Effect of parametric modulation on the onset of convection in ^{3}He-^{4}He mixtures near the λ line

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The onset of convection in ^{3}He-^{4}He mixtures is considered with the temperature difference sinusoidally modulated between the plates in the Rayleigh-Bénard geometry. The system can show stabilization or destabilization depending on the mean temperature. For oscillatory instability, parametric resonances should occur. As a preliminary step, we obtain analytic results under idealized boundary conditions for the onset of convection in the unmodulated case. The results there are in qualitative agreement with the experiments of Lee, Lucas, and Tyler.

I. INTRODUCTION

Convective motions[1] can be produced in single-component fluids by maintaining a vertical temperature gradient (heating from below when the expansion coefficient is positive and from above when it is negative) that affects the density gradient. In fluid mixtures or in a fluid with a solute (e.g., saline water), convection can occur, even if the net density decreases upwards and the system is hydrostatically stable in the sense of a single-component fluid, provided the two diffusivities (of heat and solute) differ significantly. This effect, first anticipated by Stommel *et al.*[2] and subsequently refined by Stern[3] and Veronis,[4] is called double-diffusive convection. While these early works dealt with the case of a fluid containing a dissolved solid, it was realized by Schechter, Prigogine, and Hamm[5] that the same effect could occur in binary-fluid mixtures.[6,7]

Experimental advantages such as good temperature stability and resolution have motivated the study of convection in fluids at very low temperatures in recent years. Ahlers[8] investigated the convection in a single-component fluid near the λ temperature of ^{4}He. An interesting candidate for studying double-diffusive convection at these temperatures is the mixture ^{3}He-^{4}He near the superfluid transition temperature T_λ. This mixture differs from usual binary mixtures in three striking ways: (i) the mass diffusion of ^{3}He into ^{4}He becomes high near the λ point and ultimately diverges[9] at the transition, (ii) the thermodiffusion k_T is relatively large, reaching a limiting value[10] of about 0.57 at T_λ, and (iii) at about 6 mK above the T_λ, the expansion coefficient changes sign,[11] becoming positive for $T > T_\lambda + 6$ mK. Lee, Lucas, and Tyler[12] have reported on the Rayleigh-Benard instability in this system.

Note that because of (i) and (ii) the ratio of mass diffusivity to the thermal conductivity (in the absence of mass current) can change from a number much less than unity (the usual situation) far from T_λ to a number much greater than unity very close to T_λ. This fact invalidates the usual assumption of neglecting the effect of concentration gradient on the heat current. What is essentially involved here is a more versatile form of double-diffusive convection in which the ratio of diffusivities can be parametrically varied over a wide range.

The problems of onset of convection in such systems have been treated at length by Gutkowicz-Krusin, Collins, and Ross.[13] In Sec. II we treat the problem in a simplified form by assuming idealized boundary conditions. This has the advantage of producing results in closed form—a fact which is exploited in the following sections in dealing with the parametric modulation of the temperature difference. It will be seen that results obtained on the basis of the idealized boundary conditions are in qualitative agreement with the experimental observations of Lee *et al.*[12]

An interesting problem to study is the effect of periodic modulation of the temperature difference between the plates. For the single-component fluid this modulation is known to produce dynamic stabilization and delay the onset of convection if finite-size effects are ignored.[14,15] It has recently been pointed out[16] that for a double-diffusive system such as salt solution, such parametric modulation can lead to a far richer stability pattern. In this paper we carry out a calculation of the effect of the modulation of the temperature difference on the onset of convection in the ^{3}He-^{4}He mixtures. We believe that the various possibilities of stabilization, destabilization, and resonances will stimulate experimental work in the field. Our calculation follows closely that of Venezian[17] for the single-component fluids.

In Sec. III we set up the equations of motion for the modulated system in a perturbation approach. The first correction to the critical Rayleigh number is obtained in Sec. IV and a brief summary provided in Sec. V.

