Annihilation and creation of a Korteweg–de Vries soliton

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Some new physical features illustrated by one set of new analytical Korteweg–de Vries solutions are discussed. The solutions describe the propagation of a 1-soliton through an "interaction region" in space, changing its width, amplitude, and velocity on its way. At the interaction boundary this incident soliton disappears while receiving a "message." At the other side of this boundary, a new soliton having the same shape as the incident one is created and propagates out. The characteristics of the Korteweg–de Vries soliton discovered here indicate that the previous definition of a soliton has to be revised. A hint on the unified description of particles and interaction via a nonlinear equation is mentioned.

I. INTRODUCTION

With the use of Bäcklund transformation,[1,2] we have obtained new sets of solutions to the Korteweg–de Vries (KdV) equation[3] and the nonlinear equation[4,5]

$$y_t + y_{xxx} - 6y^2 y_x + 6\lambda y_x = 0 ,$$

employing the theorems connecting the solutions of the above two equations. In this paper we follow up our investigation by analyzing one set of the new KdV solutions and emphasize the special physical aspects represented by these solutions. In particular, we have analyzed the interaction of a propagating soliton with a static interaction field. We have discussed that when the incident soliton propagates through an interaction field region, it changes its width, amplitude, and velocity, and "disappears" in a certain interaction domain. On the other hand, with the disappearance of this soliton, a new soliton is "created" on the other side of the interaction domain and propagates along the same incident direction to infinity. Such new characteristics are interesting in the theory of soliton physics. It also gives us the hint that if we take a soliton to represent a particle, our discovery appears to show that the KdV solutions give a unified description of the particle and the interaction of particles in the presence of a field.

II. ANNIHILATION AND CREATION OF A KdV SOLITON

Recently, using the Bäcklund transformation[1]

$$u^*(x,t) = -u(x,t) - y^2 + \lambda \tag{1}$$

of the KdV equation

$$u_t + u_{xxx} + 12uu_x = 0 , \tag{2}$$

where the function y satisfies

$$y_x = -2u(x,t) - y^2 + \lambda , \tag{3}$$

$$y_t = -4[u(x,t)+\lambda] y_x + 2u_{xx} - 4u_x y , \tag{4}$$

we arrive at several sets of exact solutions[2] to (2); in (1) and (3) λ is a constant.

Starting with $u(x,t)=b$, a constant, one set of those solutions appears as

$$u^* = b + (\lambda - 2b)\text{sech}^2\{\sqrt{\lambda - 2b}\,[x - 4(b+\lambda)t]\} , \tag{5}$$

where the velocity of this 1-soliton has changed from the "traditional" 4λ to

$$v = 4(b+\lambda) , \tag{6}$$

and the new parameter b has been called the vacuum parameter since as $x \to \pm\infty$, $u \to b$. We have also demonstrated in Ref. 2 that the vacuum parameter b controls the velocity of the soliton: When b takes on various values, the soliton can propagate to both left and right directions and can also be stationary.

Guided by the physical meaning of b that we discovered and the form of Bäcklund transformation, we observe from Eq. (4) that if $u(x,t)$ is an explicit function of x, then $u^*(x,t)$ given in (1) should give us new solutions whose velocity will be varying as a function of spatial coordinate x. Based on this view, we chose

$$u = -1/x^2 \tag{7}$$

and arrived at a new set of exact solutions,[3] one of which is

$$u^* = \frac{1}{x^2} + \lambda - \left[\frac{1}{x^2(\lambda x^2 - 1)^2} + \frac{2\lambda^{3/2} x}{(\lambda x^2 - 1)^2} \left[\frac{\exp(\sqrt{\lambda}\xi) - \exp(-\sqrt{\lambda}\xi)}{\exp(\sqrt{\lambda}\xi) + \exp(-\sqrt{\lambda}\xi)} \right] + \frac{\lambda^3 x^4}{(\lambda x^2 - 1)^2} \left[\frac{\exp(\sqrt{\lambda}\xi) - \exp(-\sqrt{\lambda}\xi)}{\exp(\sqrt{\lambda}\xi) + \exp(-\sqrt{\lambda}\xi)} \right]^2 \right] , \tag{8}$$

where

$$\xi = x + \frac{1}{2\sqrt{\lambda}} \ln \frac{|\sqrt{\lambda}x - 1|}{|\sqrt{\lambda}x + 1|} - 4\lambda t, \quad \lambda > 0 . \tag{9}$$

In Ref. 3, we have analyzed numerically this set of new solutions to demonstrate that indeed the velocity varies as the soliton propagates in space. Following our previous research, we would point out in the present paper several important physical aspects arising from the mathematical properties reported and demonstrated in Ref. 3.

