Glauber dynamics for one-dimensional spin models with random fields

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We present an exact solution of the long-time relaxational behavior of the magnetization in the Ising and XY chains in a quenched random field. The random field is assumed to be of infinite strength but present at only a fraction of the spin sites. We use Glauber dynamics for the Ising case, and a suitably generalized master equation for the XY case. In both models we find that for $T=0$, as time $t\to\infty$ the magnetization decays as $\exp(-Ct^{1/3})$, where C is a constant. At finite temperatures the ultimate asymptotic behavior is purely exponential.

I. INTRODUCTION

Recently, an exactly soluble one-dimensional model of Ising spins interacting with quenched random fields was introduced and solved in the static limit by Grinstein and Mukamel.[1] Subsequently, Pelcovits and Mukamel[2] studied the properties of the analogous XY model including its relaxational behavior governed by Langevin dynamics. In these models, the strength of the random fields is assumed to be infinite, but the fields are present at only a fraction p $(0\le p\le 1)$ of the sites of the chain. Because of the infinite strength of the fields, evaluation of a quenched thermodynamic quantity first involves evaluating the thermal average of the particular quantity in a *finite* chain of spins bounded by two random fields of arbitrary orientation (the directions of the fields are assumed to be isotropically distributed in space: equally up or down for the Ising case, and uniformly distributed over the unit circle in the XY case). Subsequently one averages the thermal average over the orientations of the two bounding fields and sums over all possible lengths of the chain weighted by the appropriate probability distribution [see, e.g., (2.10) below]. The relative simplicity of this calculational procedure allows one to solve for practically any static or time-dependent thermodynamic observable. This calculational power is not necessarily present in the more general one-dimensional Ising (Ref. 3) and XY (Ref. 4) models with random fields studied by other authors (where the field strength is not assumed to be infinite). The assumption of infinite field strength in the models of Refs. 1 and 2 restricts their analyses to the case of "strong pinning,"[2,5] but in a quenched system this is by no means a trivial limit.

In this paper, we calculate exactly the long-time relaxational behavior of the magnetization in the Ising and XY chains in a random field introduced in Refs. 1 and 2. We assume that the time evolution of the models is governed by a Glauber equation[6] for the Ising model and a suitably generalized master equation for the XY case. In both models we find that if the spins are fully aligned at time $t=0$, then at long times the magnetization $M(t)\sim\exp(-Ct^{1/3})$, where C is a time-dependent function of impurity dilution p and the bond strength J. At $T=0$ this behavior is true as $t\to\infty$; for $T\neq 0$ the ultimate asymptotic behavior is nonrandom in nature, i.e., it is purely exponential. Identical behavior was found in Ref. 2 for the XY model using Langevin dynamics at $T=0$, and qualitatively similar behavior was found in studies of the Ising chain with bond dilution[7] and models for particle diffusion in a one-dimensional medium with random traps.[8] As can be seen from the saddle-point analysis of $M(t)$ below [see (2.13) and (2.14)] this apparently universal behavior arises whenever a random system exhibits a particular distribution of relaxation times. Here we have chains of N spins bounded by random fields. For each chain, the relaxation time goes as N^2 and the probability of finding a chain of length N goes as $\exp N$ [see (2.12)]. In general, if the relaxation time goes as N^β and the probability of finding a chain of length N goes as $\exp N^\alpha$, then the saddle-point analysis indicates that if $\alpha/(\alpha+\beta)=\frac{1}{3}$, then $M(t)\sim\exp(-Ct^{1/3})$. A similar mechanism has been proposed by Cohen and Grest[9] to explain experimentally observed anomalous time decays in glasses.

In the next two sections of this paper we discuss the calculations leading to the above result for the Ising and XY models, respectively.

II. THE ISING MODEL

Our model for an Ising chain in a random field is defined by the Hamiltonian[1]

$$H=-J\sum_{\langle ij\rangle}\sigma_i\sigma_j-\sum_i h_i\sigma_i \ , \tag{2.1}$$

where J is the nonrandom positive bond strength, $\sigma_i = \pm 1$ and the sum is over nearest-neighbor sites. At each site $h_i = +\infty$, $-\infty$, or 0 with probabilities $p/2$, $p/2$, and $1-p$, respectively. Note that at low T an infinite field is equivalent to a field greater than $2J$ (an analogous statement *cannot* be made in the XY case due to the continuous symmetry). To calculate the quenched time-dependent magnetization we first calculate the thermal-averaged time-dependent magnetization in a chain of N spins bounded by two infinite fields. The Glauber master equation for a chain of N spins takes the form

$$\frac{d}{dt}P(\sigma_1,\sigma_2,\ldots,\sigma_N;t)=\sum_{j=1}^{N}[-w_j(\sigma_j)P(\sigma_1,\ldots,\sigma_j,\ldots,\sigma_N;t)+w_j(-\sigma_j)P(\sigma_1),\ldots,-\sigma_j,\ldots,\sigma_N;t)]\ , \tag{2.2}$$

where $w_j(\sigma_j)$ is the probability per unit time that the jth spin flips from the value of σ_j to $-\sigma_j$ [w depends on the entire spin configuration of the chain, but for simplicity, we write $w_j(\sigma_i)$]. The probability function $P(\sigma_1,\ldots,\sigma_N;t)$ gives the normalized probability of finding the system at time t in the state $(\sigma_1,\ldots,\sigma_N)$.

