Spin systems on hierarchical lattices. II. Some examples of soluble models

Miron Kaufman
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Robert B. Griffiths
Department of Physics, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213
(Received 29 February 1984)

Several examples are given of soluble models of phase-transition phenomena utilizing classical discrete spin systems with nearest-neighbor interaction on hierarchical lattices. These include critical exponents which depend continuously on a parameter, the Potts model on a lattice with two different coupling constants, surface tension, and excess free energy of a line of defects. In each case we point out similarities and differences with a corresponding Bravais-lattice model.

I. INTRODUCTION

In a previous paper[1] we introduced a general definition of a hierarchical lattice, discussed some of the topological properties of such lattices, and proved the existence of a thermodynamic limit for some classical spin models associated with a certain class of hierarchical lattices. The interest in statistical mechanics models on this sort of lattices was triggered by Berker and Ostlund's[2] observation that some renormalization-group methods, which are approximate when the models are defined as Bravais lattices, are exact if the models are considered on hierarchical lattices. In the present paper we show by means of examples how variations on this general theme can illustrate a variety of phase-transition phenomena: critical exponents which vary continuously with a parameter (Sec. II), Potts models on a lattice with two interaction constants (Sec. III), surface tension (Sec. IV), and defect free energies (Sec. V). Some of the results were announced earlier in Ref. 3.

In each case the phenomena of interest, or at least some analog of it, has been studied for the corresponding model on a two-dimensional Bravias lattice by means of complex and difficult mathematical techniques. By contrast, the exact solution on the corresponding hierarchical lattice is comparatively simple and straightforward.

We make no claim to be exhaustive. There are surely a large number of other phenomena related to phase transitions which can be studied on hierarchical lattices. Some of them have already been studied in Refs. 4–16 and references to even earlier work can be found in Ref. 1.

The question as to whether calculations on hierarchical lattices provide genuine insight into phase transitions for the corresponding models on "realistic" Bravais lattices is not easy to answer. It may be worth noting the rather close connection between hierarchical lattices and approximate real-space renormalization-group methods. Whereas the two are not identical, whatever limitations apply to one are likely to apply to the other.

II. CONTINUOUSLY VARYING CRITICAL EXPONENTS

Presently, the phenomenon of "universality" near critical points, in which the critical exponents and asymptotic equations of state are the same for a wide variety of systems, is explained in the renormalization-group context on the basis of a single fixed point, while continuously varying exponents, as in the Baxter model,[17] are ascribed to lines (or surfaces, etc.) of fixed points in a suitable space of parameters. While the existence of fixed points, lines of fixed points, etc. remains conjectural in the case of spin systems on Bravais lattices of dimension $d=2$ or more, they can often be found explicitly for the corresponding spin systems on hierarchical lattices. In particular, lines (surfaces, etc.) of fixed points in a suitable parameter space are very easy to obtain, as we shall now show by means of an example.

Figure 1 [the same as Fig. 2(c) of Ref. 1] shows the aggregation procedure for the lattice of interest: $B=4$ primitive (order zero) bonds represented by solid lines and a noniterated bond shown by a dashed line are combined to form a bond or order one. Next, B bonds of order one and a *single* noniterated bond are combined by the same procedure to form a bond of order two, Fig. 1(c), and continuing in a similar way produces bonds of arbitrarily

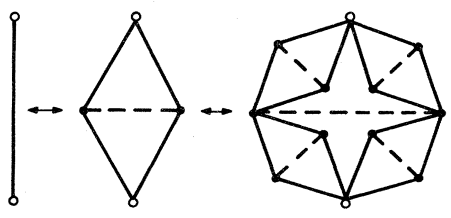

FIG. 1. Diamond hierarchical lattice with noniterated bonds (dashed lines).

high order. With site j we associate an Ising spin $\sigma_j = \pm 1$, and with a primitive bond connecting sites i and j we associate a dimensionless interaction ($-\mathcal{H}/k_BT$):

$$H^{(0)} = K_0 \sigma_i \sigma_j \ . \tag{2.1}$$

The corresponding dimensionless interaction for a noniterated bond is

$$\bar{H} = \bar{h}(\sigma_i - \sigma_j) \ , \tag{2.2}$$

i.e., a staggered magnetic field having opposite signs at the two ends of the bond. (The symmetry of the problem makes which end is positive and which negative unimportant).

