Comprehensive study of the renormalization of the theory of strong-coupling superconductors

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The renormalization of the superconducting state is presented consistently and predictions for the screened charges as well as for the mixing in the phonon-Coulomb system are derived.

I. INTRODUCTION

Field-theoretic calculations start with bare fields and bare interactions. These bare quantities cannot be directly applied for predictions of observed phenomenology. Theoretical quantities that predict experimental results can, however, be expressed in perturbation sums, which include various radiative corrections to the bare theory. In order to treat these terms consistently, the normalization method has been developed. So far, no consistent study on the renormalization of the microscopic theory of superconductivity exists, although much effort has been dedicated to the calculation of the electron self-energy and the phonon polarization.

This paper is organized in the following way: Section II introduces the bare theory. In Sec. III the coupled system of equations for the renormalized Green's functions is enumerated, and field renormalization and the screened charges are defined. Section IV deals with their actual calculation and the approximations that must be made. Finally, in Sec. V, results are discussed and experiments to test the calculations are encouraged.

II. BARE THEORY

The Lagrangian of the electron-phonon system in the two-component notation of Anderson and Nambu[1] may be decomposed into three parts: (i) the Lagrangian $\mathscr{L}_0$ of the free (noninteracting) electron, phonon, and electrostatic fields Ψ, φ, and ϕ, (ii) interaction terms $\mathscr{L}_I$ describing the electron-phonon coupling as well as electromagnetic forces approximated by static Coulomb contributions, and (iii) counterterms $\mathscr{L}_C$ that are in the bare theory but must be included to absorb radiative corrections:

$$\begin{aligned}
&\mathscr{L}=\mathscr{L}_0+\mathscr{L}_I+\mathscr{L}_C\ ,\\
&\mathscr{L}_0=\Psi^\dagger\left[i\frac{\partial}{\partial t}+\epsilon_p\tau_3\right]\Psi+\tfrac{1}{2}(\partial_\mu\varphi)^2+\tfrac{1}{2}(\nabla\phi)^2\ ,\\
&\mathscr{L}_I=\Psi^\dagger\tau_3\Psi(g_0\varphi+e_0\phi)\ ,\\
&\mathscr{L}_C=\Psi^\dagger(\Delta\tau_1+\bar{\Delta}\tau_2+\chi\tau_3)\Psi\ .
\end{aligned}\tag{2.1}$$

Electron fields $\Psi_s(x)$ (s is the spin index) are arranged in a two-component vector $\Psi(x)=(\Psi_s(x),\Psi_{-s}(x))$. The τ matrices are well known from the theory of angular momentum: $\tau_3=\mathrm{diag}(1,-1)$, τ_1 and τ_2 are normal on 1, τ_3, and each other. Together with the unity matrix they form a complete basis of the linear (2,2) space. The phonons are represented by a scalar field φ with an effective Yukawa-type coupling to the charge density of the electrons. The Coulomb term is the electrostatic vector potential $A_0(\vec{x},0)=\phi(\vec{x})$ that stems from the substitution $i\partial_\mu\rightarrow i\partial_\mu+e_0\tau_3A_\mu(\vec{x},t)$ in the free-electron Lagrangian, where only longitudinal components have been kept.

The bare Green's functions of perturbation theory may be either read off directly from the Lagrangian $\mathscr{L}_0$ or taken from the literature.[2] The bare-electron propagator is denoted by $G_0(p)$, the propagator of the acoustic phonon by $B_0(q_0)$ and the bare Coulomb line by $V_0(\vec{q})$. A factor τ_3 should be inserted at each bare vertex. For the complete composition rules the reader is referred to the literature. Factors and functional dependences entering the perturbation expansion are dropped if they are not essential to the argument.