Our u solution, Eq. (8), gives us a soliton propagating from $x \to -\infty$ to $x \to +\infty$ as time evolves. Such a feature is illustrated in Fig. 1. The superscript in u^* is omitted from now on for simplicity.

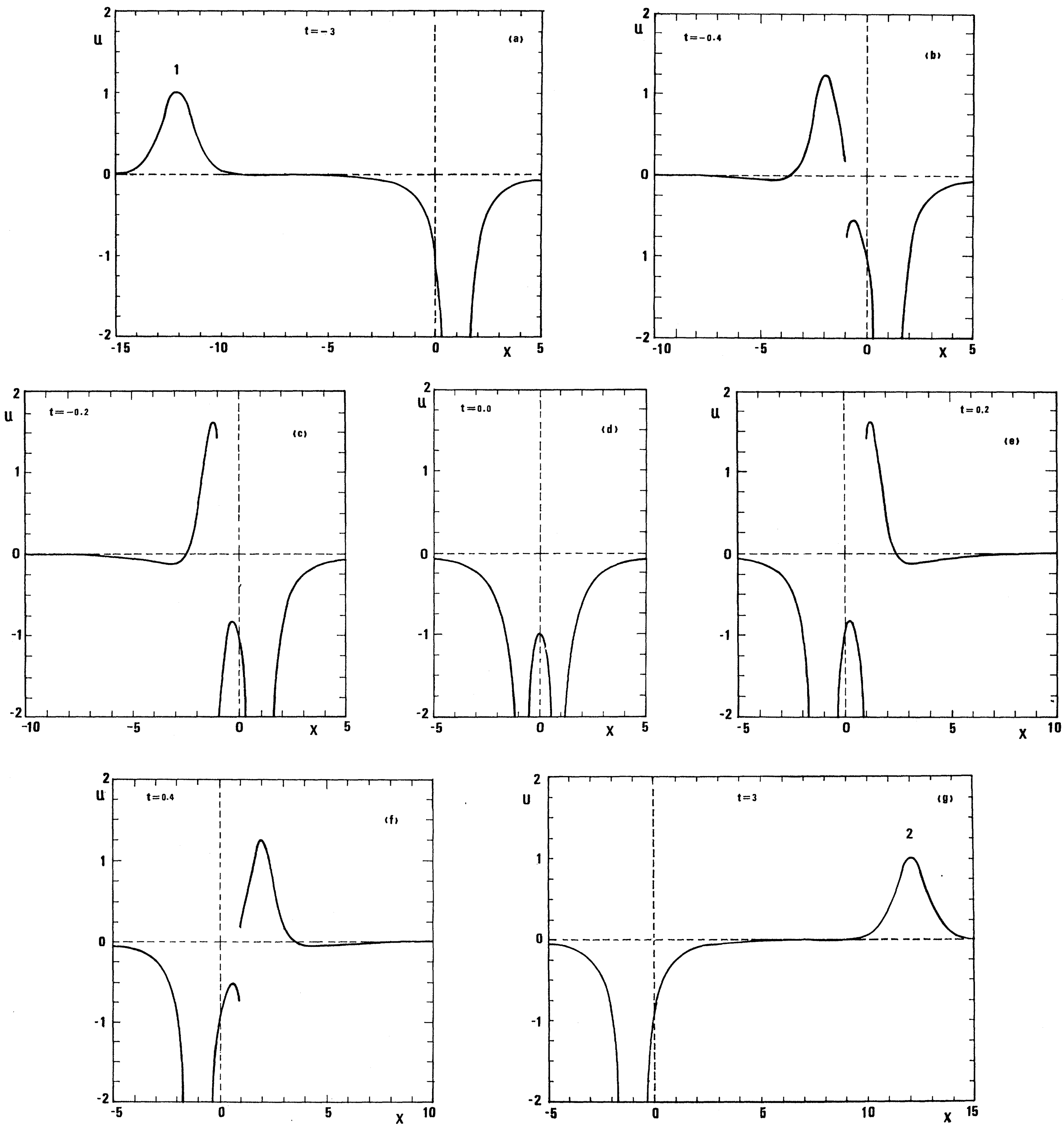

FIG. 1. 1-soliton solution u as a function of x at time (a) $t=-3$ units; (b) $t=-0.4$ units; (c) $t=-0.2$ units; (d) $t=0$ units; (e) $t=+0.2$ units; (f) $t=+0.4$ units; (g) $t=+3$ units.

The details of the propagating features in the interaction domain will be discussed later in this section and form the core of our study reported here.