The usual choice for $w_i(\sigma_i)$, namely

$$w_i(\pm\sigma_i)=\tfrac{1}{2}[1\mp\tfrac{1}{2}\sigma_i(\sigma_{i+1}+\sigma_{i-1})\gamma]\ , \tag{2.3}$$

where $\gamma=\tanh(2K)$, $K=J/k_BT$, guarantees that the system obeys the principle of detailed balance and will relax to an equilibrium state governed by a probability distribution proportional to $e^{-\beta H}$. In writing (2.3), we have assumed for simplicity that the bare relaxation time is unity.

Using (2.2) and (2.3) we find

$$\frac{d}{dt}S_i(t)=-S_i(t)+\frac{\gamma}{2}[S_{i+1}(t)+S_{i-1}(t)]\ , \tag{2.4}$$

where $S_i(t)$ is the time-dependent magnetization at site i given by

$$S_i(t)=\langle\sigma_i\rangle=\sum_{\{\sigma\}}\sigma_iP(\sigma_1,\ldots,\sigma_N;t)\ .$$

Equation (2.4) is equivalent to the following matrix equation:

$$\frac{dS_i(t)}{dt}=-\sum_{j=1}^{N}A_{ij}S_j+C_j\ , \tag{2.5}$$

where A_{ij} is an $N\times N$ Jacobi matrix given by

$$A=-\begin{bmatrix} -1 & \gamma/2 & 0 & \cdots & 0 \\ \gamma/2 & -1 & \gamma/2 & \cdots & 0 \\ 0 & \gamma/2 & -1 & \cdots & 0 \\ \vdots & & & \ddots\cdots & 0 \\ & & & -1 & \gamma/2 \\ 0 & & \cdots & \gamma/2 & -1 \end{bmatrix} \tag{2.6}$$

and $\vec{C}=(\tfrac{1}{2}\gamma\sigma_0,0,\ldots,0,\tfrac{1}{2}\gamma\sigma_{N+1})$. The boundary spins σ_0 and σ_{N+1} are fixed by the random fields.

Equation (2.5) can be integrated to yield the solution

$$S_i(t)=\sum_{l=1}^{N}\sum_{j=1}^{N}e^{-\lambda_l t}e_{li}e_{lj}S_j'+S_{eq,i}\ , \tag{2.7}$$

where λ_l and $\vec{e}_l$ denote the eigenvalues and eigenvectors of A, respectively, and are given by[10]

$$\lambda_l=1-\gamma\cos\beta_l\ , \tag{2.8a}$$

$$\vec{e}_l=\left[\frac{2}{N+1}\right]^{1/2}(\sin\beta_l,\sin(2\beta_l),\ldots,\sin(N\beta_l))\ , \tag{2.8b}$$

where $\beta_l=l\pi/(N+1)$. The vector $\vec{S}_{eq}$ represents the equilibrium (i.e., $t\to\infty$) state of $\vec{S}(t)$ and $\vec{S}\,'$ is given implicitly by the initial conditions, i.e., $\vec{S}\,'=\vec{S}(0)-\vec{S}_{eq}$. We will assume that the initial state is fully aligned so that $\vec{S}(0)=(1,1,\ldots,1)$.

The long-time relaxational behavior of $\vec{S}(t)$ is determined by the smallest eigenvalue of A, i.e., $l=1$. Hence (2.7) simplifies at long times to

$$S_i(t)\to\exp\left\{-\left[1-\gamma\cos\left[\frac{\pi}{N+1}\right]\right]t\right\}e_{1i}(\vec{e}_1\cdot\vec{S}\,')+S_{eq,i}\ \text{as}\ t\to\infty\ , \tag{2.9}$$

where we have used (2.8a).