III. RENORMALIZATION

Since both the phonon as well as the Coulomb line couple to the charge density of the electrons, a mixing of the phonon and Coulomb fields occurs due to an exchange term from the proper polarization Π with one phonon and one Coulomb leg, respectively. Therefore, they cannot be treated independently and must be arranged in a (2,2) matrix in a linear space spanned by the Coulomb and phonon fields. Taking the bare fields as the basis, the bare propagator D_0 and the proper polarization can be written in the following representation (the indices φ and ϕ refer to the outgoing legs):

$$D_0=\begin{bmatrix}B_0 & 0\\ 0 & V_0\end{bmatrix},\quad \Pi=\begin{bmatrix}\Pi_{\varphi\varphi} & \Pi_{\varphi\phi}\\ \Pi_{\phi\varphi} & \Pi_{\phi\phi}\end{bmatrix}.$$

The full Green's functions G' and D' are obtained by the summation of complete proper self-energy and polarization terms Σ and Π, a sum that can formally be written as

$$G'=G_0+G_0\Sigma G_0+\cdots\ ,\tag{3.1a}$$

$$D'=D_0+D_0\Pi D_0+\cdots\ .\tag{3.1b}$$

This should not be confused with the random-phase approximation, since it contains the complete proper terms and not only their lowest-order approximations. G' and D' in (3.1) are geometrical series that can be summed up only in the radius of convergence, although the formal

sums are assumed to be valid outside:

$$G'=(1-G_0\Sigma)^{-1}G_0 , \tag{3.2a}$$

$$D'=(1-D_0\Pi)^{-1}D_0 . \tag{3.2b}$$

The structure of the proper self-energy Σ, polarization Π, and vertex parts Γ can be investigated by the following integral equations[3] [sums in (φ,ϕ) space are not denoted explicitly]:

$$\Sigma=\int D'\Gamma G'\tau_3 , \tag{3.3}$$

$$\Pi=-\int \mathrm{Tr}(\tau_3 G'\Gamma G') , \tag{3.4}$$

$$\Gamma=\tau_3-\int \Gamma G'KG' , \tag{3.5}$$

where another proper diagram K has been introduced. K itself will not give rise to any further renormalization, since the only corrections entering are self-energies and vertex insertions in the internal lines. Σ, Π, and Γ will be the central ingredients of the renormalization program, and all other quantities in the theory such as the S matrix and other Green's functions will be expressed in terms of these three. Equations (3.1)—(3.5) are graphically represented in Fig. 1.

The central idea of renormalization is to rescale the propagators (and thus the fields), vertex functions, and charges so that near a certain renormalization point $\mathscr{M}$ and in the case of the vertex for zero momentum transfer these quantities approach the corresponding bare quantities:

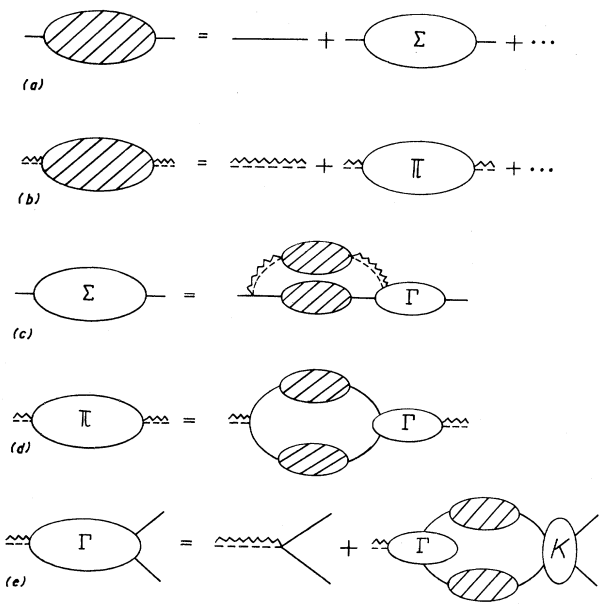

FIG. 1. The complete system of Eqs. (3.1)—(3.5) is represented by diagrams. Shaded areas indicate full Green's functions. The propagator of the phonon and the Coulomb line D is drawn as a double line. Each diagram can be associated with an equation as follows: (a) with (3.1a), (b) with (3.1b), (c) with (3.3), (d) with (3.4), and (e) with (3.5).

$$\begin{aligned} G'&=Z_2G^* , \\ B'&=Z_3^{(1)}B^* , \\ V'&=Z_3^{(2)}V^* , \\ \Gamma&=Z_1^{-1}\Gamma^* , \\ g&=g_0(Z_3^{(1)})^{1/2}Z_2Z_1^{-1} , \\ e&=e_0(Z_3^{(2)})^{1/2}Z_2Z_1^{-1} , \end{aligned} \tag{3.6a}$$

so that for $p_0\to\Delta_0=\Delta(p_0)$ and $q_0\to\mathscr{M}$

$$\begin{aligned} G^*(p)&=G_0^C(p) , \\ B^*(q)&=B_0(q) , \\ V^*(q)&=V_0(q) , \\ \Gamma^*(p,p)&=\tau_3 . \end{aligned} \tag{3.6b}$$