To provide a better sequence of thought we must remark that the value of the velocity is obtained by following the path specified by

$$\xi = x + \frac{1}{2\sqrt{\lambda}}\ln\frac{|\sqrt{\lambda}x-1|}{|\sqrt{\lambda}x+1|} - 4\lambda t = \text{const}\ , \qquad (10)$$

where ξ is the argument of the wave solution given in (8). Thus the velocity is given by differentiating Eq. (10):

$$v = \frac{dx}{dt} = 4\lambda\frac{\lambda x^2 - 1}{\lambda x^2} \quad \text{for } \lambda > 0\ . \qquad (11)$$

In Fig. 2 we show the velocity of the soliton as a function of x according to (11). Let us now observe the propagation of the soliton in a time sequence (namely, $t = -3.0, -0.4, -0.2, 0, +0.2, +0.4, +3.0$). As illustrated in Figs. 1(a) and 1(g), superficially, as if the asymptotic 1-soliton passes through the singular region unchanged in shape. However, an inspection of Fig. 2 indicates that the velocity decreases in values as the soliton propagates from $x \to -\infty$ toward $x = -1/\sqrt{\lambda}$. In fact the velocity is zero at the point $x = -1/\sqrt{\lambda}$ and becomes negative in the "interaction" domain $(-1/\sqrt{\lambda} < x < 1/\sqrt{\lambda})$. It appears at first sight that having a negative value of the velocity is ambiguous physically to the passage of the soliton through the domain $-1/\sqrt{\lambda} < x < 1/\sqrt{\lambda}$. In fact, if we follow Figs. 1(b)–1(f), the incident soliton changes its width, amplitude, and velocity. When it propagates to the interaction domain when t is about zero, the incident soliton "disappears" at $x = -1/\sqrt{\lambda}$. After $t = 0$, a "new" soliton begins to appear around $x = 1/\sqrt{\lambda}$. At any time $t = t_1$, the u solution is an image of the $t = -t_1$ solution about the "mirror" at $x = 0$. Thus the new soliton increases its width, decreases its amplitude, and increases its velocity as it propagates to $x \to \infty$. The outcoming soliton is not the same incident soliton. To see from another aspect actually what happens to the dynamics of the soliton in the sequence of time, we plot an $x-t$ graph (Fig. 3) specified by the condition $\xi = \text{const} = 0$, for example, according to Eq. (10). We have already noted that the

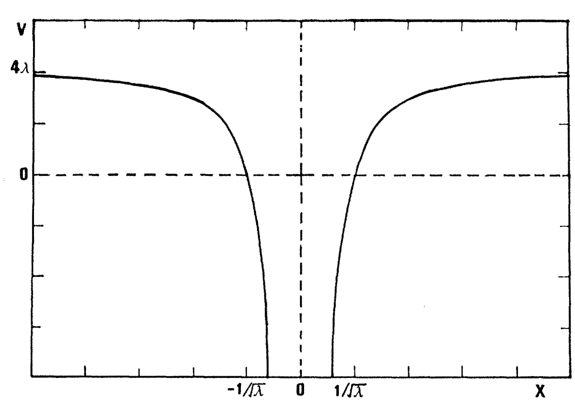

FIG. 2. Variation of the soliton velocity v in space according to relation (11).

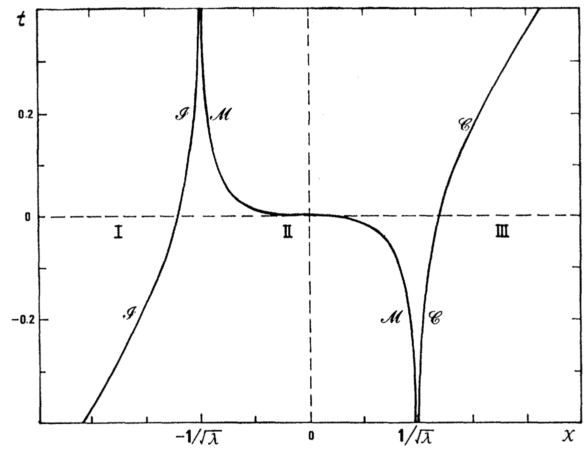

FIG. 3. Dependence of x on time according to $\xi = 0$ in Eq. (10).

condition $\xi = \text{const}$ gives us the velocity of the soliton. From Fig. 3, obviously we can separate the x space into three regions:

region I: $-\infty < x < -1/\sqrt{\lambda}$,

region II: $-1/\sqrt{\lambda} < x < 1/\sqrt{\lambda}$,

region III: $1/\sqrt{\lambda} < x < \infty$.