The partition function Z_N for a bond of order N is obtained by summing the restricted partition function

$$Z_N(\sigma_1, \sigma_2) = \exp(K_N \sigma_1 \sigma_2 + C_N) \tag{2.3}$$

over the Ising spins σ_1 and σ_2 at two surface sites (top and bottom sites in Fig. 1) of this bond. As shown in Ref. 1 [see the discussion following (4.3)], the restricted partition function of a bond of order N may be expressed in terms of that for a bond of order $N-1$, leading to the renormalization mapping of the effective interaction:

$$e^{2K'} = [1 + \cosh(2\bar{h})]^{-1}[\cosh(4K) + \cosh(2\bar{h})] \ , \tag{2.4}$$

where $K' = K_N$ when $K = K_{N-1}$.

For each choice of $\bar{h}$, (2.4) admits a single (critical) fixed point

$$K' = K = K_c(\bar{h}) \tag{2.5}$$

in the interval $0 < K < \infty$, with thermal eigenvalue

$$\left[\frac{\partial K'}{\partial K}\right]_{K=K_c} = B^{\hat{y}} = \frac{2(1+e^{4K_c})}{e^{2K_c}+e^{4K_c}} \ . \tag{2.6}$$

The exponent $\hat{y}$ is related to the usual critical exponents α and y (thermal) by

$$\hat{y} = (2-\alpha)^{-1} = y/d \ .$$

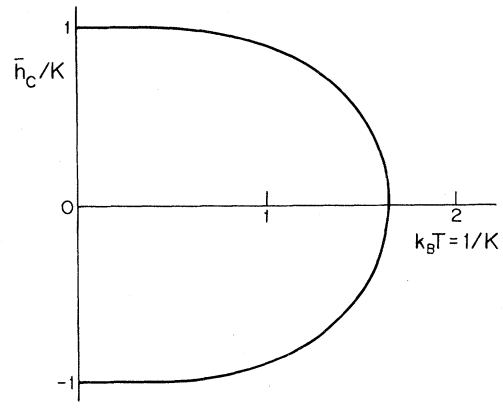

FIG. 2. The critical line of the Ising model in a staggered magnetic field on the diamond hierarchical lattice.

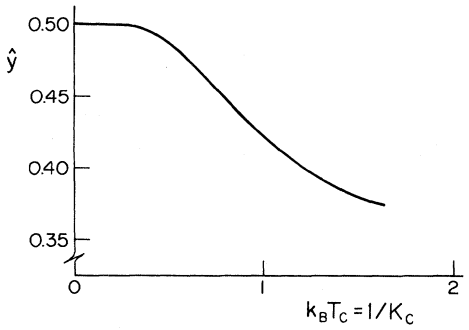

FIG. 3. Variation of the critical exponent $\hat{y} = (2-\alpha)^{-1}$ as a function of temperature along the critical line of Fig. 2 for the Ising model in a staggered field.

(As the dimensionality d of a hierarchical lattice is not well defined,[3] y is similarly ambiguous.)

Note that since K_c depends on $\bar{h}$, the same is true of $\hat{y}$ and α, i.e., one has a situation in which critical exponents depend continuously on a parameter. Figure 2 shows the inverse function of (2.5): $\bar{h}_c/K$ as a function of $1/K$, corresponding (in the usual terminology for Ising magnets) to the critical staggered magnetic field as a function of temperature. In Fig. 3, $\hat{y}$ is shown as a function of $1/K_c$. The curve in Fig. 2 is qualitatively the same as one would expect for an Ising ferromagnet on a square lattice in a staggered magnetic field (equal magnitude but opposite sign on nearest-neighbor sites), or, equivalently, the antiferromagnet in a uniform field. It is usually assumed that on a square lattice the critical singularity is a logarithm ($\alpha = 0$) independent of $\bar{h}$, corresponding to $\hat{y}$ having a constant value of $\frac{1}{2}$, and some indirect evidence in favor of this comes from Fisher's decorated antiferromagnet.[18].