G_0^C includes contributions from the counterterms in (2.1). It can be seen easily that the renormalized couplings g and e are defined to contain the renormalization factors Z_2 and Z_3 from the electron, phonon, and Coulomb lines (one-half of a phonon or Coulomb and half of each of the two electron lines; the other halves of these lines join to other vertices or are external). Therefore, at a particular momentum $\mathscr{M}$ the full Lagrangian containing the renormalized fields and charges may be written formally identical to (2.1); with the result that a tree-graph calculation of some process, based on this redefined Lagrangian, already contains all possible radiative corrections in a consistent and well-defined way.

Complications arise because of the nondiagonal form of D' in the basis where D_0 is diagonal (and vice versa). This indicates that the fields φ and ϕ are not multiplicative renormalizable, as was assumed in (3.6). Owing to the invariant form of the free Lagrangian $\mathscr{L}_0$ (since $\partial_t\phi=0$) it is possible, however, to rotate the fields such that two new fields

$$\eta=\frac{e_0\phi+g_0\varphi}{(e_0^2+g_0^2)^{1/2}}, \quad \xi=\frac{g_0\phi-e_0\varphi}{(e_0^2+g_0^2)^{1/2}}$$

decouple and become multiplicative renormalizable.

One of the formidable problems facing quantum-field theory is the proof that G^*, B^*, V^*, and Γ^* remain finite to all orders of the renormalized couplings. It is thought, however, that ultraviolet divergencies are logarithmic and therefore weak and arise from the notion of a continuous field, which is fortunately only an approximation in many-body theory, since fundamental lengths (serving as ultraviolet cutoffs) appear as Debye frequencies from band structure, etc. Therefore, the major concern is with the actual calculation of the renormalized quantities.

IV. CALCULATION OF THE RENORMALIZED QUANTITIES

A theorem by Migdal[4] (recently discussed by Allen and Mitrović[5]) states that high-order phonon corrections to the vertex function Γ are of the order of $\sqrt{m/M}$ (where m and M stand for the electron and ion masses, respectively), resulting in a suppression of approximately 10^2.

The Coulomb sector is generally more difficult to treat and is often omitted. For the calculation of the electron self-energy, a pseudopotential can be introduced that takes into account the long-range nature of the Coulomb force. In what follows it is assumed that for strong-coupling superconductors these electromagnetic interactions yield small corrections to the dominant phonon sector.

A. Electron self-energy

The sum in (3.1a) has been evaluated by Eliashberg[6] in an approximation of the self-energy that essentially decouples (3.2a) and (3.3) from the rest of the system (Fig. 2):

$$\Sigma = \int D_0 \tau_3 G' \tau_3 \ . \tag{4.1}$$

Equation (3.2a) can be written into a form often called Dyson's equation $G'^{-1} = G_0^{-1} - \Sigma$ and

$$G'^{-1} = G_0^{-1} - \int D_0 \tau_3 G' \tau_3 \ , \tag{4.2}$$

which is referred to as self-consistent perturbation theory. The approximations made are supported by Migdal's argument[4] for the electron-phonon sector that states that $\Gamma \approx \tau_3$. Coulomb contributions are more cumbersome and are bypassed by the introduction of a pseudopotential, as mentioned above. Recently the inclusion of the phonon propagator (that may show resonances due to bound states) into the system of equations has been suggested by Machida.[7] Equation (4.2) can be solved by introducing counterterms Δ, $\bar{\Delta}$, and χ into the full propagator G' and arranging it into the form

$$\begin{aligned} G'(p) &= Z_2 G^*(p) \\ &= Z_2 [p_0 1 - (\epsilon_p + \chi)\tau_3 - \Delta(p)\tau_1 - \bar{\Delta}(p)\tau_2 + i\delta]^{-1} \ . \end{aligned} \tag{4.3}$$