Actually what happens is this: When the initial soliton (i.e., soliton 1 in Fig. 1(a)) begins to propagate from $x = -\infty$, following the line $\mathscr{S}$ in region I, a disturbance $\mathscr{M}$ also transmits from $x = 1/\sqrt{\lambda}$ towards the left in region II. At the same time, another disturbance $\mathscr{C}$ propagates from $x = 1/\sqrt{\lambda}$ towards the right in region III. When $\mathscr{C}$ propagates far enough it is really an asymptotic soliton having the same shape and velocity as $\mathscr{S}$, namely as a copy of $\mathscr{S}$. The dynamics of disturbance $\mathscr{M}$, however, is more complicated because near the singular points $\pm 1/\sqrt{\lambda}$, $\mathscr{M}$ represents a certain up and down motion of the medium.[3] We would therefore interpret $\mathscr{M}$ as a "message" which actually begins to emerge at $t = -\infty$, propagating very slowly at first, increasing its speed quickly in the "middle period," and then slowing down again.

We observe that $\lambda\,\text{sech}^2[\sqrt{\lambda}(x - 4\lambda t)]$, $-1/x^2$, as well as expression (8) are all solutions to the KdV equation. We can take

$$\lambda - \frac{\lambda^3 x^4}{(\lambda x^2 - 1)^2}\left[\frac{\exp(\sqrt{\lambda}\xi) - \exp(-\sqrt{\lambda}\xi)}{\exp(\sqrt{\lambda}\xi) + \exp(-\sqrt{\lambda}\xi)}\right]^2$$

which is part of (8) and which tends to $\lambda\,\text{sech}^2[\sqrt{\lambda}(x - 4\lambda t)]$ as $x \to \pm\infty$ to represent the asymptotic 1-soliton solutions, including the incoming and outgoing solitons, as indicated in Figs. 1(a) and 1(f). As the velocity of the soliton represented by expression (8) is controlled by $u = -1/x^2$, we can view solution (8) as describing the dynamics of one soliton under the interaction specified by $u = -1/x^2$. The occurrence of the annihilation of an incoming soliton [see Fig. 1(c) and $\mathscr{S}$ in Fig. 3] and the creation of an outgoing soliton [see Fig. 1(e) and $\mathscr{C}$ in Fig. 3] is due to the singularity property of the function $-1/x^2$, which can be essentially viewed as a field. Thus the previous identity property[6] of the soliton has to be revised. Such an interesting property leads us to speculate the particle-soliton correspondence, which is briefly discussed in the next section.

III. A HINT ON THE UNIFIED DESCRIPTION OF PARTICLES AND INTERACTION VIA NONLINEAR EQUATION

There is another interesting aspect: Our analysis appears to indicate that if we view an asymptotic 1-soliton as representing a physical particle, then we may take the incoming and outgoing solitons as incident and emergent particles, respectively ($\mathscr{I}$ and $\mathscr{C}$ in Fig. 3). On the other hand, $u=-1/x^2$ may be viewed as representing the interaction on the incident particle. It is precisely due to this interaction that the "particle" changes its velocity. The incident particle stops at the boundary of the interaction region specified by $x=-1/\sqrt{\lambda}$. At the other end of the stated interaction region (namely $x=1/\sqrt{\lambda}$ in this particular case), a "new particle" is created; this particle has exactly the same face as the incident one. We see from our analysis that the solution (8) of nonlinear Eq. (1) gives a description of particles, interaction, and the behavior of the particles influenced by this interaction. It is therefore meaningful and perhaps fruitful to investigate whether interaction among particles, such as the annihilation of two particles, can be described by *one* single nonlinear equation. In other words, can one equation alone describe both particles and the corresponding interacting field? It seems that such an idea is in line with the basic concept of unified field theory. We would remark that the interaction is embedded in the solution of the KdV equation, rather than as an external imposed field. We would note also that our result arises only from a classical nonlinear equation—no quantum-mechanical ideas have been employed.

IV. CONCLUSION

We would like to conclude with the following three points.

(1) We have brought out some new soliton physical features illustrated by one set of the new analytical KdV solutions found recently. The solutions describe propagation of a 1-soliton from a remote distance to an interaction domain, changing its width, amplitude, and velocity on the way.

(2) At one boundary of the interaction region this incoming soliton stops while receiving a "message." At the other end, a new soliton having the same shape as the incoming one is created and propagates out. The characteristics of the KdV soliton discovered here indicate that the previous definition[6] of a soliton has to be revised.

(3) If we take a soliton to represent a particle, our discovery appears to indicate that the nonlinear KdV solution may be taken to describe the interaction of particles in the presence of a field.

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