II. ONSET OF CONVECTION IN ABSENCE OF MODULATION

The hydrodynamic equations for the temperature (T) and concentration (c) variables in the binary mixture have been obtained by Landau and Lifshitz[18] as

$$\frac{\partial T}{\partial t}+(\vec{v}\cdot\vec{\nabla})T=\frac{\Lambda}{C_P}\nabla^2 T+\frac{Dk_T}{\chi C_P}\nabla^2 c \ , \tag{2.1}$$

$$\frac{\partial c}{\partial t}+(\vec{v}\cdot\vec{\nabla})c=D\nabla^2 c+\frac{Dk_T}{T}\nabla^2 T \ . \tag{2.2}$$

Here $\vec{v}$ is the velocity of flow. The concentration c is the mass fraction of ^{3}He, D is the isothermal mass diffusion coefficient, Λ is the thermal conductivity in the absence of the concentration gradient, k_T is the thermodiffusion, C_P is the specific heat at constant pressure and concentration, and χ is the susceptibility $(\partial c/\partial\mu)_T$, μ being the chemical potential. The two diffusivities whose relative magnitude drives this double-diffusive convection are the mass diffusion D and the thermal diffusion in the absence of mass current $(1/C_P)(\Lambda - Dk_T^2/\chi T)$. The velocity follows the usual Navier-Stokes equations, augmented by the buoyancy term,

$$\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\frac{1}{\rho_m}\vec{\nabla}p+\vec{g}\frac{\rho}{\rho_m}+\nu\nabla^2\vec{v} \ . \tag{2.3}$$

Here ρ_m is the mean density, ρ is the position-dependent density, ν is the kinematic viscosity, and p is the pressure. The fluid is assumed to be incompressible, i.e.,

$$\vec{\nabla}\cdot\vec{v}=0 \ , \tag{2.4}$$

and the equation of state is

$$\rho=\rho_m[1-\alpha(T-T_m)-\beta(c-c_m)] \ , \tag{2.5}$$

where T_m and c_m are the mean temperature and concentration. The thermal expansion coefficient is

$$\alpha=-\frac{1}{\rho_m}\frac{\partial\rho}{\partial T} \ , \tag{2.6}$$

and

$$\beta=-\frac{1}{\rho_m}\frac{\partial\rho}{\partial c} \tag{2.7}$$

measures the change of density of mixture with change in ^3He concentration. By our definition $\beta>0$. In the steady state (shown by subscript s) the velocity field is zero everywhere and the temperature and concentration profiles are linear,

$$T_s(z)=T_0-\Delta T\frac{z}{d} \ , \tag{2.8}$$

$$c_s(z)=c_0-\Delta c\frac{z}{d} \ , \tag{2.9}$$

where T_0 and c_0 are the values at the lower plate, d is the plate separation, and ΔT and Δc are the differences in temperature and concentration between the plates. The fact that in the steady state there is no mass current implies that

$$\frac{\Delta c}{\Delta T}=-\frac{k_T}{T} \ , \tag{2.10}$$

where ΔT is positive when heated from below.

To study the onset of convection in the above system one needs the linearized equations of motion. For the velocity field w in the z direction, the deviation δT and δc of the temperature from their steady-state values, respectively, we have the equations of motion

$$\nabla^2\left[\frac{\partial}{\partial t}-\nu\nabla^2\right]w=\alpha g\nabla_1^2\delta T+\beta g\nabla_1^2\delta c \ , \tag{2.11}$$

$$\left[\frac{\partial}{\partial t}-\lambda\nabla^2\right]\delta T=\frac{\Delta T}{d}w+\frac{Dk_T}{\chi C_P}\nabla^2\delta c \ , \tag{2.12}$$

$$\left[\frac{\partial}{\partial t}-D\nabla^2\right]\delta c=\frac{\Delta c}{d}w+\frac{Dk_T}{T}\nabla^2\delta T \ , \tag{2.13}$$

where $\lambda=\Lambda/C_P$ is the thermal diffusivity. We introduce the dimensionless variables

$$l=z/d,\quad \sigma=\nu/\lambda,\quad S_1=\nu/D \ , \qquad R=\frac{\alpha g(\Delta T)d^3}{\lambda\nu},\quad \tau=\frac{\nu}{d^2}t \ , \tag{2.14}$$