The phonon propagator D_0 is written in its spectral representation which is inferred from experiment. This leads to the Eliashberg equations[6] for the renormalization factor Z_2 and the counterterms ($p_0 = \omega$):

$$Z_2^{-1}(\omega) = 1 + \frac{1}{\omega} \int_{\Delta_0}^{\omega_c} d\omega' \mathrm{Re} \left[\frac{\omega'}{[(\omega')^2 - \Delta^2(\omega')]^{1/2}} \right] \times K_-(\omega,\omega') \ , \tag{4.4a}$$

$$\Delta(\omega) = Z_2(\omega) \int_{\Delta_0}^{\omega_c} d\omega' \mathrm{Re} \left[\frac{\Delta(\omega')}{[(\omega')^2 - \Delta^2(\omega')]^{1/2}} \right] \times K_+(\omega,\omega') \ , \tag{4.4b}$$

$$\bar{\Delta}(\omega) \approx 0 \ , \tag{4.4c}$$

$$\chi(\omega) \approx \epsilon_p [Z_2(\omega) - 1] \ , \tag{4.4d}$$

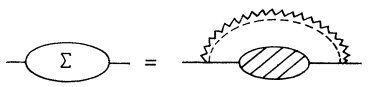

FIG. 2. Approximate calculation of the electron self-energy.

where K_- and K_+ are kernels containing the phonon and Coulomb structure. At the critical temperature [which can be obtained from (4.4b)], the gap $\Delta(\omega)$ vanishes for all frequencies ω.

B. Phonon and Coulomb polarization

An approximate calculation of the proper polarization in a superconducting medium was first performed by Schuster[8] and has been used by Balseiro and Falicov[9] to explain Raman scattering in the presence of charge-density waves. As has been pointed out earlier, the mixing between the bare phonon and Coulomb lines complicates the actual calculation of the effective charges. Since both fields couple to the electron identically with different coupling constants, the polarization can be written as

$$\Pi = \begin{bmatrix} g_0^2 & g_0 e_0 \\ e_0 g_0 & e_0^2 \end{bmatrix} P \ .$$

Physically this means that phonons are no eigenstates of the phonon-Coulomb system. Consequences can be derived by rewriting the Schrödinger equation for the phonon and the Coulomb state in the interaction picture into an integral equation that can be solved by an iteration process similar to perturbation theory.[10] If a phonon is injected it starts to oscillate, and the probability that it is in the phonon state after some time Δt is given by a second-order calculation in e_0 and g_0 [$\gamma = 2(e_0^2 + g_0^2)\mathrm{Im}(P)$, $\epsilon = (e_0^2 + g_0^2)\mathrm{Re}(P)$]:

$$\omega_\varphi(\Delta t) = \frac{e_0^4 + g_0^4 e^{-\gamma \Delta t} + 2 e_0^2 g_0^2 e^{-(1/2)\gamma \Delta t} \cos(\epsilon \Delta t)}{(g_0^2 + e_0^2)^2} \ . \tag{4.5}$$

This oscillation is observable if e_0 is not too small compared to g_0 and the linewidth γ does not dominate the expression. The full propagator D' for the phonon-Coulomb system in the basis of the bare fields can be computed from (3.2b):

$$D' = \frac{1}{1 - (g_0^2 B_0 + e_0^2 V_0)P} \times \begin{bmatrix} (1 - e_0^2 V_0 P) B_0 & e_0 g_0 B_0 P V_0 \\ e_0 g_0 B_0 P V_0 & (1 - g_0^2 B_0 P) V_0 \end{bmatrix} . \tag{4.6}$$

For actual calculations the electron-Coulomb coupling e_0 is assumed to be small compared to g_0, so that the proper polarization reduces to $\Pi = \mathrm{diag}(g_0^2, 0)P$ and (3.2b) may be evaluated for a decoupled phonon-Coulomb system [see Fig. 3(a)]:

$$B' = \frac{B_0}{1 - g_0^2 B_0 P} \ , \tag{4.7a}$$

$$V' = V_0 \ . \tag{4.7b}$$

An approximation for the proper polarization (3.4) is obtained by setting $\Gamma = \tau_3$ [Fig. 3(b)],

$$P = -\int \mathrm{Tr}(\tau_3 G' \tau_3 G') \ . \tag{4.8}$$

Since the full electron propagators appear in this expression, (4.7) and (4.8) are not complete. For practical purposes, however, G' is assumed to be known and the gap

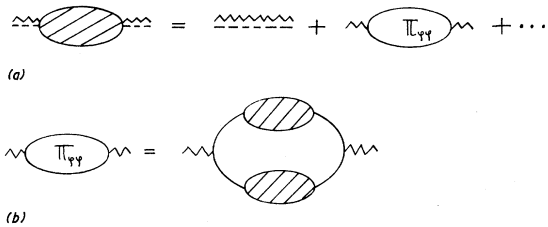

FIG. 3. Approximate calculation of the phonon and the Coulomb propagator.