The quenched magnetization per site $M(t)$ is given by the configurational average of

$$(1/N)\sum_{i=1}^{N}S_i(t)\ .$$

This averaging procedure consists of first averaging this latter quantity over the four possible configurations of the boundary spins σ_0 and σ_{N+1} and subsequently summing over all values of N weighted by the impurity probability distribution. Thus,

$$M(t)=\sum_{N=0}^{\infty}N\left[\frac{p}{2}\right]^2(1-p)^N[S_{++}(t)+S_{+-}(t)+S_{-+}(t)+S_{--}(t)]\ , \tag{2.10}$$

where, e.g.,

$$S_{+-}(t)=\frac{1}{N}\sum_{i=1}^{N}S_i(t),\ \ \sigma_0=1,\ \ \sigma_{N+1}=-1 \tag{2.11}$$

and similar expressions hold for the other quantities. Since the quenched equilibrium state is not magnetized,

$$S_{eq\,++}+S_{eq\,+-}+S_{eq\,-+}+S_{eq\,--}=0\ .$$

Using this fact, and (2.8b) and (2.9), we can write (2.10) at long times as

$$M(t)\sim\sum_{N=0}^{\infty}2p^2(1-p)^N\exp\left\{-\left[1-\gamma\cos\left[\frac{\pi}{N+1}\right]\right]t\right\}\frac{1}{(N+1)}\left[\sin\left[\frac{N\pi}{2(N+1)}\right]\csc\left[\frac{\pi}{2(N+1)}\right]\right]^2 . \tag{2.12}$$

The long-time decay of $M(t)$ is determined by the chains of long length. We can then replace the sum over N in (2.12) by an integral and obtain

$$M(t)\sim\frac{8p^2}{\pi^2}e^{-(1-\gamma)t}t^{2/3}\times\int_0^\infty x\exp-\left[\frac{\gamma}{2}\frac{\pi^2}{x^2}+x\,|\ln(1-p)|\,\right]t^{1/3}dx\,, \tag{2.13}$$

where $x=N/t^{1/3}$. A saddle-point analysis of (2.13) yields

$$M(t)\sim e^{-(1-\gamma)t}t^{1/2}e^{-Ct^{1/3}}\,, \tag{2.14}$$

where

$$C=\tfrac{3}{2}(\pi^2\gamma)^{1/3}\,|\ln(1-p)|^{2/3} .$$

This result can be expressed more physically by defining the thermal- and random-field correlation lengths ξ_T and ξ_p, respectively, as[1]

$$\xi_T=-(\ln\tanh K)^{-1}\sim[2(1-\gamma)]^{-1/2},\quad T\to 0 \tag{2.15a}$$

$$\xi_p=|\ln(1-p)|^{-1} . \tag{2.15b}$$

Thus from (2.14) we see that for times $t\ll\xi_T^3/\xi_p$, the anomalous term proportional to $t^{1/3}$ in the exponential will dominate the relaxation of $M(t)$,[11] while at times $t\gg\xi_T^3/\xi_p$, the behavior is dominated by the exponential factor, $\exp(-t/2\xi_T^2)$. At $T=0$, this latter factor will be absent completely and the decay will be asymptotically anomalous.

In their analysis of the Ising chain with bond dilution, Dhar and Barma[7] found an intermediate time regime where $M(t)\sim\exp(-t^{1/2})$. This regime is absent here presumably because the boundary conditions differ between the two problems. In the present analysis, the chains have fixed ends, i.e., the spins are pinned by the infinite fields. In the case of bond dilution where the chains of spins are bounded by missing bonds, the chains have free boundary conditions.

III. THE XY MODEL

In this section we consider the dynamics of the XY analog of (2.1), namely,[2]

$$H=-J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)-\sum_i h_i\cos(\theta_i-\phi_i)\,, \tag{3.1}$$

where θ_i is the polar angle of the spin and ϕ_i is the polar angle of the random field distributed uniformly over the unit circle. The magnitude h_i is now either ∞ or 0 with probabilities p and $1-p$, respectively. The Langevin dynamics appropriate to (3.1) were considered in Ref. 2, and the magnetization was found to decay as in (2.14) (only the $T=0$ case was considered, however). Here we consider a generalization of the Ising Glauber dynamics (2.2) to the XY case and show explicitly that these dynamical equations yield the same anomalous decay of $M(t)$ at $T=0$.

A suitable generalization of (2.2) for a chain of N XY spins takes the form

$$\frac{d}{dt}P(\theta_1,\theta_2,\ldots,\theta_N;t)=\sum_{i=1}^{N}\int_0^{2\pi}\frac{d\alpha_i}{2\pi}[-w_i(\theta_1\to\theta_i+\alpha_i)P(\theta_1,\ldots,\theta_i,\ldots,\theta_N;t)$$
$$+w_i(\theta_i+\alpha_i\to\theta_i)P(\theta_1,\ldots,\theta_i+\alpha_i,\ldots,\theta_N;t)]\,, \tag{3.2}$$

where $w_i(\theta_i\to\theta_i+\alpha_i)$ is the transition probability per unit time that the ith spin rotates by α_i. (Again we have avoided explicitly displaying the dependence of w_i on the other spins.) The detailed balance condition can be satisfied by choosing

$$w_i(\theta_i\to\theta_i+\alpha_i)=S\frac{\exp\{(K/2)[\cos(\Delta_{i,i+1}+\alpha_i)+\cos(\Delta_{i,i-1}+\alpha_i)]\}}{\exp\{(K/2)[\cos(\Delta_{i,i+1})+\cos(\Delta_{i,i-1})]\}}\,, \tag{3.3}$$

where $K=\beta J$, $\Delta_{ij}=\theta_i-\theta_j$, and S is a constant.