That α varies continuously with $\bar{h}_c$ or K_c in the case of the Ising model on a hierarchical lattice comes as no surprise when one notes that the mapping (2.4) contains $\bar{h}$ as a simple parameter, and thus the critical fixed point and all its properties depend on $\bar{h}$. By introducing other noniterated interactions one can obtain surfaces or higher-dimensional manifolds of fixed points. Similarly, whenever a hierarchical lattice contains one or more noniterated bonds one would expect to find the critical exponents dependent on the corresponding interaction parameters. Indeed this phenomenon is already visible in the Kadanoff bond-shifting scheme,[19] where the critical exponents depend on a parameter p which corresponds, at least in the case in which p is temperature independent, to a noniterated interaction on a hierarchical lattice. Still another example of this phenomenon was discussed in Ref. 20. However, noniterated bonds are not always necessary to produce continuously varying exponents, as it is shown in Sec. V below.

III. POTTS MODEL ON A NONUNIFORM LATTICE

The term "nonuniform" was used in our previous paper[1] for a lattice in which different units are assembled by different rules of aggregation. An example is shown in Fig. 4(a), which is to be interpreted as follows. There are

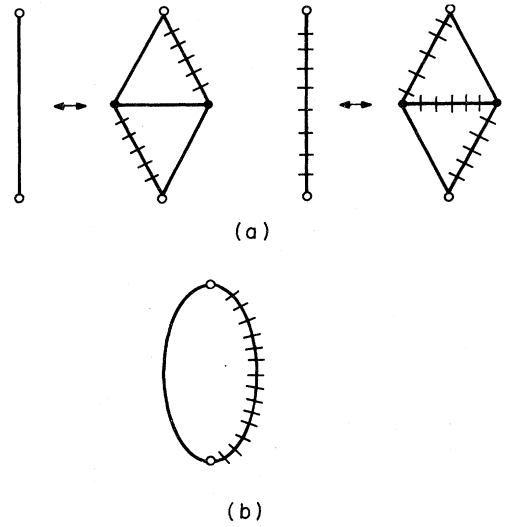

FIG. 4. Nonuniform hierarchical lattice.

two types of bond of order N: "plain" and "crossed." Each is composed of bonds of order $N-1$ in the manner indicated in the figure. If the surface vertices of a plain and a crossed bond of order N are identified, as in Fig. 4(b), the resulting graph is self-dual, with plain and crossed primitive bonds on the original lattice corresponding to their counterparts on the dual lattice. This lattice is also left invariant if the plain and crossed bonds are interchanged. In what follows we shall be interested in the free energy per bond of the lattice in Fig. 4(b) as N tends to infinity. In fact, one obtains precisely the same result from considering either a plain or a crossed bond of order N in the $N\to\infty$ limit.

With site j of the lattice we associate a Potts "spin" S_j which takes the values $1,2,\ldots q$, and with a primitive plain bond connecting sites i and j we associate a dimensionless interaction

$$H^{(0)}=2K_0\delta(S_i,S_j)\,, \tag{3.1}$$

where $\delta(\cdots,\cdots)$ is the Kronecker δ factor. The factor of 2 makes the $q=2$ case correspond with our previous notation, (2.1), for the Ising model. For a crossed primitive bond, K_0 in (3.1) is replaced by $\bar{K}_0$. For plain and crossed bonds of order N, the effective interactions are of the form (3.1) but with $\bar{K}_0$ replaced by K_N and $\bar{K}_N$, respectively. If we define

$$\omega=e^{2K},\quad \bar{\omega}=e^{2\bar{K}}\,, \tag{3.2}$$

then the two transformations analogous to (2.4) are

$$\omega'=\frac{\omega^3\bar{\omega}^2+(q-1)(\omega+2\omega\bar{\omega}+q-2)}{\omega^2+\bar{\omega}^2+2\omega^2\bar{\omega}+(q-2)(3\omega+2\bar{\omega}+q-3)} \tag{3.3}$$

and an equation for $\bar{\omega}'$ obtained by interchanging ω and $\bar{\omega}$ on the right-hand side of (3.3). That is, if ω and $\bar{\omega}$ correspond to K_{N-1} and $\bar{K}_{N-1}$ through (3.2), ω' and $\bar{\omega}'$ correspond to K_N and $\bar{K}_N$, respectively.