and eliminate δT and δc from Eqs. (2.11)–(2.13) to obtain

$$Lw=0 \ , \tag{2.15}$$

where

$$\begin{aligned}L=\nabla^2\Bigg[&\left[\sigma\frac{\partial}{\partial\tau}-\nabla^2\right]\left[S_1\frac{\partial}{\partial\tau}-\nabla^2\right]\\&-\frac{Dk_T^2}{\lambda\chi C_P T}\nabla^4\Bigg]\left[\frac{\partial}{\partial\tau}-\nabla^2\right]\\&-R\left[\nabla_1^2\left[S_1\frac{\partial}{\partial\tau}-\nabla^2\right]-\frac{\beta k_T}{\alpha T}\frac{\lambda}{D}\nabla_1^2\left[\sigma\frac{\partial}{\partial\tau}-\nabla^2\right]\right.\\&\left.+\frac{\beta}{\alpha}\frac{k_T}{T}\nabla_1^2\nabla^2-\frac{k_T^2}{\chi C_P T}\nabla_1^2\nabla^2\right] \ .\end{aligned} \tag{2.16}$$

Before proceeding any further we need to prescribe boundary conditions. Analytic answers are obtained for the idealized free boundary conditions

$$w=\frac{d^2w}{dz^2}=0 \ \text{ at } z=0 \text{ and } d \ . \tag{2.17}$$

While the quantitative results obtained from these boundary conditions can differ significantly from those obtained with more realistic ones, the qualitative features are quite accurate. Furthermore, analytic answers at this stage will simplify the study of temperature modulation in the next section. The velocity field which obeys the boundary conditions of Eq. (2.17) has the form

$$w(\vec{r})\sim\sin(n\pi z/d)e^{i(k_x x+k_y y+\omega t)} \ , \tag{2.18}$$

where n is an integer and $\vec{k}=(k_x,k_y)$ is a two-dimensional vector. Defining

$$N=n^2\pi^2+k^2d^2=n^2\pi^2+K^2 \ , \tag{2.19}$$

we find the Rayleigh number of stationary instability ($\omega=0$) to be

$$R_0=\frac{27\pi^4}{4}\frac{1-\dfrac{D}{\lambda}\dfrac{k_T^2}{\chi C_P T}}{1+\dfrac{k_T^2}{\chi C_P T}-\dfrac{\beta k_T}{\alpha T}\left[1+\dfrac{\lambda}{D}\right]} \ . \tag{2.20}$$

For the oscillatory instability, on the other hand,

$$\widetilde{R}_0=\frac{27\pi^4}{4}\frac{\left[1+\frac{1}{S_1}+\frac{k_T^2}{\chi TC_PS_1}\left[1+\frac{1}{\sigma}-\frac{k_T^2}{S_1T\chi C_P}\right]^{-1}\right]\left[1+\frac{D}{\lambda}\right]}{1-\frac{\beta}{\alpha}\frac{k_T}{T}\left[1+\frac{1}{\sigma}-\frac{k_T^2}{S_1TC_P\chi}\right]^{-1}}\,, \tag{2.21}$$

and the oscillatory frequency is

$$\omega_0^2=\frac{9\pi^4}{4}\left[1-\frac{Dk_T^2}{\lambda\chi C_PT}\right]-\frac{1}{3}\left[1+\frac{k_T^2}{\chi C_PT}-\frac{\beta k_T}{\alpha T}\left[1+\frac{\lambda}{D}\right]\right]. \tag{2.22}$$

Note that $27\pi^4/4$ is the minimum value of the quantity N^3/K^2.