The thermally averaged y component of the magnetization at site k is defined by

$$\langle\sin\theta_k(t)\rangle=\int_0^{2\pi}d\theta_j\sin\theta_k P(\theta_1,\ldots,\theta_N;t) \tag{3.4}$$

and using (3.2) we find that it obeys

$$\frac{d}{dt}\langle\sin\theta_k(t)\rangle=\int_0^{2\pi}d\theta_j\sum_{i=1}^{N}\int_0^{2\pi}\frac{d\alpha_i}{2\pi}[-\sin\theta_k w(\theta_i\to\theta_i+\alpha_i)P(\theta_1,\ldots,\theta_N;t)$$
$$+\sin\theta_k w(\theta_i+\alpha_i\to\theta_i)P(\theta_1,\ldots,\theta_i+\alpha_i,\ldots,\theta_N;t)] . \tag{3.5}$$

Using (3.3) we find after some straightforward algebra that (3.5) reduces to

$$\frac{d}{dt}\langle \sin\theta_i(t)\rangle = -S\langle \sin\theta_i e^{-KA_i/2} I_0(\tfrac{1}{2}KC_i)\rangle - \langle \sin[\tfrac{1}{2}(\theta_{i+1}+\theta_{i-1})]e^{-KA_i/2} I_1(\tfrac{1}{2}KC_i)\rangle , \tag{3.6}$$

where

$$A_i = \cos\Delta_{i,i+1} + \cos\Delta_{i,i-1} , \tag{3.7a}$$

$$C_i = 2\cos\left[\frac{\theta_{i+1}-\theta_{i-1}}{2}\right] \tag{3.7b}$$

and $I_0(z)$ and $I_1(z)$ are the modified Bessel functions of zeroth and first order, respectively. In the low-temperature limit ($K\rightarrow\infty$), (3.6) further reduces to

$$\frac{d}{dt}\langle \sin\theta_i(t)\rangle = -S\left\langle \frac{e^{-K(A_i-C_i)/2}}{\sqrt{\pi K C_i}}\{\sin\theta_i - \sin[\tfrac{1}{2}(\theta_{i+1}+\theta_{i-1})]\}\right\rangle . \tag{3.8}$$

Proceeding as in Ref. 2 we linearize (3.8) about the ground-state solution θ_i^0 in order to study the relaxational dynamics at long times at $T=0$. Thus we write

$$\theta_i = \theta_i^0 + \epsilon_i, \quad \epsilon_i \ll 1 \tag{3.9a}$$

where[2]

$$\theta_i^0 = \phi_0 + \frac{\phi_{N+1}-\phi_0}{N+1} i , \tag{3.9b}$$

assuming that the random fields at the ends of the chain have orientations given by ϕ_0 and ϕ_{N+1}. To first order in ϵ (3.8) reduces to

$$\frac{d}{dt}\langle \epsilon_i\rangle = -S\frac{1}{(2K\cos\Delta_N)^{1/2}}(\langle \epsilon_i - \tfrac{1}{2}(\epsilon_{i+1}+\epsilon_{i-1})\rangle) , \tag{3.10}$$

where $\Delta_N = (\phi_{N+1}-\phi_0)/(N+1)$. With the choice

$$S = \Gamma J\cos\Delta_N[2(2\pi K\cos\Delta_N)^{1/2}] \tag{3.11}$$

for some damping coefficient Γ, (3.10) reduces to (18) of Ref. 2. The analysis of Ref. 2 is then applicable and yields

$$M(t) \sim t^{7/6} e^{-Dt^{1/3}} , \tag{3.12}$$

where

$$D = 3.4^{-1/3}(\pi^2 J\Gamma)^{1/3} |\ln(1-p)|^{2/3} .$$

The $T\neq 0$ is more difficult to treat here though we expect behavior qualitatively similar to that of the Ising model, i.e., exponential decay at sufficiently long times. In principle, one would have to linearize (3.8) about the equilibrium state for finite T, writing

$$\langle \sin\theta_i(t)\rangle = \langle \sin\theta_i\rangle_{eq} + \epsilon_i \tag{3.13}$$

and solving the subsequent linear equation in ϵ. This procedure seems less than straightforward mathematically and we will not attempt it here.

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[11]We have assumed in deriving (2.14) that $t\gg 1$, so we must also assume $\xi_T^3 \gg \xi_p$ here.