The recursion transformation (3.3) and its counterpart for $\bar{\omega}'$ commute with the duality transformation [Eq. (2.11) of Ref. 21] for the Potts model:

$$(\omega^*-1)(\omega-1)=q=(\bar{\omega}^*-1)(\bar{\omega}-1)\,, \tag{3.4}$$

where ω^* and $\bar{\omega}^*$ correspond to the interactions K^* and $\bar{K}^*$ on the dual lattice (which, as noted above, is the same as the original lattice). Consequently, the "duality line"

$$(\omega-1)(\bar{\omega}-1)=q\,, \tag{3.5}$$

is mapped onto itself by the recursion transformations, as is the "isotropic" line

$$\omega=\bar{\omega}\,. \tag{3.6}$$

The limits $\omega=1$ ($K=0$) and $\bar{\omega}=1$ ($\bar{K}=0$) are invariant lines, as are also the lines $\omega=\infty$ and $\bar{\omega}=\infty$.

The corresponding flows are indicated schematically in Fig. 5. On the curved duality line the flows lead to the isotropic critical point

$$\omega=\bar{\omega}=1+\sqrt{q} \tag{3.7}$$

and hence all these points (with the exception of the end points where ω or $\bar{\omega}$ is infinite) are in the same universality class as the isotropic point. An analogous result is known[22] to hold for the Ising model ($q=2$) on a square (or "rectangular") lattice in which the vertical bonds have the strength (K) differing from that of the horizontal bonds ($\bar{K}$): the asymptotic critical behavior is the same whatever the ratio of K and $\bar{K}$.

On a square lattice the Potts model has a discontinuous or first-order transition for $q>4$,[23] whereas on the lattice of Fig. 4 the transition remains continuous for all q. However, as $q\to\infty$, there is a tendency towards a first-order transition in that the thermal eigenvalue $\hat{y}$ in the counterpart of (2.6) ($B=5$ in the present case) tending towards 1.[24]

IV. SURFACE TENSION

On some hierarchical lattices it is possible to define a quantity which is analogous to the surface tension for the corresponding model on a Bravais lattice. We illustrate this for the case of the "diamond" hierarchical lattice corresponding to Fig. 1 when the dashed (noniterated) bonds are deleted, corresponding to $\bar{h}=0$ in (2.2). We let γ be the surface tension and

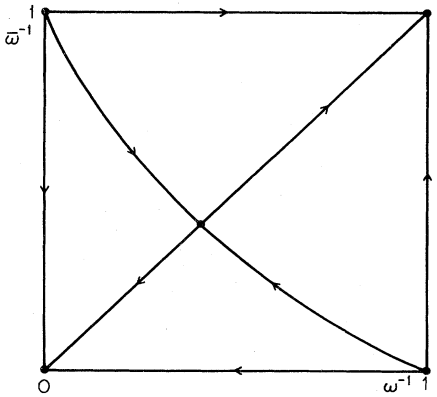

FIG. 5. Schematic renormalization-group flows for the Potts model on the lattice of Fig. 4.

$$\bar{\gamma}=\gamma/k_BT=\lim_{N\to\infty}\{L^{-N}\ln[Z_N(+,+)/Z_N(+,-)]\} \quad (4.1)$$

be its dimensionless counterpart, where L^N is the minimum number of primitive bonds which must be deleted from a bond of order N to disconnect its surface vertices. For the diamond hierarchical lattice, $L=2$. Note that $\bar{\gamma}$ is, equivalently, the difference in the dimensionless free energy between the two situations corresponding to periodic, $\sigma_1=\sigma_2$, and antiperiodic, $\sigma_1=-\sigma_2$, boundary conditions for the bond of order N (with σ_1 and σ_2 being the surface spins), divided by the "area" L^N of the interface that is produced by the antiperiodic boundary condition. It is thus analogous to one of the common definitions[25] of surface tension for an Ising model on a Bravais lattice.