The above equations have been analyzed near the λ point with the help of accurate data on the transport properties of ^{3}He-^{4}He mixtures that have recently become available.[19,20] $\widetilde{R}_0<R_0$ close to T_λ as long as $\alpha>0$. Consequently the instability is oscillatory when heated from below ($\Delta T>0$) and disappears when

$$\frac{\beta}{\alpha}\frac{k_T}{T}=1+\frac{1}{\sigma}-\frac{k_T^2}{S_1T\chi C_P}\,. \tag{2.23}$$

This is in agreement with Lee *et al.*,[12] who found that the instability is oscillatory when heated from below and stationary when heated from above. Furthermore Eq. (2.23) shows that the onset of the oscillatory instability disappears at the temperature where $\alpha\to 0$ as the concentration of ^{3}He approaches zero. This can be seen from the fact that $k_T\to 0$ as the concentration vanishes; since the right-hand side (rhs) is finite, the equality is satisfied when $\alpha\to 0$. At higher concentrations it is seen from Eq. (2.23) that this onset phenomenon occurs at higher temperatures—again in qualitative agreement with Lee *et al.*[12] When heated from above $R_0<\widetilde{R}_0$ ($\alpha<0$), consequently the instability is stationary. The occurrence of the combination $\lambda-k_T^2D/\chi C_PT$ in the expression for the Rayleigh number now makes it possible for convection to occur very close to the λ point. This is because while λ and D both diverge, this combination does not and, hence, it is possible to go very close to T_λ and yet observe the instability. For pure ^{4}He this instability is unobservable near T_λ owing to its divergent thermal conductivity. In the next section we consider the hydrodynamic equations in the presence of the modulation.

III. EQUATIONS OF MOTION UNDER MODULATION

The effect of temperature modulation on the Rayleigh number for onset of convective instability can be substantial. However, its sign (i.e., whether it will increase or decrease the Rayleigh number) is not intuitively obvious. In particular, if the system admits an oscillatory instability there is the interesting possibility of a resonance. In the double-diffusive systems, where both stationary and oscillatory instabilities are possible, the modulation therefore leads to a rich stability pattern. We proceed to analyze the effect by first establishing the equations of motion, i.e., the analogs of Eqs. (2.12), (2.13), and (2.15) under temperature modulation. The lower plate temperature T_L is considered to be modulated with frequency ω as

$$T_L=T_1+\epsilon\,\Delta T\cos(\omega t)\,, \tag{3.1}$$

where ϵ is a small parameter and $\Delta T=T_1-T_2$, T_2 being the temperature of the upper plate. To obtain the steady-state ($\vec{v}=0$) temperature and concentration profiles in this case, we need to solve Eqs. (2.1) and (2.2) under this condition and with the constraint of Eq. (2.10). Straightforward algebra yields the profile for T to be

$$T=T_1+(T_2-T_1)\frac{z}{D}+\mathrm{Re}\left[A\frac{\sinh[(i\omega/D_1)^{1/2}(d-z)]}{\sinh[(i\omega/D_1)^{1/2}d]}+B\frac{\sinh[(i\omega/D_2)^{1/2}(d-z)]}{\sinh[(i\omega/D_2)^{1/2}d]}\right]e^{i\omega t}\epsilon(\Delta T)\,, \tag{3.2}$$

where

$$D_{1,2}=\tfrac{1}{2}\left[\lambda+D\pm\left[(\lambda-D)^2+\frac{4D^2k_T^2}{\chi C_PT}\right]^{1/2}\right], \tag{3.3a}$$

$$A=\frac{1}{D_1-D_2}\left[\frac{Dk_T^2}{\chi C_PT}-D_2+\lambda\right], \tag{3.3b}$$

$$B=\frac{1}{D_1-D_2}\left[D_1-\lambda-\frac{Dk_T^2}{\chi C_PT}\right]. \tag{3.3c}$$

This leads to the gradient

$$|\vec{\nabla}T|=\frac{T_2-T_1}{d}-\epsilon(\Delta T)f(z,t)/d\,, \tag{3.4}$$

with

$$f(z,t)=\mathrm{Re}\left[A\alpha_1d\frac{\cosh[\alpha_1(d-z)]}{\sinh(\alpha_1d)}+B\alpha_2d\frac{\cosh[\alpha_2(d-z)]}{\sinh(\alpha_2d)}\right]e^{i\omega t}. \tag{3.5}$$

