T_c$ on β_λ/T_{c0} is the same as shown in Fig. 5 of Ref. 1.

V. CONCLUSIONS

We have studied the problem of anisotropic superconductors containing paramagnetic impurities. The host superconductor is described using the Eliashberg formalism and the impurities are treated within the Shiba-Rusinov theory. The analytical expressions for the transition temperature T_c and the specific-heat jump ΔC are derived by using the $\lambda^{\theta\theta}$ model of the electron-phonon interaction and our results are the generalization of those given in Ref. 1.

T_c and ΔC are discussed in Secs. III and IV, respectively. The initial depression in T_c [Eq. (3.4)], the critical value of the spin-flip scattering rate α_{cr} [Eq. (3.6)], and the initial depression in ΔC [Eq. (4.10)] depend on the microscopic parameters λ, ω_D, and μ^* of the host material. The variation of T_c/T_{c0} and $\Delta C/\Delta C_0$ with α/T_{c0} are shown in Figs. 1 and 2, respectively, and these curves also depend on the microscopic parameters (we took $a^2=0.03$). We have found that the dependence on the material parameters becomes insignificant if the above properties are plotted against $\alpha/\alpha_{\rm cr}$ or if $\Delta C/\Delta C_0$ versus T_c/T_{c0} is studied.

ACKNOWLEDGMENTS

Financial support provided by the Natural Sciences and Engineering Research Council of Canada (NSERC) is gratefully acknowledged. One of use (S.Y.) would also like to thank NSERC for an international research award. We would also like to thank Professor Jules Carbotte for sending us information (Ref. 11) prior to publication.

*Permanent address: Department of Physics, Srinakharinwirot University, Prasarnmitr, Bangkok, Thailand.

[1]Y. Okabe and A. D. S. Nagi, Phys. Rev. B **28**, 1323 (1983).

[2]H. Shiba, Prog. Theor. Phys. **40**, 435 (1968).

[3]A. I. Rusinov, Zh. Eksp. Teor. Fiz. **56**, 2047 (1969) [Sov. Phys.—JETP **29**, 1101 (1969)].

[4]A. A. Abrikosov and L. P. Gor'kov, Zh. Eksp. Teor. Fiz. **39**, 1781 (1960) [Sov. Phys.—JETP **12**, 1243 (1969)].

[5]J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[6]G. M. Eliashberg, Zh. Eksp. Teor. Fiz. **38**, 966 (1960) [Sov. Phys.—JETP **11**, 696 (1960)].

[7]A review of the Eliashberg formalism of the theory of superconductivity is given by P. B. Allen and B. Mitrović, in *Solid State Physics*, edited by H. Ehrenreich, F. Seitz, and D. Turnbull (Academic, New York, 1982), Vol. 37, pp. 1—92.

[8]P. B. Allen, in *Lecture Notes for the XVIth Karpacz Winter School of Theoretical Physics, Karpacz, Poland, 1979,* edited by J. Lopuszanski (University of Wroctaw, Wroctaw, Poland, 1979).

[9]E. Schachinger, J. M. Daams, and J. P. Carbotte, Phys. Rev. B **22**, 3194 (1980).

[10]E. Schachinger, Z. Phys. B **47**, 217 (1982).

[11]H. G. Zarate and J. P. Carbotte, J. Low Temp. Phys. **55**, 67 (1984).

[12]D. Markowitz and L. P. Kadanoff, Phys. Rev. **131**, 563 (1963).

[13]P. J. Davis, in *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965), pp. 253—294.

[14]J. D. N. Cheeke and E. Ducla-Soares, J. Low Temp. Phys. **11**, 687 (1973).

[15]In Ref. 11, ΔC has been calculated from the free-energy difference obtained by using the Bardeen-Stephen formula [J. Bardeen and M. J. Stephen, Phys. Rev. **136**, A1485 (1964)]. A proper generalization of this formula to include the effect of SR impurities does not exist at this time. The correctness of the present approach for the $\lambda^{\theta\theta}$ model can be seen by noting that in the BCS limit our results for ΔC reduce to the ones given in Ref. 1 and also, in the AG limit ($\epsilon_0=1.0$), the results of Ref. 11 are retrieved.