From (4.1) and (2.3) it follows that

$$\bar{\gamma}(K)=\lim_{N\to\infty}(2^{1-N}K_N) \quad (4.2)$$

with $K_0=K$. The results of a numerical calculation of $\gamma=k_BT\bar{\gamma}$, with k_BT defined as $1/K$, are shown in Fig. 6. Near the critical point γ varies as

$$\gamma\sim(T_c-T)^{\mu}\,, \quad (4.3)$$

where $\mu=1.3383$ satisfies

$$\mu=(\ln L/\ln B)(2-\alpha)\,, \quad (4.4)$$

which is the analog of the Widom[26] relation

$$\mu=d^{-1}(d-1)(2-\alpha) \quad (4.5)$$

on a d-dimensional Bravais lattice, since L^N is the interface area for a hierarchical lattice of "volume" B^N. For $T>T_c$ (i.e., $K<K_c$), γ vanishes, as expected. In fact, (4.3) is not strictly a power law; the numerical study shows that the amplitude multiplying the power law on the right-hand side has a small variation which is periodic in $\ln(T_c-T)$.

When K is large the recursion relation (2.4) (with $\bar{h}=0$) yields

$$K'\simeq 2K-\tfrac{1}{2}\ln 4 \quad (4.6)$$

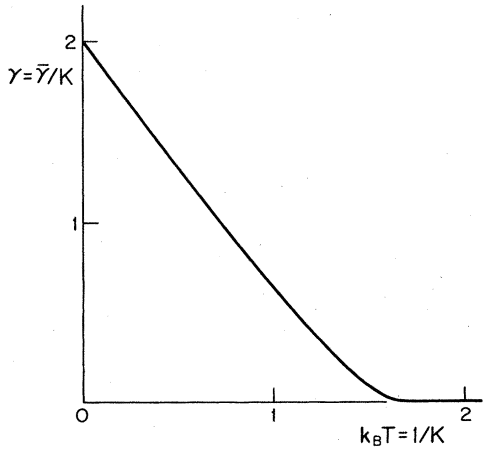

FIG. 6. Surface tension $\gamma=\bar{\gamma}/K$ dependence on temperature, $k_BT=1/K$, for the Ising model on the diamond hierarchical lattice.

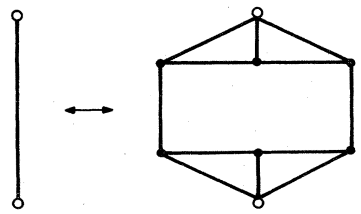

FIG. 7. Hierarchical lattice on which the Ising model exhibits zero surface entropy at zero temperature.

with corrections of order e^{-4K}, and thus, up to corrections of the same order,

$$\bar{\gamma}(K)\simeq 2K-\ln 4\,. \quad (4.7)$$

Consequently $d\gamma/dT$ approaches $-\ln 4$ as $T\to 0$, indicating a finite surface entropy at zero temperature. This entropy arises from a degeneracy of the ground state when $\sigma_1=+1$, $\sigma_2=-1$, due to the fact that there are a large number of ways to choose the L^N primitive bonds which have $\sigma_i=1$ at one end and $\sigma_j=-1$ at the other. In fact, if the degeneracy is D when $N=1$, a recursive argument shows that it is equal to

$$D^{(L^N-1)/(L-1)} \quad (4.8)$$

for order N, assuming $L>1$. Consequently, the entropy per unit area at $T=0$ is

$$(L-1)^{-1}\ln D$$

times k_B, in agreement with (4.7) in the case of the diamond hierarchical lattice, where $D=4$ and $L=2$.