Equations (2.12) and (2.13) for the fluctuating quantities $\delta T(r)$ and $\delta c(r)$ now take the form

$$\left[\frac{\partial}{\partial t}-\lambda\nabla^2\right]\delta T=\frac{\Delta T}{d}[1+\epsilon f(z,t)]w+\frac{Dk_T}{\chi C_P}\nabla^2\delta c\,,$$
$$\left[\frac{\partial}{\partial t}-D\nabla^2\right]c=-\frac{k_T}{T}\frac{\Delta T}{d}[1+\epsilon f(z,t)]w+\frac{Dk_T}{T}\nabla^2\delta T\,. \tag{3.6}$$

Eliminating δT and δc from Eqs. (2.11) and (3.6) we now obtain the equation of motion for $w(r)$ as

$$Lw=\epsilon RG(fw)\,, \tag{3.7}$$

where L is the operator defined in Eq. (2.16) and

$$G=\nabla_1^2\left[S_1\frac{\partial}{\partial\tau}-\nabla^2\right]-\frac{\beta}{\alpha}\frac{k_T}{T}\frac{\lambda}{D}\nabla_1^2\left[\sigma\frac{\partial}{\partial\tau}-\nabla^2\right]-\frac{k_T^2}{\chi C_P T}\nabla_1^2\nabla^2+\frac{\beta}{\alpha}\frac{k_T}{T}\nabla_1^2\nabla^2\,. \tag{3.8}$$

To proceed further we expand the relevant quantities w and R in powers of ϵ to

$$w=w_0+\epsilon w_1+\epsilon^2 w_2+\cdots\,,\qquad R=R_0+\epsilon R_1+\epsilon^2 R_2+\cdots\,. \tag{3.9}$$

We note that the odd corrections $R_1, R_3, \ldots$ in R must vanish. This follows from the fact that correction to Rayleigh number should not depend on the phase of the modulation and, hence, on the sign of ϵ. This argument breaks down, however, at the parametric resonance at $\omega=2\omega_0$, which is analyzed separately. Inserting the expansion (3.9) into Eq. (3.7) and equating equal powers of ϵ, we obtain

$$L_0 w_0=0\,, \tag{3.10}$$

$$L_0 w_1=R_0 G(fw_0)\,, \tag{3.11}$$

$$L_0 w_2=R_2 G w_0+R_0 G(fw_1)\,. \tag{3.12}$$

Here L_0 is the operator L of Eq. (2.16) with $R=R_0$. Integration of the above set of equations will yield the corrections to the Rayleigh number R_0 which marks the onset of convection in the absence of modulation.

Before ending this section, we introduce for later use the following Fourier transform for $f(z,t)w_0$:

$$f(z,t)w_0=\sum_{n=1}^{\infty}a_n\sin(n\pi z/d)e^{i(\vec{k}\cdot\vec{r}+\omega t)}\,, \tag{3.13}$$

for static instability, where

$$a_n=4\pi^2 n\left[A\frac{(\alpha_1 d)^2}{(n-1)^2\pi^2+\alpha_1^2 d^2}\,\frac{1}{(n+1)^2\pi^2+\alpha_1^2 d^2}+B(\alpha_1\to\alpha_2)\right]. \tag{3.14}$$

Note that

$$a_1=4\pi^2\left[\frac{A}{4\pi^2+\alpha_1^2 d^2}+\frac{B}{4\pi^2+\alpha_2^2 d^2}\right], \tag{3.15}$$

and in the limit of low frequencies, α_1 and α_2 tend to approach zero and $a_1\to 1$. For oscillatory instability we replace ω on the right-hand side of Eq. (3.13) by $\omega\pm\omega_0$, as the case may be. We have normalized the amplitude of w_0 to unity as it drops out of the subsequent analysis.