even constant, although the complete structure is much richer and can only be deduced by a simultaneous solution of (3.2)—(3.5). From the definition of the scaling laws (3.6) and from (4.7), the renormalization factors Z_3 are seen to be

$$Z_3^{(1)}=(1-g_0^2B_0P)^{-1}\,, \tag{4.9a}$$

$$Z_3^{(2)}=1\,. \tag{4.9b}$$

For the explicit calculation of P the reader is referred to the literature.[7,8]

C. The proper vertex function and charge screening

Migdal's theorem[4] states that phonon corrections to the proper vertex (3.5) can be neglected due to a strong supression of the order of 10^2. Coulomb corrections are generally omitted, consistent with the approximations made for the calculation of the self-energy and polarization terms. This reduces the proper vertex part drastically to its bare contribution $\Gamma=\tau_3$, which implies

$$Z_1=1\,. \tag{4.10}$$

We conclude with the remark that, since only the electrostatic potential has been included into the Lagrangian (2.1), a generalized Ward identity[11] that would establish $Z_1=Z_2$ for the electromagnetic corrections cannot be applied.

From the approximate calculation of the renormalization factors the renormalized charges may be obtained according to their definition in (3.6). Since the resulting theory is effectively (up to Z factors) a free theory, this rescaling is very similar to the classical definition of the static dielectric constant: *(effective potential)=(bare potential)/(dielectric constant).* The classical concept is refined by quantum theory to include certain fine structures (such as the gap) that can only be understood on the quantum level. Therefore, the identification of the dielectric constant

$$\epsilon^{(i)}=[(Z_3^{(i)})^{1/2}Z_2Z_1^{-1}]^{-2}\,, \tag{4.11}$$

where (i) denotes either g or e renormalization, is a formal one that has only an exact classical limit ($h\to 0$). If the renormalization point $\mathcal{M}$ is chosen to be identical to the energy flow Q across the vertex, all propagators reduce to a convenient bare form, and g and e may be written as

$$g^2(Q)=\frac{g_0^2}{\epsilon^{(1)}(Q)}\,, \tag{4.12a}$$

$$e^2(Q)=\frac{e_0^2}{\epsilon^{(2)}(Q)}\,, \tag{4.12b}$$

where in the approximation of McMillan[12] $Z_2\approx(1+\lambda)^{-1}$ for $Q\approx\Delta$

$$\epsilon^{(1)}(Q)=[1-g_0^2B_0(Q)P(Q)]\,(1+\lambda)^2\,, \tag{4.13a}$$

$$\epsilon^{(2)}Q=(1+\lambda)^2\,, \tag{4.13b}$$

and for $Q\gg\Delta$, $Z_2\approx 1$ (λ is the effective electron-phonon coupling that includes both the bare coupling g_0 and the phonon spectrum).

V. CONCLUDING REMARKS

We have discussed the renormalization of the microscopic theory of superconductivity. This procedure reproduces the well-known Eliashberg equations for the gap and the electron renormalization. New structures appear in the phonon and photon (Coulomb) sector, since both fields couple to the charge density of the electrons and become a hybrid state. It is also interesting to note that the electron renormalization Z_2 must be included in charge screening; an effect that is observable only in the low-energy region near the gap. The gap parameter Δ shows some similarity with the mass of relativistic field theory [the excitation spectrum of the quasiparticle states is $E_p^2=\epsilon_p^2+|\Delta|^2(p)$] and is not obtained by spontaneous or dynamical symmetry breaking (such that a scalar field similar to the phonon would acquire a nonvanishing vacuum expectation value), but rather by a self-consistent evaluation of the electron Green's function.

Some questions have been left out. It would be interesting to know the full structure of the phonon-photon system for the case when $g_0\approx e_0$. Results from relativistic field theory[13] indicate that mass counterterms for the phonon and the photon would need to be included very much like the counterterms for the electron field in $\mathcal{L}_C$.

Whereas computations of the gap have already been confirmed experimentally, an empirical examination of the renormalized charges (4.12) as well as oscillations in the phonon-Coulomb system (4.5) still remains to be done and would be welcomed as a test for the theoretical approach.

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