Not every Ising model on a hierarchical lattice has a finite surface entropy at $T=0$. For example, $D=1$ and $L=2$ for the case shown in Fig. 7. In the case of an Ising model on a square lattice the zero-temperature surface entropy is zero for interfaces perpendicular to one of the edges of the square,[22] since the ground-state degeneracy with antiperiodic boundary conditions is on the order of the linear dimension. For an interface at 45° to an edge of the square, on the other hand, the surface entropy is finite at $T=0$.[27]

V. DEFECT FREE ENERGY

Line defects in Ising models have received a fair amount of attention. In particular, the excess free energy associated with such a defect can show interesting behavior near a bulk phase transition. The same type of thing is easy to study in hierarchical models. As an example, we consider the Ising model on a lattice, Fig. 8, which is a modification of that shown in Fig. 4. In particular, a plain bond of order N is made up entirely of $B=5$ plain bonds of order $N-1$, while a defect bond (indicated by a crossed line) of order N contains $\bar{B}=3$ defect bonds and $B-\bar{B}$ plain bonds of order $N-1$. For the primitive defect bond between sites i and j we assume a dimensionless interaction

$$H_d^{(0)}=K_{d0}\sigma_i\sigma_j\,, \quad (5.1)$$

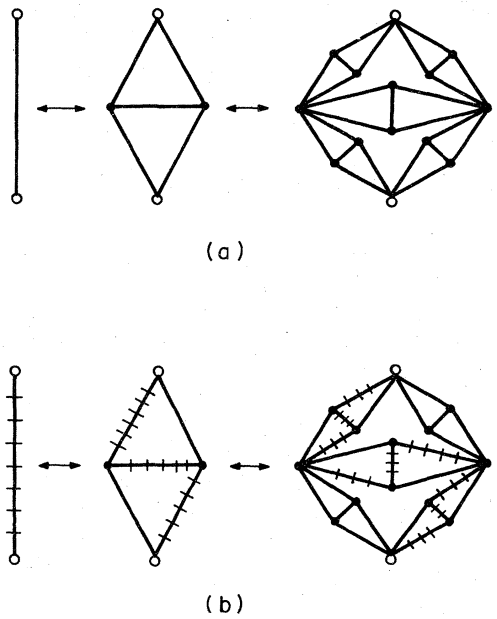

FIG. 8. Hierarchical lattice with (a) no defect bonds and (b) a line of defect bonds (crossed lines).

while for plain bonds K_0 replaces K_{d0}.

We write the restricted partition function for a defect bond of order N in the form

$$Z_{dN}(\sigma_1,\sigma_2)=\exp[K_{dN}(\sigma_1\sigma_2-1)+C_{dN}]\ , \tag{5.2}$$

while that for plain bonds is given by deleting the subscript d. This only differs from (2.3) in the convention used to define the constants C_{dN} and C_N. The form (5.2) is convenient because as long as K_{dN} is positive, the partition function is

$$Z_{dN}=\sum_{\sigma_1}\sum_{\sigma_2} Z_{dN}(\sigma_1,\sigma_2)\simeq \exp C_{dN}\ , \tag{5.3}$$

with an error of at most a factor of 4, and the analogous equation applies to Z_N if $K_N \geq 0$. In a defect bond of order N, the excess defect free energy Δf_N per primitive defect bond is given by the expression

$$\exp(\bar{B}^N \Delta f_N)=Z_{dN}/Z_N \simeq e^{\Delta C_N}\ , \tag{5.4}$$

where

$$\Delta C_N = C_{dN}-C_N\ . \tag{5.5}$$

The recursion transformation expressed in terms of the quantities

$$\omega=e^{2K},\quad \omega_d=e^{2K_d} \tag{5.6}$$

reads

$$\omega_d'=\omega_d(1+2\omega+\omega^2\omega_d^2)/(\omega^2+\omega_d^2+2\omega\omega_d^2) \tag{5.7}$$

for the defect bond; the same equation applies to a plain bond if d is deleted everywhere. That is to say, if ω and ω_d correspond to K_N and K_{dN}, then ω' and ω_d' correspond to K_{N+1} and K_{dN+1}, respectively. With the same conventions for primed and unprimed quantities, we find a recursion relation