IV. RESULTS FOR R_2

A. Stationary instability

In this case the solution w_0 is time independent. From Eq. (3.11) it is then apparent that the time dependence of w_1 is sinusoidal with frequency ω. The right-hand side of Eq. (3.12) now consists of a time-independent part and a part with frequency 2ω. The time-independent part on the (rhs) of Eq. (3.12) (which determines w_2) will lead to a singularity in w_2 since $Lw_0=0$ has a time-independent solution. This is the well-known secular term in perturbation theory,[21] and we remove it by choosing R_2 such that the time-averaged part of the rhs of Eq. (3.12) is 0. This yields

$$\begin{aligned}R_2\langle w_0\,|\,Gw_0\rangle&=-\tfrac{1}{2}R_0\langle w_0\,|\,G(fw_1)\rangle\\&=-\tfrac{1}{2}R_0\langle f^*G^\dagger w_0\,|\,w_1\rangle\\&=-\tfrac{1}{2}R_0^2\langle f^*G^\dagger w_0\,|\,Gfw_0/L\rangle\,,\end{aligned} \tag{4.1}$$

where the angular brackets stand for the usual scalar product. The functions G and L are given by

$$\begin{aligned}G(\omega)&=-K^2\left\{i\omega\left[S_1-\frac{\beta}{\alpha}\frac{k_T}{D}\frac{\lambda}{D}\sigma\right]+N\left[1+\frac{k_T^2}{\chi C_P T}-\frac{\beta}{\alpha}\frac{k_T}{T}\left[1+\frac{\lambda}{D}\right]\right]\right\}\\&=-K^2(G_1 i\omega+G_2)\,,\end{aligned} \tag{4.2}$$

and

$$L(\omega)=L_1(\omega)+iL_2(\omega)\,, \tag{4.3}$$

with

$$L_1(\omega)=N\omega^2(\sigma S_1+\sigma+S_1)-N^4\left[1-\frac{D}{\lambda}\frac{k_T^2}{\chi C_P T}\right]+R_0K^2N\left[1+\frac{k_T^2}{\chi C_P T}-\frac{\beta k_T}{T\alpha}\left[1+\frac{D}{\lambda}\right]\right], \tag{4.4}$$

$$L_2(\omega)=N\omega^3\sigma S_1-N^3\omega\left[1-\frac{D}{\lambda}\frac{k_T^2}{\chi C_P T}+\sigma+S_1\right]+\omega R_0K^2\left[S_1-\frac{\beta}{\alpha}\frac{k_T}{T}\frac{\lambda}{D}\sigma\right]. \tag{4.5}$$

We now obtain the correction R_2 as

$$\begin{aligned}R_2&=-\frac{R_0^2}{4}\mathrm{Re}\left[\sum_n\frac{|a_n|^2G(\omega)}{L_1(\omega)+iL_2(\omega)}\right]\\&=\frac{R_0^2K^2}{4}\sum_n\frac{|a_n|^2}{L_1^2+L_2^2}(G_2L_1-\omega L_2G_1)\,.\end{aligned} \tag{4.6}$$

The K appearing in the above equation is to be evaluated at its optimum value for the instability at R_0. Approximating Eq. (4.6) by the leading term ($n=1$) we obtain in the limit of zero frequency

$$R_2(\omega=0)=\frac{\frac{R_0^2K^2}{4}\left\{G_2(\sigma S_1+\sigma+S_1)-G_1\left[R_0K^2\left[S_1-\frac{\beta}{\alpha}\frac{k_T}{T}\frac{\lambda\sigma}{D}\right]-\left[1-\frac{D}{\lambda}\frac{k_T^2}{\chi C_P T}+\sigma+S_1\right]\right]\right\}}{\left[R_0K^2\left[S_1-\frac{\beta}{\alpha}\frac{k_T}{T}\frac{\lambda}{D}\sigma\right]-\left[1-\frac{D}{\lambda}\frac{k_T^2}{\chi C_P T}+\sigma+S_1\right]\right]^2} . \quad (4.7)$$

The interesting thing is the versatility of this expression since all the different quantities σ, s_1, λ, D, k_T, and α have strong temperature dependences as the temperature is varied near the λ point. Note that both the numerator and denominator are capable of having zeros. The numerator thus can be both positive and negative. Hence, at low frequencies one can expect stabilization or destabilization depending on the mean temperature of the system. The possibility of a zero in the denominator indicates that these stability effects can be made very pronounced by suitably varying temperature. At high frequencies, as is obvious from Eq. (4.6), $R_2\to 0$, which is what one would expect.

B. Oscillatory instability

We first treat the nonresonant situation $\omega\neq 2\omega_0$. For an oscillatory instability the solution w_0 has time dependence of the form $\cos(\omega_0 t)$, and hence from Eq. (3.11) w_1 will have the two frequencies $\omega\pm\omega_0$. The right-hand side of Eq. (3.12) now consists of two parts: one with frequency ω_0 and the other with frequency $2\omega\pm\omega_0$. The part with frequency ω_0 will cause a spurious resonance in the determination of w_2 since $Lw_0=0$ has a solution with frequency ω_0. This is remedied by choosing $R_2=\widetilde{R}_2$ such that the coefficient of $e^{i\omega_0 t}$ on the rhs of Eq. (3.12) vanishes. The analog of Eq. (4.6) now becomes

$$\widetilde{R}_2=\frac{R_0^2}{8}K^2\sum_n\frac{|a_n|^2[G_2(\omega-\omega_0)L_1(\omega-\omega_0)-(\omega-\omega_0)L_2(\omega-\omega_0)G_1(\omega-\omega_0)]}{L_1^2(\omega-\omega_0)+L_2^2(\omega-\omega_0)}+\{\omega\to-\omega\} . \quad (4.8)$$

Keeping only the first term in the expansion, we find after straightforward algebra

$$\widetilde{R}_2\propto[(\omega-\omega_0)^2-\omega_0^2]^{-1} . \quad (4.9)$$

Thus we have a parametric resonance at $\omega=2\omega_0$. This is probably the most striking feature of a double-diffusive system under parametric modulation of the temperature difference. The infinity in the above expression at $\omega=2\omega_0$ is an artifact of dropping the first correction, R_1. The parametric resonance produces a correction at $O(\epsilon)$. Our previous argument for $R_1=0$ is not valid at $\omega=2\omega_0$, and Eq. (3.11) must be modified to

$$L_0w_1=R_0G(fw_0)+R_1Gw_0 . \quad (4.10)$$

Since $w_0\sim\cos(\omega_0 t)$ and $f\sim e^{2i\omega_0 t}$, the right-hand side has the frequencies ω_0 and $3\omega_0$. The frequency ω_0 gives the secular term which now must be removed by choice of R_1. This yields

$$R_1=-\tfrac{1}{2}R_0\mathrm{Re}\left[\frac{\langle w_0|Gfw_0\rangle}{\langle w_0|Gw_0\rangle}\right] , \quad (4.11)$$

where the factor $\frac{1}{2}$ comes from time-averaging. The integration over z yields

$$R_1=-R_0\mathrm{Re}(a_1/2)=-\frac{R_0}{2}\left[\frac{A}{1+\omega_0^2d^4/4\pi^2D_1^2}+\frac{B}{1+\omega_0^2d^4/4\pi^2D_2^2}\right] . \quad (4.12)$$

Thus for the resonant case at $\omega=2\omega_0$, the correction occurs at $O(\epsilon)$. When one takes into account this nonvanishing R_1, the divergence at $O(\epsilon^2)$ for $\omega=2\omega_0$ in Eq. (4.9) is removed.[16] The width of the resonance region is of $O(\epsilon\omega_0)$.[16] This concludes our discussion of the stability pattern.

IV. SUMMARY

We have derived analytic expressions for the critical Rayleigh number for the double-diffusive convection in ^{3}He-^{4}He mixtures by using the idealized free boundary conditions. Although this makes the results lose some quantitative accuracy, the qualitative features should be reliable and seem to agree with the experiments of Lee *et al.*[12] The onset of convection is then studied by modulating sinusoidally the temperature of the lower plate. The first nonvanishing correction to the Rayleigh number has been obtained for both stationary and oscillatory instabilities. For stationary instability the correction can be positive or negative, depending upon the mean temperature of the system which determines the values of the various transport coefficients. For the oscillatory instability parametric resonance occurs for $\omega=2\omega_0$, where ω_0 is the frequency of the oscillation of the velocity and temperature fields. It should be possible to carry out experimental studies on the system and observe the various stabilization and destabilization patterns.

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