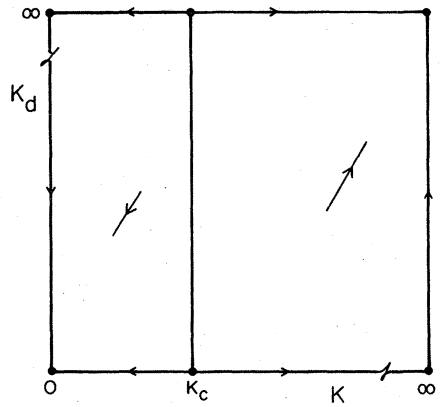

FIG. 9. Schematic renormalization-group flows for the Ising model on the lattice of Fig. 8(b). The line $K=K_c$ is a line of fixed points.

$$\Delta C'=\Delta\psi'+\bar{B}\,\Delta C\ , \tag{5.8}$$

where

$$\exp(\Delta\psi')=\frac{\omega^2}{\omega_d^2}\,\frac{1+2\omega+\omega^2\omega_d^2}{1+2\omega+\omega^4}\ . \tag{5.9}$$

Consequently the defect free energy is

$$\Delta f(K,K_d)=\lim_{N\to\infty}\Delta f_N=\sum_{N=0}^{\infty}\bar{B}^{-N}\Delta\psi_N \tag{5.10}$$

with

$$K_0=K,\quad K_{d0}=K_d,\quad \Delta\psi_0=K_d-K\ . \tag{5.11}$$

Whereas we assumed that K and K_d were non-negative in order to derive (5.10), this equation remains valid for $K_d<0$. Alternatively, one can use the symmetry

$$\Delta f(K,-K_d)=\Delta f(K,K_d) \tag{5.12}$$

to reexpress Δf in terms of positive K_d.

The flows corresponding to (5.7) and its counterpart with d deleted are indicated schematically in Fig. 9. At

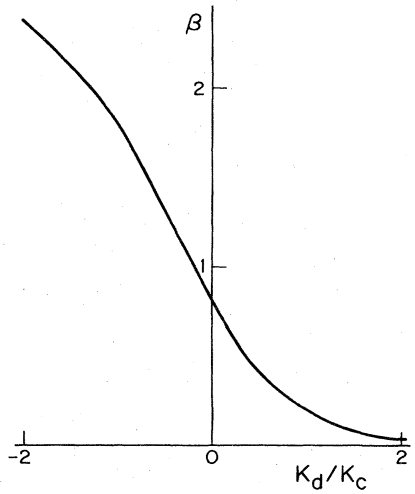

FIG. 10. Dependence of the defect exponent β on the ratio K_d/K, at the bulk critical temperature ($K=K_c$).

$K=0$ one finds an exact renormalization transformation for a one-dimensional Ising model with $b=\bar{B}=3$, and associated fixed points at $K_d=0$ and ∞. With $K_d=0$ (K arbitrary) the lattice forms two noninteracting pieces, analogous to what happens if a connected staircase of successive horizontal and vertical bonds are removed from a square lattice. On the invariant line $K_d=0$, there are fixed points at

$$K=0, \quad K=K_c=\tfrac{1}{2}\ln(1+\sqrt{2}), \quad K=\infty \ . \tag{5.13}$$

The last of these is a first-order phase transition in the defect free energy as K_d goes from negative to positive values through zero [the corresponding fixed-point eigenvalue for the flow (5.7) is $\bar{B}=3$], and thus $\partial\Delta f/\partial K_d$ is discontinuous at $K_d=0$ for all $K>K_c$.

The line $K=K_c$ is a line of fixed points of (5.7) and thus one is not surprised to find that the critical exponents of the defect free energy depend continuously on K_d. The dependence of the defect β on K_d/K_c is shown in Fig. 10. The square Ising model with a line of defects shows[28] an analogous dependence of the defect exponent β on K_d.

ACKNOWLEDGMENTS

We thank A. N. Berker for useful observations. Financial support of this research by the National Science Foundation, through Grant Nos. DMR-81-19295 to Massachusetts Institute of Technology and DMR-81-08310 to Carnegie-Mellon University is gratefully acknowledged. One of us (M.K.) is the recipient of a Bartrell Fellowship.

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